Pseudo-Anosov flows and incompressible tori

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Abstract. We study incompressible tori in 3-manifolds supporting pseudo-Anosov flows and more generally $\mathbf{Z}\oplus\mathbf{Z}$ subgroups of the fundamental group of such a manifold. If no element in this subgroup can be represented by a closed orbit of the pseudo-Anosov flow, we prove that the flow is topologically conjugate to a suspension of an Anosov diffeomorphism of the torus. In particular it is non singular and is an Anosov flow. It follows that either a pseudo-Anosov flow is topologically conjugate to a suspension Anosov flow, or any immersed incompressible torus can be realized as a free homotopy from a closed orbit of the flow to itself. The key tool is an analysis of group actions on non Hausdorff trees, also known as $\mathbf{R}$-order trees $-$ we produce an invariant axis in the free action case. An application of these results is the following: suppose the manifold has an $\mathbf{R}$-covered foliation transverse to a pseudo-Anosov flow. If the flow is not an $\mathbf{R}$-covered Anosov flow, then it follows that the manifold is atoroidal.

1 Introduction

This paper deals with the relationship between pseudo-Anosov flows in 3-manifolds and incompressible tori. Roughly a pseudo-Anosov flow is one that has finitely many singular closed orbits with p-prong type and has Anosov (or hyperbolic) behavior everywhere else, see detailed definition in section 2. Here we include the case without singularities, namely an Anosov flow [An, An-Si]. Another well known example of pseudo-Anosov flow is the suspension of a pseudo-Anosov homeomorphism of a closed surface of genus ≥ 2 [Th2, FLP, Ca-Th]. It turns out that pseudo-Anosov flows are very common: any closed, irreducible, atoroidal, orientable 3-manifold with non trivial second homology admits pseudo-Anosov flows [Mos2]. Also foliations coming from irreducible "slitherings" as defined by Thurston admit transverse pseudo-Anosov flows [Th4]. In fact every $\mathbf{R}$-covered foliation (defined below) in an aspherical, atoroidal manifold admits a transverse pseudo-Anosov flow [Fe8, Cal]. Finally pseudo-Anosov flows are ubiquitous because they survive under most Dehn surgeries on closed orbits of the flow [Fr] and also after branched covers on closed orbits. Notice that there is no known example of a closed hyperbolic 3-manifold which does not admit a pseudo-Anosov flow.

In 3-manifold theory it is extremely important to understand the prime and torus decompositions of a manifold [He, Ja-Sh, Jo]. Manifolds that are prime and atoroidal are the building blocks of all 3-manifolds. When the manifold supports a pseudo-Anosov flow, then in the universal cover the lifted flow has orbit space homeomorphic to the plane [Fe-Mo] and all orbits are homeomorphic to the real line [Mos1]. Hence the universal cover is a line bundle over the plane and is homeomorphic to $\mathbf{R}^3$. Consequently the manifold is irreducible, that is, every embedded sphere bounds a ball [He]. Therefore the manifold itself is the only prime factor.

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On the other hand many manifolds supporting pseudo-Anosov flows contain incompressible tori. For example if the manifold supports a suspension Anosov flow, then the fiber of a fibration of the manifold over the circle is an incompressible torus which is in fact transverse to the flow. In the case of geodesic flows in the unit tangent bundle of a closed surface of negative curvature (hereby called geodesic flows), there are many incompressible tori: just take a closed geodesic in the surface and rotate the tangent vectors to it by 2π degrees to produce a free homotopy from the orbit to itself and an incompressible torus. Furthermore any intransitive pseudo-Anosov flow has a transverse torus [Sm, Mos1] which is then incompressible [Mos1] - see the examples of intransitive Anosov flows constructed by Franks and Williams [Fr-Wi]. Finally there are many classes of transitive Anosov flows, which are not topologically conjugate to suspensions, but which admit incompressible transverse tori [Bo-La, Br, Ba5]. One might argue that many of the recent constructions of pseudo-Anosov flows in manifolds start with an atoroidal manifold in order to produce the pseudo-Anosov flow [Mos2, Th4, Fe8, Cal]. However as we mentioned before pseudo-Anosov flows survive after most Dehn surgeries on closed orbits [Fr] and also after branched coverings $-$ but the resulting manifold may be toroidal. For example consider Anosov flows $-$ there are many examples in atoroidal and even hyperbolic manifolds [Go]. Any such flow has a surface of section [Fr] and after finitely many Dehn surgeries on closed orbits it becomes a suspension Anosov flow, resulting in a toroidal manifold. Therefore pseudo-Anosov flows and incompressible tori can coexist quite generally.

It is therefore very important to understand how pseudo-Anosov flows interact with incompressible tori. First consider the (much simpler) smooth setting. In that case the study of incompressible tori and Anosov flows was previously done in [Ba1, Ba3, Fe4].

The purpose of this article is to analyse the situation for general pseudo-Anosov flows. We explain below how this differs from the Anosov case. Here we also consider immersed π_1-injective tori. Consider the rank two free abelian subgroup $\mathbf{A} \cong \mathbf{Z} \oplus \mathbf{Z}$ associated to the fundamental group of the π_1-injective torus. This acts in the universal cover of the manifold by covering translations and so acts in the orbit space of the lifted flow. As explained above this orbit space is homeomorphic to the plane $\mathbf{R}^2$ and is denoted by $\mathcal{O}$. Suppose first that some non trivial element of $\mathbf{A}$ does not act freely in $\mathcal{O}$. Then this element of $\mathbf{A}$ leaves invariant an orbit of the lifted flow acting by translations and is therefore associated to a closed orbit of the flow in the manifold. In this case we show that the torus associated to $\mathbf{A}$ can be put in the form of a free homotopy from this closed orbit to itself. The remaining case in the analysis of $\mathbf{Z} \oplus \mathbf{Z}$ actions in $\mathcal{O}$ is that $\mathbf{A}$ acts freely in $\mathcal{O}$ (except for the identity element in $\mathbf{A}$). The stable foliation of the flow lifts to a foliation in the universal cover with leaf space denoted by $\mathcal{H}^s$ and similarly define $\mathcal{H}^u$. The space $\mathcal{H}^s$ is a 1-dimensional object which is simply connected, usually not Hausdorff [Fe5]. In addition because of the singularities, $\mathcal{H}^s$ may have non manifold points too - with tree like behavior near the singular points. Let g be a non trivial element of $\mathbf{A}$ $-$ then g acts freely in $\mathcal{H}^s$. We look for an axis of this action. This is a very natural point of view, because whenever a homeomorphism acts freely on a simply connected 1-dimensional manifold or an $\mathbf{R}$-tree one looks for an axis of the action [Gh, Ba1, MS1], with many important consequences. Hence we study the action of g on $\mathcal{H}^s$. This is more difficult because $\mathcal{H}^s$ really is neither a manifold nor a tree. This hybrid object we will call here a *non Hausdorff tree*. For example one of main complications introduced by the singularities is that usually $\mathcal{H}^s$ is not orientable - which occurs if and only if there are singularities of Φ with an odd number of prongs. The analysis of actions on simply connected 1-manifolds [Ba1, Ba3] does not work here $-$ in fact some previous properties are not true in the more general setting: some of the many equivalent definitions of an invariant axis in [Ba3, Ba4] are not equivalent in general and do not work, see section 3. Notice that Barbot [Ba3, Ba4] assumes that not only g acts freely in $\mathcal{H}^s$ but also that it separates points $-$ which is relevant as $\mathcal{H}^s$ may be non Hausdorff. We do not assume that g separates points, only that it acts freely $-$ so our analysis gives new information even in the case of action on simply connected

1-manifolds. One cannot apply the results of group actions on $\mathbf{R}$-trees [MS1] either because the leaf spaces usually are not Hausdorff. In fact the $\mathbf{R}$-tree case is quite simple compared to the general case. In section 3 of this paper we give a natural definition for the invariant axis in the general case and show they exist:

Theorem A - Let $\mathcal{H}$ be a non Hausdorff tree and let g be a homeomorphism of $\mathcal{H}$ without fixed points. Then g has a non empty invariant axis $\mathcal{A}$ where it acts by translations.

Showing that the invariant axis is non empty in general turns out to be very subtle and involves a substantial part of this article. We mention that non Hausdorff trees were also considered in the context of essential laminations in 3-manifolds by Gabai and Kazez in [Ga-Ka]. They used the terminology *order trees*. Group actions in order trees are also studied by Roberts and Stein in [Ro-St].

Then we use the invariant axis to study $\mathbf{Z} \oplus \mathbf{Z}$ actions on the leaf space $\mathcal{H}^s$. The axis has excellent properties which allow us to start the analysis when $\mathbf{A}$ acts freely in $\mathcal{H}^s$. Our main result is:

Main theorem. Let Φ be a pseudo-Anosov flow in M^3 closed and let $\mathbf{A}$ be a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$. If a non trivial element of $\mathbf{A}$ is associated to a closed orbit of Φ, then $\mathbf{A}$ can be geometrically represented as a free homotopy from this closed orbit to itself. Otherwise it follows that Φ is topologically conjugate to a suspension of an Anosov diffeomorphism of the torus and in particular it is non singular.

The topology of the pseudo-Anosov foliations $\mathcal{F}^s, \mathcal{F}^u$ as developed in [Fe5, Fe6] is fundamental for the proof of this result. Roughly the proof goes as follows: one uses the invariant axis for $\mathbf{A}$ acting on $\mathcal{H}^s, \mathcal{H}^u$ to construct the joint topological structure of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ in $\widetilde{M}$. The resulting topological picture can only occur for flows topologically conjugate to suspension Anosov flows.

We present one application of this theorem in the case the pseudo-Anosov flow Φ is transverse to a foliation $\mathcal{G}$. There are various constructions of pseudo-Anosov flows transverse to foliations: 1) flows transverse to fibrations with pseudo-Anosov monodromy [Th1, Th2, Th3], 2) finite depth foliations [Ga1, Ga2, Ga3, Mos2], 3) slitherings of M over S^1 - equivalently uniform foliations [Th4], 4) any $\mathbf{R}$-covered foliation in an aspherical, atoroidal manifold [Fe8]. Recall that a foliation $\mathcal{G}$ is $\mathbf{R}$-covered if the leaf space of the lifted foliation in the universal cover is Hausdorff and homeomorphic to the set of real numbers [Pl1, Fe1]. In 1), 3) and 4) above the foliations are $\mathbf{R}$-covered and in 2) they are usually not $\mathbf{R}$-covered. Our result helps to study general pseudo-Anosov flows transverse to $\mathbf{R}$-covered foliations. Recall also that an $\mathbf{R}$-covered Anosov flow is one for which the stable and unstable foliations are $\mathbf{R}$-covered [Ba2, Fe1].

Theorem B Let Φ be a pseudo-Anosov flow in M^3 closed. Suppose that Φ is transverse to an $\mathbf{R}$-covered foliation $\mathcal{G}$ and that Φ is not an $\mathbf{R}$-covered Anosov flow. Then M is atoroidal, that is, there are no $\mathbf{Z} \oplus \mathbf{Z}$ subgroups of $\pi_1(M)$.

Since M is atoroidal, this implies that Φ is transitive [Mos1]. In the proof of theorem B we need to use the results from [Fe7].

The paper is organized as follows: In the next section we review background material about pseudo-Anosov flows. Section 3 contains the study of group actions on non-Hausdorff trees and proves theorem A. The following section reviews the needed results about the topological structure of pseudo-Anosov flows. The main theorem is proved in sections 5 through 8. Theorem B is proved in section 9.

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2 Pseudo-Anosov flows

Pseudo-Anosov flows are a generalization of suspension flows of pseudo-Anosov surface homeomorphisms. These flows behave much like Anosov flows, but they may have finitely many singular orbits which are periodic and have a prescribed behavior. In order to define pseudo-Anosov flows, first we recall singularities of pseudo-Anosov surface homeomorphisms.

Given $n \geq 2$, the quadratic differential $z^{n-2}dz^2$ on the complex plane $\mathbf{C}$ (see [St] for quadratic differentials) has a horizontal singular foliation f^u with transverse measure μ^u, and a vertical singular foliation f^s with transverse measure μ^s. These foliations have n-pronged singularities at the origin, and are regular and transverse to each other at every other point of $\mathbf{C}$. Given $\lambda > 1$, there is a homeomorphism $\psi : \mathbf{C} \to \mathbf{C}$ which takes f^u and f^s to themselves, preserving the singular leaves, stretching the leaves of f^u and compressing the leaves of f^s by the factor λ. Let R_θ be the homeomorphism $z \to e^{2\pi\theta}z$ of $\mathbf{C}$. If $0 \leq k < n$ the map $R_{k/n} \circ \psi$ has a unique fixed point at the origin, and this defines the local model for a *pseudohyperbolic fixed point*, with n-prongs and rotation k. This map is everywhere smooth except at the origin. Let $d_{\mathbf{E}}$ be the singular Euclidean metric on $\mathbf{C}$ associated to the quadratic differential $z^{n-2}dz^2$, given by

$$d_{\mathbf{E}}^2 = \mu_u^2 + \mu_s^2$$

Note that

$$(R_{k/n} \circ \psi)^* d_{\mathbf{E}}^2 = \lambda^{-2}\mu_u^2 + \lambda^2\mu_s^2$$

The mapping torus $N = \mathbf{C} \times \mathbf{R}/(z, r+1) \sim (R_{k/n} \circ \psi(z), r)$ has a suspension flow Ψ arising from the flow in the $\mathbf{R}$ direction on $\mathbf{C} \times \mathbf{R}$. The suspension of the origin defines a periodic orbit γ in N, and we say that (N, γ) is the local model for a *pseudohyperbolic periodic orbit*, with n prongs and with rotation k. The suspension of the foliations f^s, f^u define 2-dimensional foliations on N, singular along γ, called the *local weak stable and unstable foliations.*

Note that there is a singular Riemannian metric ds on $\mathbf{C} \times \mathbf{R}$ that is preserved by the gluing homeomorphism $(z, r+1) \sim (R_{k/n} \circ \psi(z), r)$, given by the formula

$$ds^2 = \lambda^{-2t}\mu_u^2 + \lambda^{2t}\mu_s^2 + dt^s$$

The metric ds descends to a metric on N denoted ds_N.

Let Φ be a flow on a closed, oriented 3-manifold M. We say that Φ is a *pseudo-Anosov flow* if the following are satisfied:

- For each $x \in M$, the flow line $t \to \Phi(x, t)$ is C^1, it is not a single point, and the tangent vector bundle $D_t\Phi$ is C^0.

- There is a finite number of periodic orbits $\{\gamma_i\}$, called *singular orbits*, such that the flow is smooth off of the singular orbits.

- Each singular orbit γ_i is locally modelled on a pseudo-hyperbolic periodic orbit. More precisely, there exist n, k with $n \geq 3$ and $0 \leq k < n$, such that if (N, γ) is the local model for an pseudo-hyperbolic periodic orbit with n prongs and with rotation k, then there are neighborhoods U of γ in N and U_i of γ_i in M, and a diffeomorphism $f \colon U \to U_i$, such that f takes orbits of the semiflow $R_{k/n} \circ \psi \mid U$ to orbits of $\Phi \mid U_i$.

- There exists a path metric d_M on M, such that d_M is a smooth Riemannian metric off of the singular orbits, and for a neighborhood U_i of a singular orbit γ_i as above, the derivative of the map $f \colon (U - \gamma) \to (U_i - \gamma_i)$ has bounded norm, where the norm is measured using the metrics ds_N on U and d_M on U_i.

- On $M - \bigcup \gamma_i$, there is a continuous splitting of the tangent bundle into three 1-dimensional line bundles $E^u \oplus E^s \oplus T\Phi$, each invariant under Φ, such that $T\Phi$ is tangent to flow lines, and for some constants $\nu > 1, \theta > 1$ we have

1. If $v \in E^u$ then $|D\Phi_t(v)| \leq \theta \nu^t |v|$ for $t < 0$
2. If $v \in E^s$ then $|D\Phi_t(v)| \leq \theta \nu^{-t} |v|$ for $t > 0$

where norms of tangent vectors are measured using the metric d_M.

- In a neighborhood U_i of a singular orbit γ_i as above, $Df(E^s)$ is tangent to the local weak stable foliation and similarly for $Df(E^u)$.

With this definition, pseudo-Anosov flows are a generalization of Anosov flows in 3-manifolds [An, An-Si]. The entire theory of Anosov flows can be mimicked for pseudo-Anosov flows [Mos1, Mos2]. In particular, a pseudo-Anosov flow Φ has a singular 2-dimensional weak unstable foliation $\mathcal{F}^u$ which is tangent to $E^u \oplus T\Phi$ away from the singular orbits. A complete leaf of this foliation is called a *regular* leaf of $\mathcal{F}^u$. A non complete leaf can be completed by adding a singular orbit α. The union of α and the non complete leaves abutting α forms a *singular* leaf of $\mathcal{F}^u$ containing α. Similarly there is a 2-dimensional weak stable foliation $\mathcal{F}^s$ tangent to $E^s \oplus T\Phi$. These foliations are singular along the singular orbits of Φ, and regular everywhere else. In the neighborhood U_i of an n-pronged singular orbit γ_i, the images of $\mathcal{F}^s$ and $\mathcal{F}^u$ in the model manifold N are identical with the local weak stable and unstable foliations.

The pseudo-Anosov flow also has singular 1-dimensional strong foliations $\mathcal{F}^{ss}, \mathcal{F}^{uu}$. Outside the singular orbits, leaves of $\mathcal{F}^{ss}$ are obtained by integrating E^s. If $x \in \alpha$ and α is a singular orbit of Φ then in the local model $N = \mathbf{C} \times \mathbf{R}/\sim$, the point x corresponds to (O, t), where O is the origin in $\mathbf{C}$. Then $W^{ss}_{loc}(x)$ is $\zeta \times \{t\}$, where ζ is the singular leaf of f^s (which contains O). The $\{W^{ss}_{loc}(x)\}$, x in singular orbit glue up with the leaves of $\mathcal{F}^{ss}$ outside singular orbits to form a singular foliation $\mathcal{F}^{ss}$. The foliation $\mathcal{F}^{ss}$ is flow invariant, that is, for any leaf ζ_1 of $\mathcal{F}^{ss}$ and any real t, $\Phi_t(\zeta_1)$ is a leaf of $\mathcal{F}^{ss}$. Furthermore for $t > 0$ Φ_t exponentially contracts distances along leaves of $\mathcal{F}^{ss}$. Similarly for $\mathcal{F}^{uu}$.

Notation/definition: The discussion above applies equally well to the lifted singular foliations $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u, \widetilde{\mathcal{F}}^{ss}, \widetilde{\mathcal{F}}^{uu}$ in $\widetilde{M}$. If $x \in M$ let $W^s(x)$ denote the leaf of $\mathcal{F}^s$ containing x. Similarly one defines $W^u(x), W^{ss}(x), W^{uu}(x)$ and in the universal cover $\widetilde{W}^s(x), \widetilde{W}^u(x), \widetilde{W}^{ss}(x), \widetilde{W}^{uu}(x)$. Similarly if α is an orbit of Φ define $W^s(\alpha)$, etc... Let also $\widetilde{\Phi}$ be the lifted flow to $\widetilde{M}$.

In figure 1 we highlight the difference between non Hausdorff behavior in the leaf space of $\widetilde{\mathcal{F}}^s$ and the splitting (or branching) of leaves associated to singular orbits of $\widetilde{\Phi}$. In part (a) the leaves F, L of $\widetilde{\mathcal{F}}^s$ are not separated from each other in the leaf space of $\widetilde{\mathcal{F}}^s$. Notice that the sequence F_i converges to F and L. In fig 1 part (b) we sketch a singular leaf S with 3 prongs. Even though S separates $\widetilde{M}$ into 3 or more regions, non Hausdorffness is not involved. The leaves S_i converge only to S. In this article, except for the next section, all pictures of leaves of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ will describe them as subsets of $\widetilde{M}$, rather than in the leaf space of $\widetilde{\mathcal{F}}^s$.

3 Group actions on non Hausdorff trees

In this section we will study group actions on the leaf spaces $\mathcal{H}^s$ of $\widetilde{\mathcal{F}}^s$ and $\mathcal{H}^u$ of $\widetilde{\mathcal{F}}^u$. These leaf spaces are examples of what we call non Hausdorff trees, defined as follows. A *segment* is a set which admits a linear order making it isomorphic to an interval in $\mathbf{R}$: $[0,1], [0,1), (0,1)$ or $[0,0]$. This gives the type of the segment. Type $(0,1)$ is called an *open* segment and type $[0,0]$ is a *degenerate* segment. A *closed* segment is one of type either $[0,0]$ or $[0,1]$. A *half open* segment is one of type $[0,1)$.

Definition 3.1. *(non Hausdorff tree) A non Hausdorff tree is a set $\mathcal{H}$ satisfying:*

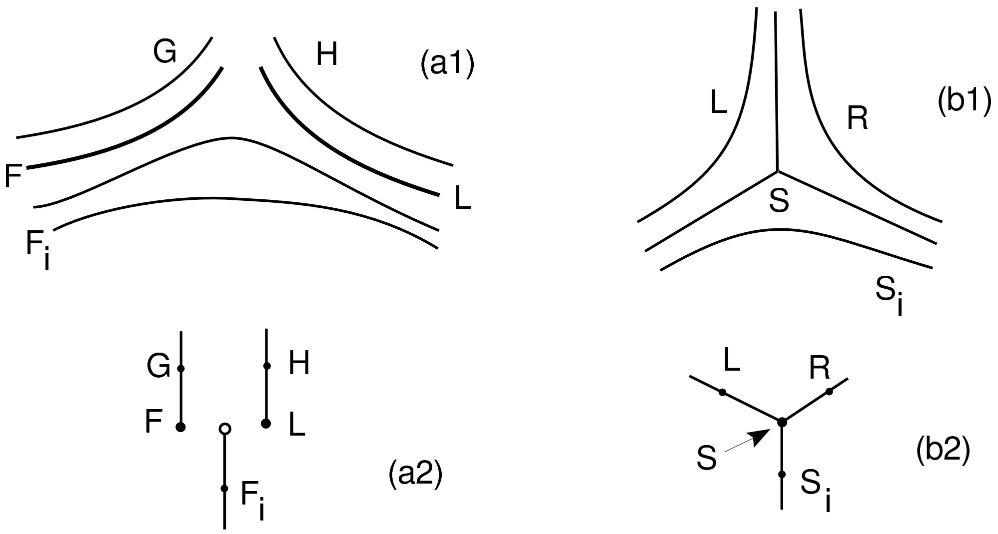

Figure 1: *a. Non Hausdorff behavior in the leaf space of $\widetilde{\mathcal{F}}^s$: (a1) F, L non separated from each other, as seen in $\widetilde{M}$, (a2) the corresponding picture in the leaf space $\mathcal{H}^s$; (b) A singular leaf of $\widetilde{\mathcal{F}}^s$: (b1) as seen in $\widetilde{M}$, (b2) as seen in $\mathcal{H}^s$.*

1) $\mathcal{H}$ is a union of open segments,

2) for each $x, y \in \mathcal{H}$, there is a finite chain of segments $I_1, ..., I_n$ with $x \in I_1, y \in I_n$ and $I_i \cap I_{i+1} \neq \emptyset$ for any $1 \leq i < n$,

3) for any $x \in \mathcal{H}$ and I_1, I_2 distinct prongs at x the following happens: Given $y_1 \in I_1 - \{x\}, y_2 \in I_2 - \{x\}$, then any finite chain of segments from y_1 to y_2 (as in (2) above) must contain x in at least one of the segments.

If I_1, I_2 are two segments with $I_1 \cap I_2$ a single point which is an endpoint of both I_1 and I_2, then $I_1 \cup I_2$ admits a natural linear order isomorphic to a segment in $\mathbf{R}$, hence we say that $I_1 \cup I_2$ is a segment. A *prong* at x is a segment I in $\mathcal{H}$ of type $[0,1)$ or $[0,1]$ with $x \in I$ corresponding to 0. Two prongs I_1, I_2 at x are distinct if $I_1 \cap I_2 = \{x\}$, or equivalently they do not share a subprong at x. Notice that a priori there may be infinitely or even uncountably many distinct prongs at x.

Definition 3.2. *(topology of $\mathcal{H}$) Let $\mathcal{H}$ be a non Hausdorff tree. Define the topology of $\mathcal{H}$ as follows: Let A be a subset of $\mathcal{H}$. Then A is open if for any $x \in A$ and any prong I at x, there is a subprong I' at x ($I' \subset I$) so that $I' \subset A$. Intuitively A contains all sufficiently small subprongs at x.*

Condition (2) means that $\mathcal{H}$ is arcwise connected. It follows from condition 3) that if I_1 and I_2 are two segments, then $I_1 \cap I_2$ is either empty or is a subsegment of both I_1, I_2. The intersection may be a degenerate segment, that is a point. Condition (3) is essentially saying that $\mathcal{H}$ is one dimensional and simply connected. Also (3) states that points completely separate $\mathcal{H}$.

A point $x \in \mathcal{H}$ is a *regular* if given any two open segments I_1, I_2 with $x \in I_1 \cap I_2$, then $I_1 \cap I_2$ is an open segment in $\mathcal{H}$. Otherwise x is *singular* and $\mathcal{H}$ is "treelike" in x. Equivalently a point is regular if there are only two distinct prongs at x, any third prong at x will share a non degenerate segment with one of first two prongs.

Definition 3.3. *(finite prong condition) A non Hausdorff tree satisfies the finite prong condition if for each $x \in \mathcal{H}$, there is an integer $p \geq 2$ so that there are at most p distinct prongs at x. If there are p distinct prongs at x then x is said to have p prongs.*

Non Hausdorff trees are hybrid generalizations of arcwise connected trees and (possibly non Hausdorff) simply connected one manifolds. There are many examples of non Hausdorff simply connected one dimensional manifolds coming from leaf spaces of stable foliations of Anosov flows [Fe5, Ba4, Ba5, Fr-Wi]. Also **R**-trees as defined by Morgan and Shalen [MS1] are examples of non Hausdorff trees. Non Hausdorff trees also occur naturally in the context of essential laminations in 3-manifolds, where they were called **R**-*order trees* or more generally *order trees* by Gabai and Oertel [Ga-Oe]. They are an important tool to produce a Palmeira theorem [Pa] for essential laminations of 3-manifolds [Ga-Oe, Ga-Ka] and to completely classify essential laminations of the plane [Ga-Ka]. More importantly for us, if Φ is a pseudo-Anosov flow in a closed 3-manifold, then the leaf spaces $\mathcal{H}^s, \mathcal{H}^u$ of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ are non Hausdorff trees. A regular leaf of $\widetilde{\mathcal{F}}^s$ corresponds to a regular point of $\mathcal{H}^s$ and a singular p-prong leaf of $\widetilde{\mathcal{F}}^s$ produces a point in $\mathcal{H}^s$ with p prongs, hence $\mathcal{H}^s$ and also $\mathcal{H}^u$ satisfy the finite prong condition.

Remark More generally one can define a segment to be just a linearly ordered set. This is the approach taken by Gabai-Kazez in [Ga-Ka] producing order trees. The results in this section work in the more general setting.

The reader should note that the local structure of non Hausdorff trees may be quite complex, even with the finite prong condition. For instance if x is a point where the finite prong condition holds, it does not follow a priori that x must have a neighborhood in $\mathcal{H}$ which is homeomorphic to a p-prong in the plane: even when two (non distinct) prongs ζ_1, ζ_2 at x share a subprong at x, the splitting point between ζ_1 and ζ_2 may be arbitrarily close to x. This is what happens for the leaf spaces of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$.

Unlike in trees, usually there is not a single path between points. This is depicted for instance in figure 1 a2: the points F, L are non separated from each other. There are many distinct paths from F to L, none of which is embedded.

For our results it will be fundamental to understand group actions on non Hausdorff trees. Group actions on simply connected one dimensional spaces have been widely studied and applicable:

- In the case of **R**-trees there is the work of Tits [Ti] and Morgan and Shalen [MS1]. This had deep applications to the study of 3-manifolds and showing the compactness of the space of hyperbolic structures in important settings [MS2, MS3].

- In the case of simply connected non Hausdorff one manifolds, group actions were analysed first by Ghys [Gh] who considered Anosov flows in Seifert spaces and analysed the corresponding space $\mathcal{H}^s$. In a seminal paper in the field, he showed that $\mathcal{H}^s$ is homeomorphic to **R** and the flow is topologically conjugate to a geodesic flow. This was extended by Barbot who used such group actions to analyse the structure of the torus decomposition with respect to Anosov flows [Ba1, Ba3] and derive important consequences in wide classes of 3-manifolds including graph manifolds [Ba4, Ba5]. Barbot also used this to study general codimension foliations in 3-manifolds [Ba6].

- Group actions in order trees are also studied by Roberts and Stein [Ro-St] in the context of essential laminations, with applications to actions on Seifert fibered spaces. There are additional conditions concerning separation of points.

We need to understand the structure of $\mathcal{H}$. Given $x \neq y$ then for any prong at y there is a subprong disjoint from x, hence contained in $\mathcal{H} - \{x\}$. It follows that $\mathcal{H} - \{x\}$ is an open set in $\mathcal{H}$ and therefore points are closed in $\mathcal{H}$, that is, $\mathcal{H}$ satisfies the T_1 property of topological spaces [Ke]. Notice that in general $\mathcal{H}$ does not satisfy the Hausdorff property $= T_2$ [Ke].

Given $x \in \mathcal{H}$ and I a prong at x let

$$A_I \ = \ \{ \ y \in \mathcal{H} - \{x\} \ \mid \ \text{there is a path} \ \ \gamma \subset \mathcal{H} - \{x\} \ \ \text{from } y \text{ to some point in } I \ \}.$$

Clearly A_I is arcwise connected. If I, J are prongs at x which share a subprong then it is easy to see that $A_I = A_J$. If I, J are distinct prongs at x then $I \cup J$ is a segment of $\mathcal{H}$ with x in the interior of the segment. If there is a path $\gamma \subset \mathcal{H} - \{x\}$ from some $y \in A_I$ to some $z \in A_J$ then one constructs a path γ contained in $\mathcal{H} - \{x\}$ from some $y' \in I$ to some $z' \in J$. This contradicts condition (3) of the definition of non Hausdorff tree. Hence $A_I \cap A_J = \emptyset$ and the collection

$$\{A_I\}, \ I \text{ distinct prongs at } x,$$

is the collection of arcwise connected components of $\mathcal{H} - \{x\}$.

In addition given $y \in A_I$ and J a prong at y, there is a subprong $J' \subset \mathcal{H} - \{x\}$. Clearly $J' \subset A_I$. This implies that A_I is open in $\mathcal{H}$ and so this collection is also the collection of connected components of $\mathcal{H} - \{x\}$. It follows that distinct prongs at x are in one to one correspondence with components of $\mathcal{H} - \{x\}$. For instance x has p prongs if and only if $\mathcal{H} - \{x\}$ has p components.

The following definitions will be necessary. Let $\mathcal{H}$ be a non Hausdorff tree. Given $x, y \in \mathcal{H}$ which are not separated from each other in $\mathcal{H}$ we write $x \approx y$.

One says that z *separates* x from y if x, y are in distinct components of $\mathcal{H} - \{z\}$.

Given any two $x, y \in \mathcal{H}$ there is a continuous path $\alpha(t), 0 \leq t \leq 1$ from x to y. Define

$$(x, y) \quad = \quad \{ \ z \in \mathcal{H} \ \mid \ z \text{ separates from } y \ \}$$

which we call the *open block* of $\mathcal{H}$ with endpoints x, y. Let

$$[x, y] \quad = \quad (x, y) \cup \{x\} \cup \{y\},$$

the *closed block* of $\mathcal{H}$ with endpoints x, y.

Lemma 3.4. *$[x, y]$ is the intersection of all continuous paths in $\mathcal{H}$ from x to y.*

Proof. Let B be the intersection of all paths from x to y. If $z \in [x, y]$ then clearly any path from x to y must contain z or else z does not separate x from y. Hence $z \in B$.

Conversely let $z \notin [x, y]$. Then $z \neq x, y$ and z does not separate x from y. Hence x and y are in the same component of $\mathcal{H} - \{z\}$. As seen above components of $\mathcal{H} - \{z\}$ are the same as arcwise components of $\mathcal{H} - \{z\}$, hence there is a path γ from x to y avoiding z. It follows that $z \notin B$. This finishes the proof. ◻

Remark - We use the notation $[x, y]$ for the closed block of $\mathcal{H}$ with endpoints x, y. When x, y are the endpoints of a segment I of $\mathcal{H}$, the notation $[x, y]$ also suggests the segment I from x to y (there is a unique such segment). In fact I and $[x, y]$ are the same: First, by definition of non Hausdorff tree, any $z \in I - \{x, y\}$ separates x from y hence $z \in (x, y)$. This shows that $I \subset [x, y]$. On the other hand I is a path in $\mathcal{H}$ from x to y, so by the previous lemma $[x, y] \subset I$ and consequently $I = [x, y]$. So the notation $[x, y]$ matches with the established convention of segments between points, whenever they are connected by a segment. We will also use the notation $(x, y]$ for half open segments.

As $\mathcal{H}$ may not be Hausdorff it may be that $[x, y]$ is not connected. It turns out that $[x, y]$ is a union of finitely many closed segments of $\mathcal{H}$ homeomorphic to either $[0, 0]$ or $[0, 1]$:

Lemma 3.5. *For any $x, y \in \mathcal{H}$ then there are $x_i, y_i \in \mathcal{H}$ with:*

$$[x, y] \quad = \quad \bigcup_{i=1}^{n} \, [x_i, y_i], \quad x_1 = x, \ \ y_n = y,$$

a disjoint union, where $[x_i, y_i]$ are segments in $\mathcal{H}$. In addition $y_i \approx x_{i+1}$ for any $1 \leq i \leq n-1$, see fig. 2. Notice that some or all segments $[x_i, y_i]$ may be degenerate, that is, points.

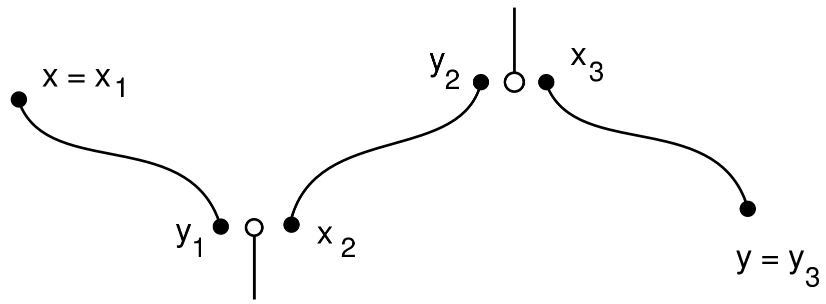

Figure 2: *Interval of leaves between any two leaves. For simplicity we describe the intervals $[x_i, y_i]$ without singularities. The points x_{i+1} and y_i are non separated from each other.*

Usually there may be many singular points z in the block $[x, y]$. If such a singular point z is in the interior of a segment in $[x, y]$, then the block $[x, y]$ will pick two prongs at z.

Proof. Recall that $\mathcal{H}$ is arcwise connected. Given $x, y \in \mathcal{H}$ let

$$\mathcal{I} \;=\; \{\; I_k = [z_k, w_k] \;\}, 1 \leq k \leq n, \quad z_1 = x, \;\; w_n = y \;\; \text{and} \;\; w_k = z_{k+1}, 1 \leq k < n,$$

be a chain of segments $[z_k, w_k]$ from x to y. Assume that $\mathcal{I}$ has the minimum number of segments among all such chains from x to y.

If $n = 1$ there is a segment from x to y and clearly this is $[x, y]$. Otherwise consider $I_1 \cap I_2$ which is a subsegment of I_1 and I_2 which contains $w_1 = z_2$. Considering $I_1 \cap I_2$ as a subsegment of I_1 there is $u_1 \in I_1$ so that the intersection is either $(u_1, w_1]$ or $[u_1, w_1]$.

Suppose first that $I_1 \cap I_2 = [u_1, w_1]$. In this case let J_1 be the closed subsegment of I_1 from z_1 to u_1 and J_2 the closed subsegment of I_2 from u_1 to w_2, see fig. 3 a.

By construction $J_1 \cap J_2 = u_1$ which is an endpoint of both J_1 and J_2. Therefore $J_1 \cup J_2$ is a segment from z_1 to w_2. Then $J_1 \cup J_2, I_3, ..., I_n$ is a chain from x to y with fewer segments than $\mathcal{I}$, contradiction to hypothesis. We conclude that $I_1 \cap I_2 = (u_1, w_1]$.

In the same way there is $u_2 \in I_2$ with $I_1 \cap I_2 = (u_2, w_2]$ as a subsegment of I_2. It now follows that u_1 and u_2 are not separated from each other (see fig. 3 b) because there are $v_i \in I_1 \cap I_2$ with $v_i \to u_1$ and $v_i \to u_2$ also. Since I_2 is a segment in $\mathcal{H}$ then $u_1 \notin I_2$ because $u_1 \approx u_2$ and $u_2 \in I_2$. If $u_1 \in I_k$ for some $k \geq 3$ then one could decrease the number of segments from the chain, contradiction. Hence $u_1 \notin \cup_{k \geq 2} I_k$ and this union is contained in a component of $\mathcal{H} - \{u_1\}$ different from the one containing z_1. It follows that u_1 separates $x = z_1$ from $y = w_n$. If $t \in (z_1, u_1)$ then u_1 and z_1 are in different components of $\mathcal{H} - \{t\}$. If t does not separate z_1 from y then y and z_1 are in the same component of $\mathcal{H} - \{t\}$. As u_1 is in another component of $\mathcal{H} - \{t\}$, then u_1 would not separate z_1 from y, contradiction. We conclude that $(z_1, u_1) \subset (x, y)$ and so $[z_1, u_1] \subset [x, y]$.

Notice that given a point $t \in I_1 - [z_1, u_1]$ we can pull the point $w_1 = z_2$ closer to u_1 in I_1 and closer to u_2 in I_2 producing a path from x to y not containing t. Hence $t \notin (x, y)$ and $I_1 - [z_1, w_1]$ is disjoint from $[x, y]$. Hence $[x, y] \cap [z_1, w_1] = [z_1, u_1]$. Let $x_1 = z_1, y_1 = u_1$.

Similarly the subsegment $[z_2, u_2)$ of I_2 is disjoint from $[x, y]$. If $u_2 = y$ then the previous lemma implies that $[x, y] = [x_1, u_1] \cup \{y\}$. In that case let $x_2 = y_2 = y$ and $n = 2$ finishing the proof.

Otherwise $u_2 \neq y$. We claim that u_2 separates x from y. Suppose that is not true. Let W be the component of $\mathcal{H} - \{u_2\}$ containing x, y. The path $\cup_{k \geq 3} I_k$ starts in w_2 and goes to y in W. But w_2 is either u_2 or is in another component W' of $\mathcal{H} - \{u_2\}$. In particular $n \geq 3$. Also u_2 completely separates $\mathcal{H}$. Therefore $\cup_{k \geq 3} I_k$ has one segment which contains a prong J at u_2, so that this prong defines the component W of $\mathcal{H} - \{u_2\}$. Since u_1 is in W and u_1 is not separated from u_2, the prong

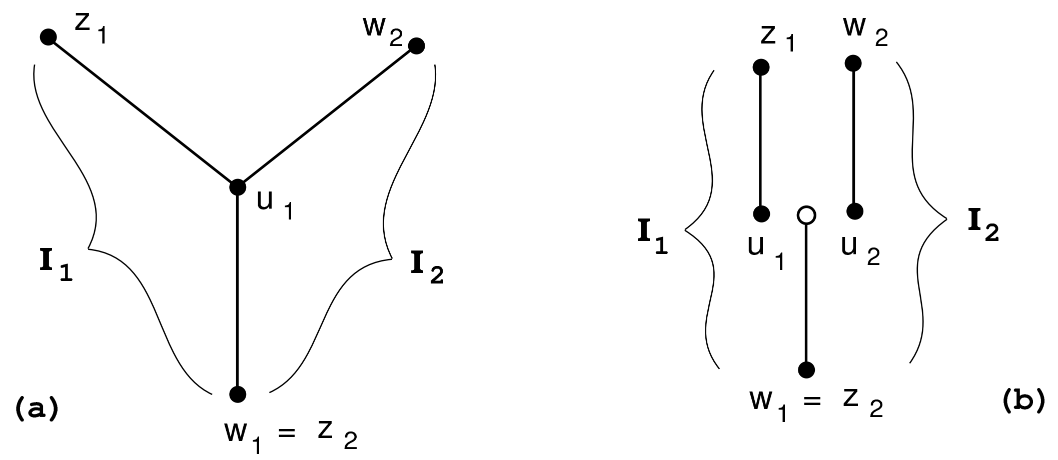

Figure 3: *a. The intersection $I_1 \cap I_2$ is a closed subsegment of I_1, b. The intersection $I_1 \cap I_2$ is a half open subsegment of I_1 (and also of I_2).*

J contains points in $(u_1, w_1] \subset I_1$. In that case we can decrease the number of segments in the chain from x to y, by eliminating at least the segment I_2. This is a contradiction to hypothesis. We conclude that u_2 separates x from y, so $u_2 \in (x, y)$.

Any path from x to y must pass through u_2 and a minimal chain from u_2 to y has exactly $n-1$ segments because it can be augmented to a chain from x to y with one more segment (this uses the fact that u_2 separates x from y). We can now restart the problem with u_2 in place of x and use $n-1$ segments from u_2 to y again a minimal number. Let $x_2 = u_2$. The argument above produces y_2 in I_2 with $[x_2, y_2]$ contained in $[x, y]$ and $I_2 \cap [x, y] = [x_2, y_2]$. Either $n = 2$ and we are finished or there is also $x_3 \in I_3$ with $x_3 \approx y_2$ and $x_3 \in [x, y]$. By induction we conclude that there are $x_i, y_i, 1 \leq i \leq n$ with $[x_i, y_i]$ segments in $\mathcal{H}$ which are disjoint, $y_i \cong x_{i+1}$ and

$$[x, y] \quad = \quad \bigcup_{1 \leq i \leq n} [x_i, y_i].$$

Notice that some or all segments may be degenerate. This finishes the proof of the lemma. □

A fundamental property is that each block $[x, y]$ has a linear ordering: any $z \in (x_i, y_i)$ separates $[x_i, y_i]$ into two components and any $z \in [x_i, y_i]$ separates the union $\cup_{j<i}[x_j, y_j]$ from the union $\cup_{k>i}[x_k, y_k]$.

There is a natural pseudo distance in $\mathcal{H}$:

$$d(x, y) \quad = \quad \#(\text{components } [x, y]) - 1,$$

So $d(x, y) = 0$ means there is a segment from x to y. Also $d(x, y)$ is the minimum number of non immersed points of any path from x to y. These definitions and arguments are the analogues of those for the non Hausdorff, simply connected 1-manifold case by Barbot [Ba1, Ba4]. Here again the non Hausdorffness is the important feature. Going through singularities is no problem in this particular analysis - the singularities only make the definitions more complicated.

We are now ready to study group actions on non Hausdorff trees. Let γ be a homeomorphism of $\mathcal{H}$. We say that γ *separates points* if $\gamma(x)$ is separated from x for any $x \in \mathcal{H}$, that is, they have disjoint neighborhoods in $\mathcal{H}$. In particular γ acts freely in $\mathcal{H}$. In [Ba1, Ba4], Barbot studied the non singular case and constructed a fundamental axis $\mathcal{A}(\gamma)$ in the case γ separates points in $\mathcal{H}$. In that case $\mathcal{H}$ is a simply connected 1-dimensional manifold and hence is orientable. He then gives various characterizations for a point x to be in $\mathcal{A}(\gamma)$ [Ba1, Ba4]:

1) $x \in \mathcal{A}(\gamma)$ if and only if $d(x, \gamma(x))$ is even;

2) $x \in \mathcal{A}(\gamma)$ if and only if $\gamma(x) \in [x, \gamma^2(x)]$.

3) under a convenient orientations of $\mathcal{H}$, x is in the back side of $\gamma(x)$ and $\gamma(x)$ is in the front side of x.

4) $[x, \gamma(x)] \cap [\gamma(x), \gamma^2(x)] = \{\gamma(x)\}$.

5) $x \in \mathcal{A}(\gamma)$ if and only if the function $d(y, \gamma(y))$ in $\mathcal{H}$ attains the minimum in x.

In our situation with singularities in $\mathcal{H}$, condition 3) does not make sense because of lack of a local and hence global orientation in $\mathcal{H}$. But there is still a local orientation in some cases as we will see. Also in general properties 1), 4) and 5) do not hold in general, see counterxamples in the proof of theorem 3.8. Condition 2) is the most natural one, hence our definition:

Definition 3.6. *(fundamental axis) Let γ be a homeomorphism of a non Hausdorff tree $\mathcal{H}$ so that γ has no fixed points. The fundamental axis of γ, denoted by $\mathcal{A}(\gamma)$ is*

$$\mathcal{A}(\gamma) \quad = \quad \{ \ x \in \mathcal{H} \ \mid \ \gamma(x) \in [x, \gamma^2(x)] \ \},$$

or equivalently $\gamma(x)$ separates x from $\gamma^2(x)$.

This is the condition that also works for group actions on **R**-trees [MS1].

One simple fact that will be used throughout is that separation properties are invariant under homeomorphisms of the non Hausdorff tree. If $\gamma(x)$ is not separated from x in $\mathcal{H}$, we say that x is an *almost invariant* point under γ. We need a preliminary result:

Lemma 3.7. *Let γ be a homeomorphism of a non Hausdorff tree $\mathcal{H}$ without fixed points. Then $x \in \mathcal{A}(\gamma)$ if and only if there is a component U to $\mathcal{H} - \{x\}$ so that $\gamma(U) \subset U$.*

Proof. Suppose that $x \in \mathcal{A}(\gamma)$ and let U be the component of $\mathcal{H} - \{x\}$ containing $\gamma(x)$. Suppose that $\gamma^2(x)$ is in another component Z of $\mathcal{H} - \{x\}$. There is a prong J at x with $(J - \{x\}) \subset Z$ and in addition Z is arcwise connected. Hence there is a path in Z from $\gamma^2(x)$ to a point in J and together with J, this produces a path β in $Z \cup \{x\}$ from $\gamma^2(x)$ to x. Then $\gamma(x) \notin \beta$. The same is true if $\gamma^2(x) = x$. But this contradicts the fact that $\gamma(x)$ separates x from $\gamma^2(x)$. Hence $\gamma^2(x) \in U$. Notice that $\gamma(U)$ is the component of $\mathcal{H} - \{\gamma(x)\}$ containing $\gamma^2(x)$. Since $\gamma(x)$ separates x from $\gamma^2(x)$, it follows that $x \notin \gamma(U)$. Since $\gamma(U)$ is arcwise connected and $x \notin \gamma(U)$ then $\gamma(U)$ is contained in a component W of $\mathcal{H} - \{x\}$. But $\gamma^2(x) \in \gamma(U) \subset W$ and $\gamma^2(x) \in U$, both components of $\mathcal{H} - \{x\}$. It follows that $W = U$ and so $\gamma(U) \subset U$, proving one implication.

For the converse, suppose there is a component U of $\mathcal{H} - \{x\}$ so that $\gamma(U) \subset U$. We first show that $\gamma(x) \in U$. Assume that is not the case. Given a prong I at x with $I - \{x\} \subset U$, then $\gamma(I)$ is a prong at $\gamma(x)$. As $x \neq \gamma(x)$, there is a subprong I' of I with $x \notin \gamma(I')$. Then $\gamma(I')$ is contained in a component of $\mathcal{H} - \{x\}$, which is disjoint from U, since $\gamma(x) \notin U$ and U is arcwise connected. But $(\gamma(I') - \gamma(x)) \subset \gamma(U)$, contradicting $\gamma(U) \subset U$. We conclude that $\gamma(x) \in U$. Now $x \notin U$, hence $x \notin \gamma(U)$. But $\gamma(x) \in U$, so $\gamma^2(x) \in \gamma(U)$, a component of $\mathcal{H} - \{\gamma(x)\}$. Therefore x and $\gamma^2(x)$ are in different components of $\mathcal{H} - \{\gamma(x)\}$, or equivalently $\gamma(x)$ separates x from $\gamma^2(x)$ and $\gamma(x) \in (x, \gamma^2(x))$. This finishes the proof of the lemma. ☐

Notice that the proof shows that if a component Z of $\mathcal{H} - \{x\}$ satisfies $\gamma(Z) \subset Z$, then $\gamma(x) \in Z$.

The main result about group actions on non Hausdorff trees is the following:

Theorem 3.8. *Let γ be a homeomorphism of a non Hausdorff tree $\mathcal{H}$ without fixed points. Then $\mathcal{A}(\gamma)$ is non empty.*

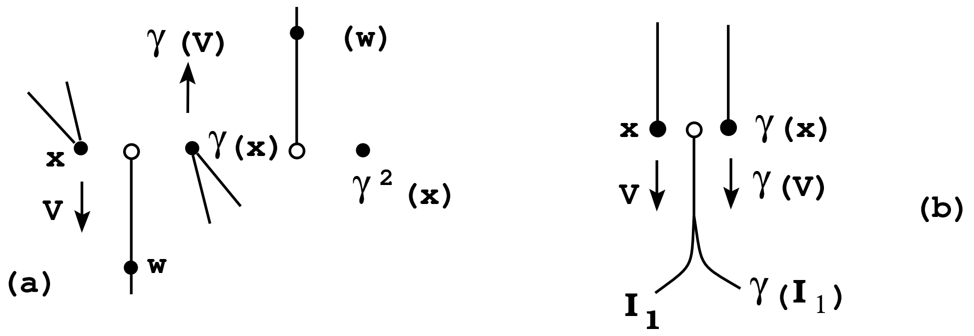

Figure 4: *a. Producing invariant axis:* $\gamma(x)$ *separates* x *from* $\gamma^2(x)$*, b. Preservation of local orientation producing* I_1 *and* $\gamma(I_1)$ *intersectin in a subsegment.*

Proof. In the non singular setting, Barbot [Ba1] shows that if x attains a minimum value of $d(y, \gamma(y)), y \in \mathcal{H}$, then this minimum value is even and x is in the fundamental axis. He restricted attention to those γ preserving orientation of $\mathcal{H}$. In the more general setting of non Hausdorff trees it does not make sense to talk about orientation. It turns out that in general, in some cases the points attaining the minimum of $d(y, \gamma(y))$ will <u>not</u> be in the fundamental axis of γ, see explanation below.

Even though $\mathcal{H}$ is in general not orientable, there are many relevant subsets of $\mathcal{H}$ which are orientable and the orientation will be useful for our purposes. For instance it turns out that the fundamental axis $\mathcal{A}(\gamma)$ admits a natural linear order.

<u>Case I</u> - γ does not separate points.

There is x with $x, \gamma(x)$ not separated from each other. Then no point $z \in \mathcal{H}$ separates x from $\gamma(x)$, so $[x, \gamma(x)] = \{x, \gamma(x)\}$. We can find I_1, I_2 closed segments in $\mathcal{H}$, with $I_1 = [x, z]$ and $I_2 = [\gamma(x), z]$, so that $I_1 \cap I_2 = (x, z]$ as a subset of I_1 and $I_1 \cap I_2 = (\gamma(x), z]$ as a subset of I_2. Notice that $d(x, \gamma(x)) = 1$. Let V be the component of $\mathcal{H} - \{x\}$ containing $\gamma(x)$.

<u>Case I.1</u> - Suppose first that $\gamma(V)$ is not the component of $\mathcal{H} - \{\gamma(x)\}$ containing x, see fig. 4, a.

Then $\gamma(x)$ separates $\gamma(V)$ from x and consequently $\gamma(V) \subset V$. By lemma 3.7, it follows that $x \in \mathcal{A}(x)$ and the proof is finished. As remarked before $d(x, \gamma(x)) = 1$, so it is odd, failing condition 1) of Barbot [Ba1]. In addition if $w \in I_1 \cap V$ then $\gamma(w) \in \gamma(I_1) \cap \gamma(V)$ and so there is a segment from w to $\gamma(x)$ and another from $\gamma(x)$ to $\gamma(w)$, both intersecting only in the common endpoint $\gamma(x)$. Hence their union is a segment of $\mathcal{H}$ and $d(w, \gamma(w)) = 0$. This shows that $x \in \mathcal{A}(\gamma)$ does not achieve the minimum of $d(y, \gamma(y))$ over all $y \in \mathcal{H}$. This shows that condition 5) of Barbot may also fail in general.

<u>Case I.2</u> - The second possibility is that $\gamma(V)$ is the component of $\mathcal{H} - \{\gamma(x)\}$ which contains x, see fig. 4, b.

Notice that I_2 is a prong at $\gamma(x)$ with $I_2 - \{\gamma(x)\}$ contained in the component of $\mathcal{H} - \{\gamma(x)\}$ containing x. By assumption $\gamma(I_1)$ is a prong at $\gamma(x)$ with $\gamma(I_1) - \{x\}$ contained in the same component of $\mathcal{H} - \{\gamma(x)\}$ as above. As components of $\mathcal{H} - \{\gamma(x)\}$ are in one to one correspondence with distinct prongs at $\gamma(x)$ it follows that I_2 and $\gamma(I_1)$ share a subprong.

But I_2 and I_1 share a subsegment, therefore $E = \gamma(I_1) \cap I_1 \neq \emptyset$ and is a segment of $\mathcal{H}$ so that $E \cup \{x\}$ is a prong at x and $E \cup \{\gamma(x)\}$ is a prong at $\gamma(x)$. The interval I_1 has a local orientation which induces a local orientation in $\gamma(I_1)$. The hypothesis about V and $\gamma(V)$ implies that the induced orientations in E by I_1 and $\gamma(I_1)$ agree. Let $z \in E$. Then $\gamma^{-1}(z) \in I_1$. The half open subsegment $(x, \gamma^{-1}(z)]$ of I_1 is taken to the half open interval $(x, z]$ of I_1 by γ and orientations are preserved. From the point of view of I_1 one interval is taken strictly into the other by either γ or γ^{-1}. For instance if $(x, \gamma^{-1}(z)] \subset (x, z]$, then $\gamma^{-1}((x, z])) = (x, \gamma^{-1}(z)] \subset (x, z]$ and $\gamma^{-1}(z)$ is closer to x than z in I_1. Applying γ^{-1} again we obtain that $\gamma^{-1}(z)$ separates z from $\gamma^{-2}(z)$ in $(x, z]$ and therefore in $\mathcal{H}$. It follows that $\gamma^{-2}(z) \in \mathcal{A}(\gamma)$ and we are done. This finishes the proof in the case γ does not separate points.

Notice that in this last situation $\gamma^2(x)$ is also non separated from $x, \gamma(x)$. Therefore $(x, \gamma^2(x)) = \emptyset$ and $\gamma(x)$ does not separate x from $\gamma^2(x)$, so $x \notin \mathcal{A}(\gamma)$. On the other hand if $x \neq \gamma^2(x)$ (which occurs in many examples), then

$$[x, \gamma(x)] \cap [\gamma(x), \gamma^2(x)] \;=\; \{x, \gamma(x)\} \cap \{\gamma(x), \gamma^2(x)\} \;=\; \{\gamma(x)\}.$$

So x satisfies property 4) of Barbot's list but $x \notin \mathcal{A}(\gamma)$. This shows that property 4) is not equivalent to being an element of the fundamental axis.

<u>Case II</u> - We assume from now on that γ separates points.

Our approach will be very similar to looking for invariant axes of actions on trees.

Let $x \in \mathcal{H}$. If $\gamma(x)$ separates x from $\gamma^2(x)$, then $x \in \mathcal{A}(\gamma)$ and we are done. So assume that $\gamma(x)$ does not separate x from $\gamma^2(x)$. Suppose first that $x = \gamma^2(x)$. Then $\gamma([x, \gamma(x)]) = [\gamma(x), x]$. If $[x, \gamma(x)]$ is a single segment in $\mathcal{H}$, then γ acts as an orientation reversing homeomorphism of this segment, hence it has a fixed point. This contradicts the hypothesis of the theorem. Otherwise

$$[x, \gamma(x)] \;=\; \bigcup_{i=1}^{n} [x_i, y_i], \quad y_i \approx x_{i+1}, \ 1 \leq i < n.$$

Since $\gamma([x, \gamma(x)]) = [\gamma(x), x]$, then $\gamma([x_i, y_i]) = [y_{n+1-i}, x_{n+1-i}]$. If n is odd then

$$\gamma\Big([x_{\frac{n+1}{2}}, y_{\frac{n+1}{2}}]\Big) \;=\; [y_{\frac{n+1}{2}}, x_{\frac{n+1}{2}}],$$

so as seen before γ has a fixed point in this segment, contradiction. Otherwise

$$\gamma\Big([x_{\frac{n}{2}}, y_{\frac{n}{2}}]\Big) \;=\; [y_{\frac{n}{2}+1}, x_{\frac{n}{2}+1}],$$

so $\gamma(y_{\frac{n}{2}}) = x_{\frac{n}{2}+1}$. But since $x_{\frac{n}{2}+1} \approx y_{\frac{n}{2}}$, then γ would have almost invariant points and we should be in case I.

We conclude that $x \neq \gamma^2(x)$ and the points $x, \gamma(x), \gamma^2(x)$ are all distinct from each other. Let

$$A \;=\; [x, \gamma(x)] \cap [x, \gamma^2(x)], \quad B \;=\; [\gamma(x), \gamma^2(x)] \cap [x, \gamma^2(x)].$$

See fig. 5 for a simple example of what A and B could look like when $\gamma(x)$ does not separate x from $\gamma^2(x)$.

Lemma 3.9. *$A \cap B$ can have at most one point.*

Proof. Suppose on the contrary there are $c, d \in A \cap B$. Since $c, d \in [x, \gamma^2(x)]$, assume without loss of generality that $c \in [x, d)$ - recall that $[x, \gamma^2(x)]$ admits a linear order.

Notice that $c \neq \gamma(x)$ since $\gamma(x) \notin [x, \gamma^2(x)]$. The set $[x, \gamma(x)]$ has a linear ordering $<_1$ with $x <_1 \gamma(x)$. Conceivably $c = \gamma^2(x)$. But then $\gamma^2(x) \in [x, \gamma(x)]$ so $[x, \gamma(x)]$ is sent into itself by γ. Then $\gamma^2([x, \gamma(x)]) \subset [x, \gamma(x)]$ and γ^2 preserves $<_1$. Hence $(\gamma^2)^n(x)$ is monotone increasing in $[x, \gamma(x)]$ and bounded above by $\gamma(x)$, hence it converges to $y \in [x, \gamma(x)]$ with $\gamma^2(y) = y$. Then $\gamma([y, \gamma(y)]) = [\gamma(y), y]$ and as seen above, this produces either a fixed point of γ or an almost invariant point of γ, both disallowed. We conclude that $c \neq \gamma^2(x)$. If $c = x$ then $x \in [\gamma(x), \gamma^2(x)]$ and a similar argument shows this is not possible. Hence c is none of $x, \gamma(x), \gamma^2(x)$. In fact this argument shows that $\gamma^2(x) \notin [x, \gamma(x)]$ and $x \notin [\gamma(x), \gamma^2(x)]$. Let

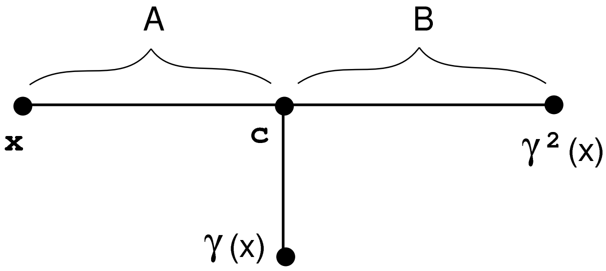

Figure 5: *A simple situation where $\gamma(x)$ does not separate x from $\gamma^2(x)$. For simplicity blocks are drawn without singularities (except of course for the singular point c) and without non separated points. Here c separates any two of the 3 points $x, \gamma(x), \gamma^2(x)$.*

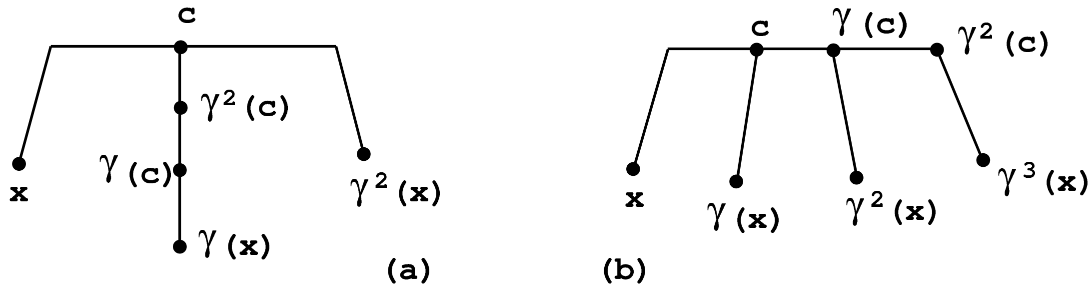

Figure 6: *a. Impossible configuration when $A \cap B = \{c\}$, The correct picture in case $A \cap B = \{c\}$.*

- $D_1 =$ component of $\mathcal{H} - \{c\}$ containing x,
- $D_2 =$ component of $\mathcal{H} - \{c\}$ containing $\gamma(x)$,
- $D_3 =$ component of $\mathcal{H} - \{c\}$ containing $\gamma^2(x)$.

Since $c \in [x, \gamma(x)]$ then $D_1 \neq D_2$. In the same way $D_2 \neq D_3$ and $D_1 \neq D_3$. As $d \in B$, then $d \in [x, \gamma^2(x)]$ and since $c \in [x, d)$ then $d \in D_3$. Also $d \in A$, so $d \in [x, \gamma(x)]$ and again as $c \in [x, d)$ then $d \in D_2$. This contradicts $D_2 \cap D_3 = \emptyset$ and proves the lemma. □

Now there are two possibilities for the intersection $A \cap B$:

Case II.1 - $A \cap B \neq \emptyset$ so $A \cap B = \{c\}$.

Consider $\gamma(c) \in [\gamma(x), \gamma^2(x)]$. Suppose first that $\gamma(c) \in [\gamma(x), c]$, see fig. 6,a, where for simplicity we draw $[x, \gamma(x)]$, etc.. as arcwise connected sets. Then we have $c, \gamma(c) \in [x, \gamma(x)]$ and c separates x from $\gamma(c)$, see fig. 6, a. Apply γ to get $\gamma(c), \gamma^2(c) \in [\gamma(x), \gamma^2(x)]$ and $\gamma(c)$ separates $\gamma(x)$ from $\gamma^2(c)$, see fig. 6, a. If $\gamma^2(c) \in [\gamma(x), c]$ then $[\gamma(c), \gamma^2(c)] \subset [c, \gamma(c)]$ and if $\gamma^2(c) \in [c, \gamma^2(x)]$ then $[c, \gamma(c)] \subset [\gamma(c), \gamma^2(c)]$. For simplicity we assume the first option, see fig. 6, a. Then $\gamma^2([c, \gamma(c)]) \subset [c, \gamma(c)]$ and γ^2 preserves a linear ordering in $[c, \gamma(c)]$. As seen before this produces either a fixed point of γ in $[c, \gamma(c)]$ or an almost invariant point of γ, both disallowed options. We conclude that the situation $\gamma(c) \in [\gamma(x), c]$ cannot occur.

Therefore $\gamma(c) \in (c, \gamma^2(x)]$, see fig. 6, b. The block $[\gamma(x), \gamma^3(x)]$ has a linear ordering $<_2$ with $\gamma(x) <_2 \gamma^3(x)$ in this block. Apply γ to $\gamma(x), c, \gamma(c), \gamma^2(x)$. Then

$$\gamma(c), \gamma^2(c) \;\in\; [\gamma^2(x), \gamma^3(x)] \quad \text{and} \quad \gamma^2(c) \;\in\; [\gamma(c), \gamma^3(x)].$$

But

$$\gamma(c) \;\in\; [\gamma(x), \gamma^3(x)], \quad \text{so} \;\; \gamma^2(c) \;\in\; [\gamma(x), \gamma^3(x)], \quad \text{with} \;\; \gamma(c) <_2 \gamma^2(c)$$

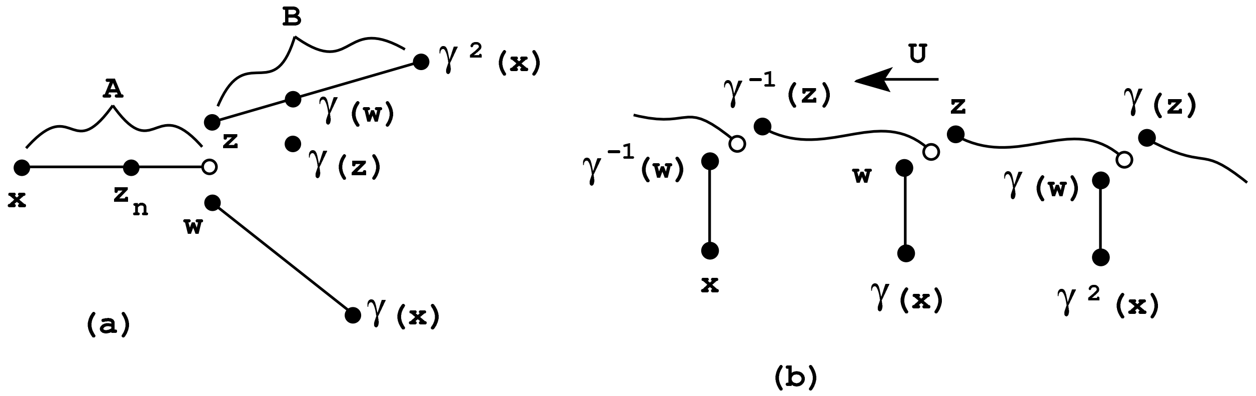

Figure 7: *The intersection $A \cap B$ can be empty. a. The first possibility is supp$(A) \in B$, b. The point z is in the invariant axis when supp$(A) \in B$.*

in the block $[\gamma(x), \gamma^3(x)]$. Also

$$c \in [\gamma(x), \gamma(c)] \subset [\gamma(x), \gamma^3(x)]$$

and so $c <_2 \gamma(c)$. Therefore $c, \gamma(c), \gamma^2(c)$ are in $[\gamma(x), \gamma^3(x)]$ and

$$c \;<_2\; \gamma(c) \;<_2\; \gamma^2(c),$$

so $\gamma(c)$ separates c from $\gamma^2(c)$. By definition $c \in \mathcal{A}(\gamma)$. This finishes the proof of case II.1.

<u>Case II.2</u> - $A \cap B = \emptyset$.

A priori this case can happen. For instance if $A = [x, a]$ and $B = [b, \gamma^2(x)]$ with a, b distinct but not separated from each other and $\gamma(x)$ in the component of $\mathcal{H} - \{a, b\}$ not containing either x or $\gamma^2(x)$, see fig. 8, a.

Notice that in any case $A \cup B = [x, \gamma^2(x)]$ because given any $z \in [x, \gamma^2(x)]$, if $z \notin A \cup B$ then $z \notin [x, \gamma(x)]$ so there is a path from x to $\gamma(x)$ not passing through z and also $z \notin [\gamma(x), \gamma^2(x)]$ so also a path from $\gamma(x)$ to $\gamma^2(x)$ not passing through z. Joining the two paths together one goes from x to $\gamma^2(x)$ without passing through z, contradiction to $z \in [x, \gamma^2(x)]$.

Put a linear ordering $<_3$ in $[x, \gamma^2(x)]$ so that $x <_3 \gamma^2(x)$. Since A and B are subsegments of $[x, \gamma^2(x)]$, it follows that for any $z \in A$, $y \in B$, then $z <_3 y$. So in particular *supp*$(A) \geq_3 z$ for any $z \in A$, where *supp*$(A) \in [x, \gamma^2(x)]$ is computed in the linear order $<_3$. The *supp* exists because $[x, \gamma^2(x)]$ is an ordered finite union of closed segments of $\mathcal{H}$.

<u>Case II.2.1</u> - $z = supp(A) \notin A$.

This implies $z \in B \subset [\gamma(x), \gamma^2(x)]$. Let $z_n \in A$ with $z_n \to z$. We may assume that z_n are increasing in $<_3$. Set $[x, \gamma^2(x)] = \cup_{i=1}^{i_0} [u_i, v_i]$ and let $1 \leq j \leq i_0$ with $z \in [u_j, v_j]$. For simplicity assume z_n all in a fixed $[u_m, v_m]$. If $m \neq j$, then the z_n cannot converge to z.

The set $[x, \gamma(x)]$ also has a linear order, hence in this set the z_n converge to w. Since $z \notin [x, \gamma(x)]$, then $z \neq w$ and z, w are not separated from each other in $\mathcal{H}$, see fig. 7, a.

We claim that these conditions imply that $w \in [\gamma(x), \gamma^2(x)]$. If $w = \gamma(x)$ this is obvious, so assume that $w \neq \gamma(x)$. Since z_n converges to both z and w in $\mathcal{H}$ (and maybe other points as well), it follows that z is in the same component of $\mathcal{H} - \{w\}$ which contains z_n for n sufficiently big. But w separates z_n from $\gamma(x)$, hence w separates z from $\gamma(x)$, that is, $w \in (z, \gamma(x))$. If $z = \gamma^2(x)$ then we are done. Otherwise z separates z_n from $\gamma^2(x)$ and since z_n converges to both z and w then z separates w from $\gamma^2(x)$. Hence $\gamma^2(x)$ is in the same component of $\mathcal{H} - \{w\}$ as z. By the above it follows that w separates $\gamma(x)$ from $\gamma^2(x)$, so $w \in (\gamma(x), \gamma^2(x))$.

Therefore z, w are in $[\gamma(x), \gamma^2(x)]$. Since z, w are not separated from each other in $\mathcal{H}$, then the description in lemma 3.5 of $[\gamma(x), \gamma^2(x)]$ as a finite union of disjoint segments implies that

$$[\gamma(x), \gamma^2(x)] \;\; = \;\; [\gamma(x), w] \; \cup \; [z, \gamma^2(x)].$$

As $w \in [x, \gamma(x)]$, then $\gamma(w) \in [\gamma(x), \gamma^2(x)]$. Suppose first that

$$\gamma(w) \;\in\; [\gamma(x), w] \;\subset\; [\gamma(x), \gamma^2(x)].$$

Then apply γ to obtain $\gamma^2(w) \in [\gamma^2(x), \gamma(w)) \;\subset\; [\gamma(x), \gamma^2(x)]$, so the 3 points $w, \gamma(w)$ and $\gamma^2(w)$ are in $[\gamma(x), \gamma^2(x)]$ and as before either

$$\gamma([w, \gamma(w)]) \;\subset\; [w, \gamma(w)] \quad \text{or} \quad [w, \gamma(w)] \;\subset\; \gamma([w, \gamma(w)]).$$

As in case II.1, this leads to either a fixed point of γ or an almost invariant point of γ, both contradiction to hypothesis in this case.

Suppose now that $\gamma(w) \in [z, \gamma^2(x)]$, see fig. 7, b. Notice that $\gamma(w) \neq z$ because z, w are not separated from each other and use the running hypothesis in case II. Let U be the component of $\mathcal{H} - \{z\}$ containing x. Then $z_n \in U$ and since $z_n \to w$, also $w \in U$. Since w separates z from $\gamma(x)$ then $\gamma(x) \in U$. As U is arcwise connected then $[x, \gamma(x)] \subset U$. As $\gamma(w) \in (z, \gamma^2(x)]$ then z separates x from $\gamma(w)$. Also $\gamma(z), \gamma(w)$ are not separated from each other, therefore z separates $\gamma(z)$ from x, so $\gamma(z) \notin U$. As $z \in [\gamma(x), \gamma^2(x)]$, then $\gamma^{-1}(z) \in [x, \gamma(x)] \subset U$. Putting it all together, $\gamma^{-1}(z) \in U$ and $\gamma(z) \notin U$, so z separates $\gamma^{-1}(z)$ from $\gamma(z)$. Therefore $\gamma^{-1}(z) \in \mathcal{A}(\gamma)$ and the proof is finished.

In fig. 7, b we describe a possible configuration in this case. The case $inf(B) \notin B$ is treated analogously. The final case to be considered is the following:

<u>Case II.2.2</u> - $supp(A) = a \in A$ and $inf(B) = b \in B$.

The only way this can happen is as follows: the union $A \cup B = [x, \gamma^2(x)] = \cup_{i=1}^{i_0} [u_i, v_i]$ and for any $x \in A, y \in B$ then $x <_3 y$. This implies that a, b are the endpoints of some intervals $[u_i, v_i]$ and a and b are non separated from each other, see fig. 8, a. Then $A \cup B$ splits $[x, \gamma^2(x)]$ nicely into a disjoint union of intervals.

Let $s \in (a, \gamma(x))$. If $s = b$, then $b \in (a, \gamma(x)) \subset [x, \gamma(x)]$. As $b \in [x, \gamma^2(x)]$ then $b \in A$, contradicting the hypothesis. Hence $s \neq b$. If $s \notin (b, \gamma(x))$, then s does not separate b from $\gamma(x)$ and there is a path from b to $\gamma(x)$ in $\mathcal{H} - \{s\}$. Since a and b are not separated from each other and $s \neq a, s \neq b$, there is also a path from a to b in $\mathcal{H} - \{s\}$, producing a path from a to $\gamma(x) \in \mathcal{H} - \{s\}$. This contradicts $s \in (a, \gamma(x))$. Hence $(a, \gamma(x)) \subset (b, \gamma(x))$ and the reverse inclusion is proven in the same way (using $a \notin B$). Consequently $(a, \gamma(x)] = (b, \gamma(x)]$.

Since $b \in [\gamma(x), \gamma^2(x)]$, then $\gamma^{-1}(b) \in [x, \gamma(x)]$.

The first option is that $\gamma^{-1}(b) \in [a, \gamma(x)]$, see fig. 8, a. We will show that this case does not occur, given the running hypothesis in case II.

Notice that $\gamma^{-1}(b) \neq a$ since a, b are non separated from each other. Hence

$$\gamma^{-1}(b) \;\in\; (a, \gamma(x)] \;\; = \;\; (b, \gamma(x)] \;\subset\; [\gamma(x), \gamma^2(x)].$$

Now apply the homeomorphism γ to the points $x, a, \gamma^{-1}(b), \gamma(x)$, which are linearly ordered in $[x, \gamma(x)]$, to obtain $\gamma(x), \gamma(a), b, \gamma^2(x)$, so

$$\gamma(a) \;\in\; [\gamma(x), b) \;\subset\; [\gamma(x), \gamma^2(x)],$$

see fig. 8, a. Put a linear order $<_4$ in $[\gamma(x), b] \subset [\gamma(x), \gamma^2(x)]$ so that $\gamma(x) <_4 \; b$.

<u>Claim</u> - $\gamma(b) \notin [\gamma(x), b]$.

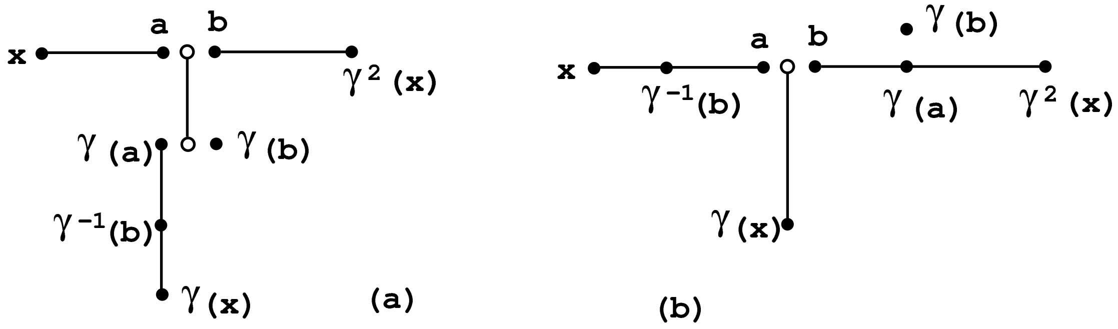

Figure 8: *The second possibility: supp*$(A) \in A$, Supp$(B) \in B$, *which splits* $[x, \gamma^2(x)]$ *nicely into disjoint intervals, a. The situation* $\gamma^{-1}(b) \in [a, \gamma(x)]$ *leads to a contradiction, b. The points* a, b *are in the invariant axis of* γ.

Suppose that $\gamma(b) \in [\gamma(x), b]$. Then $b, \gamma^{-1}(b), \gamma(b)$ are all in $[b, \gamma(x)]$. Hence either $[b, \gamma(b)] \subset [b, \gamma^{-1}(b)]$ or the reverse inclusion. Without loss of generality assume the first inclusion. Then $\gamma([b, \gamma^{-1}(b)]) \subset [b, \gamma^{-1}(b)]$ and so

$$\gamma^2([b, \gamma^{-1}(b)] \ \subset \ [b, \gamma^{-1}(b)],$$

preserving the linear order in $[b, \gamma^{-1}(b)]$. An argument in the proof of lemma 3.9 produces v with $\gamma(v)$ not separated from v - disallowed by hypothesis in case II. We conclude that $\gamma(b) \notin [\gamma(x), b]$, proving the claim.

Let U be the component of $\mathcal{H} - \{a\}$ containing b. As $a \notin B = [\gamma(x), \gamma^2(x)] \cap [x, \gamma^2(x)]$, there is a path in $\mathcal{H} - \{a\}$ from $\gamma(x)$ to $\gamma^2(x)$ and this path has to pass through b. Therefore $\gamma(x) \in U$. Also $\gamma^{-1}(b) \in [\gamma(x), \gamma^2(x)] \subset U$, hence $b \in \gamma(U)$ and $U \cap \gamma(U) \neq \emptyset$.

This implies that $(b, \gamma(a)) = (b, \gamma(b))$. To see this let $t \in (b, \gamma(b))$. Notice first that $t \neq \gamma(a)$ because $\gamma(U)$, which is a component of $\mathcal{H} - \{\gamma(a)\}$, contains both b and $\gamma(b)$. If $t \notin (b, \gamma(a))$, then there is a path in $\mathcal{H} - \{t\}$ from b to $\gamma(a)$ (using that $t \neq \gamma(a)$) and since $t \neq \gamma(a), \gamma(b)$ and these last two points are not separated from each other, there is a path from $\gamma(a)$ to $\gamma(b)$ in $\mathcal{H} - \{t\}$. This produces a path from b to $\gamma(b)$ in $\mathcal{H} - \{t\}$, contradicting $t \in (b, \gamma(b))$. We conclude that $t \in (b, \gamma(a))$ and this shows that $(b, \gamma(b)) \subset (b, \gamma(a))$. Let now $t \in (b, \gamma(a))$. Recall that $\gamma(a) \in [\gamma(x), b)$. If $t = \gamma(b)$ then

$$\gamma(b) \ \in \ (b, \gamma(a)) \ \subset \ (b, \gamma(x)],$$

which is disallowed by the claim. Hence $t \neq \gamma(b)$ and the same argument as above shows that $(b, \gamma(a)) \subset (b, \gamma(b))$, producing equality of these two sets.

Now both $\gamma^{-1}(b), \gamma(a)$ are in $[\gamma(x), b]$. If $\gamma^{-1}(b) \leq_4 \gamma(a)$ in $[\gamma(x), b]$, then

$$(b, \gamma(b)) \ = \ (b, \gamma(a)) \ \subset (b, \gamma^{-1}(b)),$$

so $\gamma(b, \gamma^{-1}(b)) \subset (b, \gamma^{-1}(b))$. If on the other hand $\gamma(a) <_4 \gamma^{-1}(b)$ then

$$(b, \gamma^{-1}(b)) \ \subset \ (b, \gamma(a)) \ = \ (b, \gamma(b)),$$

so $\gamma^{-1}(b, \gamma(b)) \subset (b, \gamma(b))$.

Without loss of generality assume the first option. Notice that γ reverses the linear order $<_4$ in the blocks. For $y \in (\gamma^{-1}(b), b)$ near $\gamma^{-1}(b)$, then

$$\gamma(y) \in (b, \gamma(b)) \subset (\gamma^{-1}(b), b) \quad \text{and} \quad \gamma(y) >_4 y,$$

because $\gamma(y)$ is near b. For $y \in (\gamma^{-1}(b), b)$ near b then $\gamma(y) \in (\gamma^{-1}(b), b)$ near $\gamma(b)$ - so near $\gamma(a)$, and therefore $\gamma(y) <_4 y$. Since $(b, \gamma^{-1}(b))$ is an (open) block of $\mathcal{H}$, this reversal of order in $(b, \gamma^{-1}(b))$ implies that there is $z \in (\gamma^{-1}(b), b)$ with either $\gamma(z) = z$ or $\gamma(z)$ non separated from z. Both of these are disallowed by the running hypothesis in case II. This shows that $\gamma^{-1}(b) \in [a, \gamma(x)]$ cannot occur.

As $\gamma^{-1}(b) \in [x, \gamma(x)]$, the last possible option is that $\gamma^{-1}(b) \in [x, a)$. In addition $\gamma(a) \in [\gamma(x), \gamma^2(x)]$ and in the same way as above one shows that $\gamma(a) \in (b, \gamma^2(x)]$. Since $\gamma(b)$ is non separated from $\gamma(a)$ it follows that $\gamma(b)$ is in the component of $\mathcal{H} - \{b\}$ which contains $(b, \gamma^2(x)]$. Therefore a separates $\gamma^{-1}(b)$ from $\gamma(b)$ and hence b separates $\gamma^{-1}(b)$ from $\gamma(b)$. Consequently $\gamma^{-1}(b)$ is in the fundamental axis of γ, see fig. 8, b.

This finishes the analysis of case II.2.2 and the proof of theorem 3.8. □

Remark: This proof works almost verbatin in the case of essential laminations. Let $\mathcal{L}$ be an essential lamination in a closed 3-manifold M. First replace each isolated leaf of $\mathcal{L}$ by an interval of leaves - a well known operation in lamination theory [Ga-Oe, Ga-Ka]. This produces an essential lamination $\mathcal{L}_1$ which is basically the same as $\mathcal{L}$. Then let $\mathcal{H}_1$ be the leaf space of the lifted lamination $\widetilde{\mathcal{L}}_1$. Then $\mathcal{H}_1$ is an order tree as defined by Gabai-Oertel [Ga-Oe] and since $\widetilde{\mathcal{L}}_1$ has no isolated leaves, then $\mathcal{H}_1$ is an $\mathbf{R}$-order tree [Ga-Oe]. But $\mathbf{R}$-order trees are exactly the same as non Hausdorff trees as defined here. Theorem 3.8 can then be used to analyse group actions in the leaf spaces of essential laminations.

We now study some properties of the invariant axis and give a description of its structure. It is obvious that $\mathcal{A}(\gamma)$ is invariant under γ for if $\gamma(x)$ separates x from $\gamma^2(x)$, then $\gamma(\gamma(x))$ separates $\gamma(x)$ from $\gamma^2(\gamma(x))$. Also notice that $x \in \mathcal{A}(\gamma^{-1})$ if and only if $\gamma^{-1}(x)$ separates x from $\gamma^{-2}(x)$. Applying γ^2, this occurs if and only if $\gamma(x)$ separates $\gamma^2(x)$ from x, which is the definition of $x \in \mathcal{A}(\gamma)$. Therefore $\mathcal{A}(\gamma) = \mathcal{A}(\gamma^{-1})$.

Proposition 3.10. *For any $x \in \mathcal{A}(\gamma)$, then $\mathcal{A}(\gamma) = \cup_{i \in \mathbf{Z}} [\gamma^i(x), \gamma^{i+1}(x)]$.*

Proof. Given $x \in \mathcal{A}(\gamma)$, let

$$\mathcal{A}_x = \bigcup_{i \in \mathbf{Z}} [\gamma^i(x), \gamma^{i+1}(x)].$$

First choose $y \in (x, \gamma(x))$. Let U be the component of $\mathcal{H} - \{x\}$ containing $\gamma(x)$ and V the component of $\mathcal{H} - \{y\}$ containing $\gamma(x)$. Since y separates x from $\gamma(x)$ and $\gamma(x) \in V$ then $x \notin V$. Also V is arcwise connected so V is contained in a component W of $\mathcal{H} - \{x\}$. But $\gamma(x) \in V \cap U$, so in fact $W = U$, that is $V \subset U$.

In addition y separates x from $\gamma(x)$, hence y, x are in the same component of $\mathcal{H} - \{\gamma(x)\}$. Since $x \notin \gamma(U)$ (because $\gamma(U) \subset U$), then $y \notin \gamma(U)$. So $\gamma(U)$ is contained in a component of $\mathcal{H} - \{y\}$. Choose a prong I at x with $I - \{x\} \subset U$. Since $y \neq \gamma(x)$, there is subprong I' of I with $y \notin \gamma(I')$. But $\gamma(I' - \{x\}) \subset \gamma(U)$. Also $\gamma(x) \in \gamma(I')$ which is then contained in the component V of $\mathcal{H} - \{y\}$. It follows that $V \cap \gamma(U) \neq \emptyset$ and as a result $\gamma(U) \subset V$. Therefore

$$\gamma(U) \subset V \subset U \quad \text{so} \quad \gamma(V) \subset \gamma(U) \subset V.$$

Lemma 3.7 applied to the component V of $\mathcal{H} - \{y\}$ implies that $y \in \mathcal{A}(\gamma)$. Hence $[x, \gamma(x)] \subset \mathcal{A}(\gamma)$ and by definition of $\mathcal{A}_x$ and γ invariance of $\mathcal{A}(\gamma)$ it follows that $\mathcal{A}_x \subset \mathcal{A}(\gamma)$.

To prove the converse inclusion we use the following: Given $z \in \mathcal{A}(\gamma)$ define U_z to be the component of $\mathcal{H} - \{z\}$ which contains $\gamma(z)$ and V_z to be the component of $\mathcal{H} - \{z\}$ with $\gamma^{-1}(z) \in V_z$.

By lemma 3.7, U_z is the unique component of $\mathcal{H} - \{z\}$ with $\gamma(U_z) \subset U_z$ and similarly $\gamma^{-1}(V_z) \subset V_z$. In addition γ invariace implies that $\gamma(U_z) = U_{\gamma(z)}$. Also since $\gamma(U_z) \subset U_z$ then $z \notin \gamma(U_z) = U_{\gamma(z)}$ – in fact $\gamma(z)$ separates $U_{\gamma(z)}$ from z.

Let now $y \in \mathcal{A}(\gamma)$ and suppose that $y \notin \mathcal{A}_x$. We want to show that this is impossible. Since $y \in \mathcal{A}(\gamma)$ then U_y, V_y are defined. There are 3 possibilities for the relative position of x. Notice first that $x \neq y$ since $y \notin \mathcal{A}_x$.

<u>Case 1</u> - $x \in U_y$.

As $\gamma(y) \notin \mathcal{A}_x$ then $\gamma(y) \notin (x, \gamma(x))$, so $\gamma(y)$ does not disconnect x from $\gamma(x)$. But $\gamma(x) \in \gamma(U_y) = U_{\gamma(y)}$, therefore $x \in U_{\gamma(y)}$ also. As $y \notin U_{\gamma(y)}$, then $\gamma(y)$ separates x from y, or $\gamma(y) \in (x, y) \subset [x, y]$. By induction assume that $\gamma^i(y) \in (x, \gamma^{i-1}(y))$ and $x \in U_{\gamma^i(y)}$. Then

$$\gamma(x) \ \in \ \gamma(U_{\gamma^i(y)}) \ = \ U_{\gamma^{i+1}(y)}.$$

But $y \notin \mathcal{A}_x$, hence $\gamma^{i+1}(y) \notin (x, \gamma(x))$, so as before $x, \gamma(x)$ are in the same component of $\mathcal{H} - \{\gamma^{i+1}(y)\}$, and hence $x \in U_{\gamma^{i+1}(y)}$. But $\gamma^i(y) \notin U_{\gamma^{i+1}(y)}$. We conclude that $\gamma^{i+1}(y)$ separates x from $\gamma^i(y)$ or

$$\gamma^{i+1}(y) \ \in \ (x, \gamma^i(y)).$$

This works for all $i \geq 0$. Consequently all $\gamma^i(y)$, $i \geq 0$ are in $[x, y]$ and are distinct and monotone in $[x, y]$. Recall that

$$[x, y] \ = \ \bigcup_{1}^{n_0} \ [u_i, v_i],$$

segments in $\mathcal{H}$. Hence there is j_0 so that eventually all $\gamma^i(y)$ are in $[u_{j_0}, v_{j_0}]$, monotonic and distinct. Hence in the segment $[u_{j_0}, v_{j_0}]$, the $\gamma^i(y)$ converge to z as $i \to +\infty$. Notice that z is either x or z separates x from all $\gamma^i(y), i \geq 0$. But also $\gamma^{i+1}(y) = \gamma(\gamma^i(y)) \to \gamma(z)$ as $i \to +\infty$. We can see that $z, \gamma(z), \gamma^2(z)$ are all non separated from each other. If $x = z$ this immediately is a contradiction, because then $\gamma(x) = \gamma(z)$ does not separate $x = z$ from $\gamma^2(x) = \gamma^2(z)$, contradicting $x \in \mathcal{A}(\gamma)$.

Suppose then that $x \neq z$ and let T be the component of $T - \{z\}$ containing x. Notice that $\gamma^i(y) \notin T$ for any i. Also $\gamma(T)$ is the component of $\mathcal{H} - \{\gamma(z)\}$ containing $\gamma(x)$ and does not contain any $\gamma^i(y)$. It follows that $\gamma(z)$ separates $\gamma(T)$ from T, therefore $\gamma(T) \cap T = \emptyset$ and so $\gamma(T) \cap \gamma^2(T) = \emptyset$ also. Let α be a path contained in $T \cup Z$ from x to z. As

$$\gamma(x) \in \gamma(T) \ \text{ and } \ \gamma(T) \cap T = \emptyset, \ \text{ then } \ \gamma(x) \notin \alpha.$$

Also $\gamma^2(\alpha)$ is a path from $\gamma^2(x)$ to $\gamma^2(z)$ contained in $\gamma^2(T) \cup \gamma^2(x)$, so again $\gamma(x) \notin \gamma^2(\alpha)$. For i big $[z, \gamma^i(y)]$ is a prong at z disjoint from T and also from $\gamma(T)$. The same is true for $\gamma^2([z, \gamma^i(y)])$. Using this two prongs one can construct a path β from z to $\gamma^2(z)$ not containing $\gamma(x)$. Then

$$\alpha \ * \ \beta \ * \ (\gamma^2(\alpha))^{-1},$$

is a path from x to $\gamma^2(x)$ not containing $\gamma(x)$. This contradicts $x \in \mathcal{A}(\gamma)$. We conclude that this case cannot happen.

<u>Case 2</u> - $x \in V_y$.

Do the same proof as in case 1, using γ^{-1} instead of γ. This leads to a contradiction as in case 1.

The remaining case is:

<u>Case 3</u> - There is a component W of $\mathcal{H} - \{y\}$ with $U_y \neq W \neq V_y$ and $x \in W$.

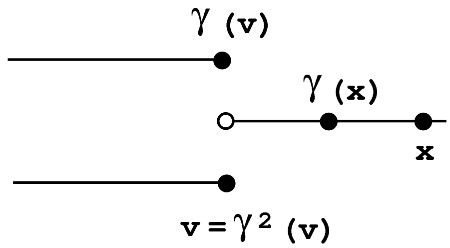

Figure 9: *The homeomorphism γ acts freely, but γ^2 does not.*

This case can happen because y may be a singular point, so $\mathcal{H} - \{y\}$ may have more than 2 components. As $y \in \mathcal{A}(\gamma)$, V_y is the unique component of $\mathcal{H} - \{y\}$ containing $\gamma^{-1}(y)$. Hence $\gamma(V_y)$ is the unique component of $\mathcal{H} - \{\gamma(y)\}$ containing y. Since $\gamma(W) \neq \gamma(V_y)$, then $y \notin \gamma(W)$ and so $\gamma(y)$ separates $\gamma(W)$ from y and $\gamma(W)$ is contained in a component of $\mathcal{H} - \{y\}$. As there is a prong at $\gamma(y)$ contained in $\gamma(y) \cup \{\gamma(y)\}$, it follows that this component of $\mathcal{H} - \{y\}$ has to be U_y, because $\gamma(y) \in U_y$. Therefore $\gamma(W) \subset U_y$. But also $W \neq U_y$ so in fact $W \cap U_y = \emptyset$. We conclude that $W \cap \gamma(W) = \emptyset$. In particular y separates W from $\gamma(W)$ so $y \in (x, \gamma(x))$. In the same way $\gamma(y) \in (x, \gamma(x))$ and in fact $x, y, \gamma(y), \gamma(x)$ are linearly ordered in $[x, \gamma(x)]$. Let now Z be the component of $\mathcal{H} - \{x\}$ containing $\gamma(x)$. Then $y \in Z$ and $\gamma(Z)$ is the component of $\mathcal{H} - \{\gamma(x)\}$ containing $\gamma(y)$. But then as $\gamma(y)$ separates x from $\gamma(x)$, it follows that $x \in \gamma(Z)$. But $x \notin Z$, consequently $\gamma(Z) \not\subset Z$. The analysis of lemma 3.7 implies that $x \notin \mathcal{A}(\gamma)$, contradiction to hypothesis. Hence case 3) cannot happen either.

We conclude that the assumption $y \in \mathcal{A}(\gamma)$ but $y \notin \mathcal{A}_x$ is impossible. Hence $\mathcal{A}(\gamma) = \mathcal{A}_x$. Since x is arbitrary in $\mathcal{A}(\gamma)$, this finishes the proof of the proposition. □

Using the characterization of $\mathcal{A}(\gamma)$ as $\mathcal{A}_x$ for $x \in \mathcal{A}(\gamma)$ it follows that $\mathcal{A}(\gamma)$ has a linear ordering: Recall the component U_x of $\mathcal{H} - \{x\}$ with $\gamma(x) \in U_x$. Since for any $n \geq 1$:

$$\gamma^n(U_x) \;=\; U_{\gamma^n(x)} \;\subset\; U_{\gamma(x)},$$

then $\gamma^n(x)$ separates $\cup_{i>n}[\gamma^i(x), \gamma^{i+1}(x)]$ from any $y \in (x, \gamma(x))$. By γ invariance it follows that any $y \in (\gamma^n(x), \gamma^{n+1}(x))$ separates $\cup_{i<n}[\gamma^i(x), \gamma^{i+1}(x)]$ from $\cup_{i>n}[\gamma^i(x), \gamma^{i+1}(x)]$. The linear order $<_5$ can be defined as follows. Put an order in $[x, \gamma(x)]$ so that $x <_5 \gamma(x)$. For any $n \in \mathbf{Z}$, γ^n then induces an order in $[\gamma^n(x), \gamma^{n+1}(x)]$. Finally if $z \in [\gamma^i x, \gamma^{i+1}(x))$ and $w \in [\gamma^j x, \gamma^{j+1}(x))$ (half open intervals), let $z <_5 w$ if and only if $i < j$. This is a linear order in $\mathcal{A}(\gamma)$.

Remark: In general it is not true that if γ acts freely in $\mathcal{H}$, then powers of γ also do. For example let γ have an almost invariant point v with $\gamma(v) \neq v$, but $\gamma^2(v) = v$ and so that $\mathcal{A}(\gamma)$ is a segment, see fig. 9.

Let $x \in \mathcal{A}(\gamma)$. If $d(x, \gamma(x)) = 0$, then $x, \gamma(x)$ are connected by a segment in $\mathcal{H}$. Since $\gamma(x)$ separates x from $\gamma^2(x)$ it follows that $[x, \gamma(x)] \cup [\gamma(x), \gamma^2(x)] = [x, \gamma^2(x)]$ is a segment of $\mathcal{H}$. It follows that $\mathcal{A}(\gamma)$ is an open segment of $\mathcal{H}$, hence homeomorphic to $\mathbf{R}$. If $d(x, \gamma(x)) > 0$, then x and $\gamma(x)$ are connected by a chain of closed segments. It is easy to see that

$$\mathcal{A}(\gamma) \;=\; \bigcup_{n \in \mathbf{Z}} \; [z_i, w_i],$$

where w_i is not separated from z_{i+1}. Then γ acts as a translation in the set of segments, that is, there is $k \in \mathbf{Z}$, so that $\gamma([z_i, w_i]) = [z_{i+k}, w_{i+k}]$ for any $i \in \mathbf{Z}$. We abuse notation and say that γ acts in $\mathbf{Z}$.

Notice that if γ acts freely and γ leaves invariant an open segment I of $\mathcal{H}$, then $\mathcal{A}(\gamma) = I$. This is because for any $z \in I$, $\gamma(x)$ separates x from $\gamma^2(x)$ (free action in I), so $x \in \mathcal{A}(\gamma)$ and hence $I \subset \mathcal{A}(\gamma)$. But then $\mathcal{A}(\gamma) = \cup_{n \in \mathbf{Z}}[\gamma^n(x), \gamma^{n+1}(x)]$ so $I = \mathcal{A}(\gamma)$.

Lemma 3.11. *Let γ, α be two commuting homeomorphisms of $\mathcal{H}$ which act freely. Then $\mathcal{A}(\gamma) = \mathcal{A}(\alpha)$.*

Proof. Let $x \in \mathcal{A}(\gamma)$, then $\gamma(x)$ separates x from $\gamma^2(x)$. Applying α: $\alpha(\gamma(x)) = \gamma(\alpha(x))$, separates $\alpha(x)$ from $\alpha(\gamma^2(x)) = \gamma^2(\alpha(x))$. Hence $\alpha(x) \in \mathcal{A}(\gamma)$. Therefore $\alpha(\mathcal{A}(\gamma)) \subset \mathcal{A}(\gamma)$. In the same way applying α^{-1} then $\alpha^{-1}(\mathcal{A}(\gamma)) \subset \mathcal{A}(\gamma)$ or $\mathcal{A}(\gamma) \subset \alpha(\mathcal{A}(\gamma))$. So $\alpha(\mathcal{A}(\gamma)) = \mathcal{A}(\gamma)$.

Put a linear order $<_6$ in $\mathcal{A}(\gamma)$. Suppose that α does not preserve the linear order $<_6$. If $\mathcal{A}(\gamma)$ is an open interval then α would have a fixed point in the interval $\mathcal{A}(\gamma)$, contradiction. If $\mathcal{A}(\gamma) = \cup_{i \in \mathbf{Z}}[z_i, w_i] = \cup_{i \in \mathbf{Z}} I_n$, then α has to preserve the collection of intervals, so it acts as a map on $\mathbf{Z}$ reversing the ordering in $\mathbf{Z}$. Since γ acts on $\mathbf{Z}$ preserving the ordering and freely, it follows that α and γ could not commute, contradiction. Hence α preserves $<_6$.

Therefore for any $x \in \mathcal{A}(\gamma)$, the points $x, \alpha(x), \alpha^2(x)$ satisfy $x \;<_6\; \alpha(x) \;<_6\; \alpha^2(x)$. Hence $\alpha(x)$ separates x from $\alpha^2(x)$ so $x \in \mathcal{A}(\alpha)$. We conclude that $\mathcal{A}(\gamma) \subset \mathcal{A}(\alpha)$ and in the same way $\mathcal{A}(\alpha) \subset \mathcal{A}(\gamma)$. Therefore $\mathcal{A}(\alpha) = \mathcal{A}(\gamma)$. □

Notice one has to assume that both γ and α act freely – for example consider γ and γ^2 in the example above.

4 Topological theory of pseudo-Anosov flows

Let Φ be a pseudo-Anosov flow in M^3 closed. We review the results about the topology of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ which will be needed in the following sections to prove the main theorem. We refer to [Fe5, Fe6] for detailed proofs. The orbit space of $\widetilde{\Phi}$ in $\widetilde{M}$ is homeomorphic to the plane $\mathbf{R}^2$ [Fe-Mo] and is denoted by $\mathcal{O} \cong \widetilde{M}/\widetilde{\Phi}$. Let $\Theta : \widetilde{M} \to \mathcal{O} \cong \mathbf{R}^2$ be the projection map. If L is a leaf of $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$, then $\Theta(L) \subset \mathcal{O}$ is a tree which is either homeomorphic to $\mathbf{R}$ if L is regular, or is a union of p-rays all with the same starting point if L has a singular p-prong orbit.

Definition 4.1. *A line leaf L' of $L \in \widetilde{\mathcal{F}}^s$ is the boundary of a component of $M - L$. Then L' is regular on the side of the corresponding sector (there are two such sides if L itself is non singular). If L is a leaf of $\widetilde{\mathcal{F}}^s$ or a line leaf of a leaf of $\widetilde{\mathcal{F}}^s$, then a half leaf of L is a connected component A of $L - \gamma$, where γ is any full orbit in L. The closure is denoted by $\overline{A} = A \cup \gamma$ and its boundary is $\partial A = \gamma$. If ζ is an open, relatively compact, connected subset of $\Theta(L)$, it defines a* flow band *L_1 of L by $L_1 = \Theta^{-1}(\zeta)$.*

If $F \in \widetilde{\mathcal{F}}^s$ and $G \in \widetilde{\mathcal{F}}^u$ then F and G intersect in at most one orbit. Also suppose that a leaf $F \in \widetilde{\mathcal{F}}^s$ intersects two leaves $G, H \in \widetilde{\mathcal{F}}^u$ and so does $L \in \widetilde{\mathcal{F}}^s$. Then F, L, G, H form a *rectangle* in $\widetilde{M}$ and there is no singularity in the interior of the rectangle [Fe6].

Definition 4.2. *([Fe2, Fe6]) Perfect fits - Two leaves $F \in \widetilde{\mathcal{F}}^s$ and $G \in \widetilde{\mathcal{F}}^u$, form a perfect fit if $F \cap G = \emptyset$ and there are line leaves F_0, G_0 of F, G respectively and half leaves F_1 of F_0 and G_1 of G_0 and also flow bands $L_1 \subset L \in \widetilde{\mathcal{F}}^s$ and $H_1 \subset H \in \widetilde{\mathcal{F}}^u$, so that F_0 is regular on the side containing L, G_0 is regular on the side containing H and:*

$$\overline{L}_1 \cap \overline{G}_1 = \partial L_1 \cap \partial G_1, \quad \overline{L}_1 \cap \overline{H}_1 = \partial L_1 \cap \partial H_1, \quad \overline{H}_1 \cap \overline{F}_1 = \partial H_1 \cap \partial F_1,$$
$$\text{with} \quad \overline{L}_1 \cap \overline{G}_1 \neq \emptyset, \quad \overline{L}_1 \cap \overline{H}_1 \neq \emptyset \quad \text{and} \quad \overline{H}_1 \cap \overline{F}_1 \neq \emptyset.$$

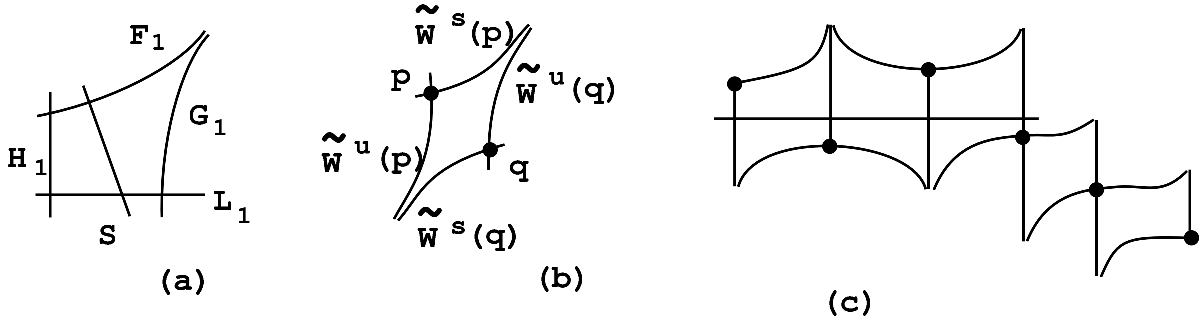

Figure 10: *a. Perfect fits in $\widetilde{M}$, b. A lozenge, c. A chain of lozenges.*

Furthermore

$$\forall\ S \in \widetilde{\mathcal{F}}^u, \quad S \cap L_1 \neq \emptyset\ \Rightarrow S \cap F_1 \neq \emptyset \quad (1)$$

$$\text{and}\ \ \forall\ E \in \widetilde{\mathcal{F}}^s, \quad E \cap H_1 \neq \emptyset\ \Rightarrow E \cap G_1 \neq \emptyset \quad (2).$$

We refer to fig. 10, a for perfect fits. Implications $(1), (2)$ force equivalences (that is $S \cap L_1 \neq \emptyset\ \Leftrightarrow S \cap F_1 \neq \emptyset$ and the same for (2)). The set $\overline{F}_1 \cup \overline{H}_1 \cup \overline{L}_1 \cup \overline{G}_1$ separates $\widetilde{M}$. Let A be the complementary region which does not contain $F - F_1$ in its closure. An important fact is that there are singularities of $\widetilde{\Phi}$ in A. Perfect fits produce "ideal" rectangles, in the sense that even though F and G do not intersect, there is a product structure (of $\widetilde{\mathcal{F}}^s$ and $\widetilde{\mathcal{F}}^u$) in the interior of A.

Definition 4.3. *[Fe2, Fe6] Given $p \in \widetilde{M}$ (or $p \in \mathcal{O}$), and a half leaf H of $\widetilde{W}^u(p)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(p)$, let*

$$\mathcal{J}^u(H) \ = \ \{F \in \mathcal{H}(\widetilde{\mathcal{F}}^s) \mid F \cap H \neq \emptyset\} \ \subset \ \mathcal{H}(\widetilde{\mathcal{F}}^s).$$

Notice that $\widetilde{W}^s(p) \notin \mathcal{J}^u(H)$. Let also

$$\mathcal{L}^u(H) \ = \ \bigcup \{\ p\ \in \widetilde{M}\ \mid\ p\ \in\ F\ \in\ \mathcal{J}^u(H)\ \}\ \ \subset\ \ \widetilde{M}.$$

Then $\mathcal{L}^u(H) \subset \widetilde{M}$ and $\widetilde{W}^s(p) \subset \partial \mathcal{L}^u(H)$. Similarly define $\mathcal{J}^s(L), \mathcal{L}^s(L)$ for a stable half leaf L.

Definition 4.4. *([Fe2, Fe6]) Lozenges - Let $p, q \in \widetilde{M}$ and half leaves L_p, H_p of $\widetilde{W}^s(p), \widetilde{W}^u(p)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(p)$, half leaves L_q, H_q of $\widetilde{W}^s(q), \widetilde{W}^u(q)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(q)$ so that:*

$$\mathcal{L}^u(L_p)\ \cap\ \mathcal{L}^s(H_q) \ = \ \mathcal{L}^u(L_q)\ \cap\ \mathcal{L}^s(H_p) \ \ \subset\ \widetilde{M}$$

Then this intersection is called a lozenge $\mathcal{A}$ in $\widetilde{M}$, see fig. 10, b. The corners of the lozenge are $\widetilde{\Phi}_{\mathbf{R}}(p)$ and $\widetilde{\Phi}_{\mathbf{R}}(q)$ and $\mathcal{A}$ is a subset of $\widetilde{M}$. The sides of $\mathcal{A}$ are L_p, H_p, L_q, H_q. The sides are not contained in the lozenge, but are in the boundary of the lozenge.

There are no singularities in the lozenges, which implies that $\mathcal{A}$ is an open region in $\widetilde{M}$. There may be singular orbits on the sides of the lozenge and the corner orbits. The definition of a lozenge implies that L_p, H_q form a perfect fit and so do L_q, H_p. This is an equivalent way to define a lozenge with corners $\widetilde{\Phi}_{\mathbf{R}}(p), \widetilde{\Phi}_{\mathbf{R}}(q)$.

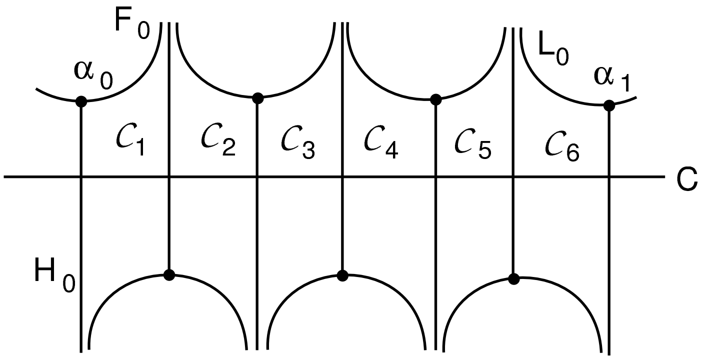

Figure 11: *The correct picture between non separated leaves of $\widetilde{\mathcal{F}}^s$. Here there are 6 lozenges $\mathcal{C}_1, ..., \mathcal{C}_6$.*

Two lozenges are *adjacent* if they share a corner and there is a stable or unstable leaf intersecting both of them, see fig. 10, c. Therefore they share a side. A *chain of lozenges* is a collection $\{\mathcal{C}_i\}, i \in I$, where I is an interval (finite or not) in $\mathbf{Z}$; so that if $i, i+1 \in I$, then $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ share a corner, see fig. 10, c. Consecutive lozenges may be adjacent or not. The chain is finite if I is finite.

Definition 4.5. *Suppose $\zeta \subset E \in \widetilde{\mathcal{F}}^s$ is a (possibly infinite) strong stable segment so that for each $p \in \zeta$ there is a half leaf H_p of $\widetilde{W}^u(p)$ defined by $\widetilde{\Phi}_{\mathbf{R}}(p)$ so that*

$$\forall\ p, q \in \zeta, \quad \mathcal{J}^u(H_p) = \mathcal{J}^u(H_q). \quad \textit{In that case let } \mathcal{P} = \bigcup_{p \in \zeta} H_p.$$

Then $\mathcal{P} \subset \widetilde{M}$ is called an unstable product region with base segment ζ. Similarly define stable product regions.

The main property of product regions is the following: for any $F \in \widetilde{\mathcal{F}}^s, G \in \widetilde{\mathcal{F}}^u$ so that (i) $F \cap \mathcal{P} \neq \emptyset$ and (ii) $G \cap \mathcal{P} \neq \emptyset$, then $F \cap G \neq \emptyset$. There are no singular orbits of $\widetilde{\Phi}$ in $\mathcal{P}$.

We will also denote by rectangles, perfect fits, lozenges and product regions the projection of these regions to $\mathcal{O} \cong \mathbf{R}^2$.

A leaf L of $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$ is called *periodic* if there is a non trivial covering translation g of $\widetilde{M}$ with $g(L) = L$. This is equivalent to $\pi(L)$ containing a periodic orbit of Φ. In the same way an orbit γ of $\widetilde{\Phi}$ is *periodic* if $\pi(\gamma)$ is a periodic orbit of Φ.

We say that two orbits γ, α of $\widetilde{\Phi}$ (or the leaves $\widetilde{W}^s(\gamma), \widetilde{W}^s(\alpha)$) are connected by a chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, if γ is a corner of $\mathcal{C}_1$ and α is a corner of $\mathcal{C}_n$.

Theorem 4.6. *[Fe6] Let Φ be a pseudo-Anosov flow in M^3 closed and let $F_0 \neq F_1 \in \widetilde{\mathcal{F}}^s$. Suppose that there is a non trivial covering translation g with $g(F_i) = F_i, i = 0, 1$. Let $\alpha_i, i = 0, 1$ be the periodic orbits of $\widetilde{\Phi}$ in F_i so that $g(\alpha_i) = \alpha_i$. Then α_0 and α_1 are connected by a finite chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$ and g leaves invariant each lozenge $\mathcal{C}_i$ as well as their corners.*

A chain from α_0 to α_1 is called *minimal* if all lozenges in the chain are distinct. Exactly as proved in [Fe3] for Anosov flows, it follows that there is a unique minimal chain from α_0 to α_1 and also all other chains have to contain all the lozenges in the minimal chain.

The main result concerning non Hausdorff behavior in the leaf spaces of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$ is the following:

Theorem 4.7. *[Fe6] Let Φ be a pseudo-Anosov flow in M^3. Suppose that $F \neq L$ are not separated in the leaf space of $\widetilde{\mathcal{F}}^s$. Then F is periodic and so is L. Let F_0, L_0 be the line leaves of F, L which are not separated from each other. Let V_0 be the sector of F bounded by F_0 and containing L. Let*

α be the periodic orbit in F_0 and H_0 be the component of $(\widetilde{W}^u(\alpha) - \alpha)$ contained in V_0. Let g be a non trivial covering translation with $g(F_0) = F_0$, $g(H_0) = H_0$ and g leaves invariant the components of $(F_0 - \alpha)$. Then $g(L_0) = L_0$. This produces closed orbits of Φ which are freely homotopic in M. Theorem 4.6 then implies that F_0 and L_0 are connected by a finite chain of lozenges $\{\mathcal{C}_i\}, 1 \leq i \leq n$, all contained in $\mathcal{L}^u(H_0)$. Consecutive lozenges are adjacent. There is an even number of lozenges in the chain, see fig. 11. In addition let $\mathcal{B}_{F,L}$ be the set of leaves non separated from F and L. Put an order in $\mathcal{B}_{F,L}$ as follows: Let $C \in \widetilde{\mathcal{F}}^s$ not singular so that $C \cap H_0 \neq \emptyset$. Put an orientation in $\zeta_1 = \widetilde{W}^{ss}(a)$ where $a \in C$. If $R_1, R_2 \in \mathcal{B}_{F,L}$ let α_1, α_2 be the respective periodic orbits in R_1, R_2. Then $\widetilde{W}^s(\alpha_i) \cap C \neq \emptyset$ and let $a_i = \widetilde{W}^u(\alpha_i) \cap \zeta_1$. We define $R_1 < R_2$ in $\mathcal{B}_{F,L}$ if a_1 precedes a_2 in the orientation of ζ_1. Then $\mathcal{B}_{F,L}$ is either order isomorphic to $\{1, ..., n\}$ for some $n \in \mathbf{N}$; or $\mathcal{B}_{F,L}$ is order isomorphic to the integers $\mathbf{Z}$. In addition if there are $Z, S \in \widetilde{\mathcal{F}}^s$ so that $\mathcal{B}_{Z,S}$ is infinite, then there is an incompressible torus in M transverse to Φ. In particular M cannot be atoroidal. Finally up to covering translations, there are only finitely many non Hausdorff points in the leaf space of $\widetilde{\mathcal{F}}^s$.

For detailed explanations and proofs, see [Fe6]. Finally notice that product regions are very rare:

Theorem 4.8. *[Fe6] Let Φ be a pseudo-Anosov flow in M^3. If there is a product region in $\widetilde{M}$, then Φ is topologically conjugate to a suspension Anosov flow.*

5 Non free actions of $\mathbf{Z} \oplus \mathbf{Z}$ subgroups

We now analyse a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$ where M supports a pseudo-Anosov flow Φ. Let $\mathbf{A} \cong \mathbf{Z} \oplus \mathbf{Z}$ be such a subgroup. Then $\mathbf{A}$ acts by homeomorphisms in the plane $\mathcal{O}$. Recall that $\mathbf{A}$ acts freely in $\mathcal{O}$ if any non trivial element in $\mathbf{A}$ does not have fixed points.

Suppose $\mathbf{A}$ does not act freely in $\mathcal{O}$ and let $f \in \mathbf{A} - \{id\}$ not acting freely in $\mathcal{O}$. Then $f(\gamma) = \gamma$ for some orbit γ of $\widetilde{\Phi}$. Let S_f be the graph whose vertices are the fixed points of f in $\mathcal{O}$ and edges corresponding to a lozenge in $\mathcal{O}$ with corners at two fixed points of f. By theorem 4.6 any two β_1, β_2 with $f(\beta_1) = \beta_1$, $f(\beta_2) = \beta_2$ are connected by a chain of lozenges, hence S_f is connected. In addition any chain from β_1 to β_2 has to contain the minimal chain from β_1 to β_2, so any path in S_f from β_1 to β_2, has to contain the path associated to the minimal chain. It follows that S_f is a tree. Two of the sides of a lozenge with corner β are in $\widetilde{W}^s(\beta), \widetilde{W}^u(\beta)$. Therefore each vertex of S_f has valence $\leq 2k_0$, where k_0 is the maximum number of prongs at a singularity of Φ. This means that S_f is a locally finite simplicial tree. The following result is similar to the one in [Ba3] for Anosov flows.

Lemma 5.1. *If g acts freely in $\mathcal{O}$, then g^n acts freely in $\mathcal{O}$ for any $n \in \mathbf{Z} - \{0\}$.*

Proof. Suppose g^n does not act freely in $\mathcal{O}$, with $n \geq 2$. Let $h = g^n$. Then g sends a fixed point of h to another one, so g acts on the tree S_h. This action is torsion because $h = g^n$ acts by identity on S_h. Look at the orbit of a vertex $p \in S_h$. The convex hull of this orbit is the smallest subtree of S_h containing this orbit. It is a finite simplicial tree, which is invariant by g. The theory of group actions on finite trees implies that either g fixes a vertex or has an invariant edge, flipping the endpoints.

The second option is equivalent to $g(\mathcal{B}) = \mathcal{B}$, where $\mathcal{B}$ is a lozenge and g switches the corners β, γ of $\mathcal{B}$. Then g acts as an orientation reversing homeomorphism is the set of stable leaves in $\mathcal{B}$, hence g leaves invariant a stable leaf L in $\mathcal{B}$. But g^n fixes the corners of $\mathcal{B}$, so g^n would fix two orbits β and $\widetilde{W}^u(\beta) \cap L$ in $\widetilde{W}^u(\beta)$ contradiction. The remaining option is that g fixes a vertex in S_h and so g has a fixed point in $\mathcal{O}$ as desired. This finishes the proof. $\square$

By the lemma we may assume that if $f \in \mathbf{A}$ does not act freely in $\mathcal{O}$, then f is indivisible in $\mathbf{A}$. Let $g \in \mathbf{A}$ so that f, g form a basis of $\mathbf{A}$. If $g(\gamma) = \gamma$, then $\mathbf{A}$ leaves γ invariant. But the stabilizer of an orbit of $\widetilde{\Phi}$ is at most infinite cyclic, because leaves of $\mathcal{F}^s$ are either simply connected or with fundamental group isomorphic to $\mathbf{Z}$. Therefore this is a contradiction and $g(\gamma) = \beta \neq \gamma$. Since

$$f(\beta) \ = \ f(g(\gamma)) \ = \ g(f(\gamma)) \ = \ g(\gamma) \ = \ \beta,$$

then $f(\beta) = \beta, f(\gamma) = \gamma$. Theorem 4.6 implies that γ and β are connected by a chain of lozenges $\mathcal{C}_1, ..., \mathcal{C}_n$ with γ corner of $\mathcal{C}_1$ and β corner of $\mathcal{C}_n$. This produces a free homotopy in M from $\pi(\gamma)$ to $\pi(\beta) = \pi(\gamma)$, that is an immersed, incompressible torus which represents $\mathbf{A}$. Using the same techniques as in [Ba3], we can then homotope the torus to be a finite union of closed annuli A_i so ∂A_i is a pair of closed orbits of Φ and the interior of A is transverse to Φ. Each annulus lifts to an infinite strip in $\widetilde{M}$ which is contained in a lozenge with the boundary in the corners of the lozenge. The boundary orbits project to closed orbits in M and the interior of the strip is transverse to $\widetilde{\Phi}$. Barbot calls such a torus a *quasi-transverse* torus [Ba3]. This is the canonical form of a free homotopy from a closed orbit of Φ to itself. In certain situations one can homotope the torus to be transverse to Φ. This finishes the analysis of the action of $\mathbf{A}$ in the non free case.

6 The free case - Almost invariant situation

From now on assume that $\mathbf{A}$ acts freely in $\mathcal{O}$. The proof is similar to the situation of Anosov flows [Fe4], but various steps are different because of singularities. In addition theorem 4.1 of [Fe4] which was needed for the argument in [Fe4] was incorrect as stated and in section 8 we provide a corrected version of that result.

Recall that $\mathcal{H}^s$ is the leaf space of $\widetilde{\mathcal{F}}^s$ and that $\mathcal{H}^s$ is a non Hausdorff tree. Similarly for $\mathcal{H}^u$, hence the results of section 3 can be applied. The following abuse of notation will be used: if $f \in \pi_1(M)$, think of f as a covering tranlsation of $\widetilde{M}$. The same notation f will be used for the induced actions in $\mathcal{O}$, $\mathcal{H}^s$, $\mathcal{H}^u$ and any invariant subsets thereof. The context will make it clear where the action is taking place.

Given any non trivial $f \in \mathbf{A}$, it acts freely in $\mathcal{H}^s$, a non Hausdorff tree. Theorem 3.8 shows that f has an invariant axis in $\mathcal{H}^s$. Since all non trivial elements of $\mathbf{A}$ commute with each other and act freely in $\mathcal{H}^s$, lemma 3.11 implies that the invariant axis is the same for all elements of $\mathbf{A}$ and this common axis is denoted by $\mathcal{I}^s \subset \mathcal{H}^s$.

Lemma 6.1. *The invariant axis $\mathcal{I}^s$ is an open segment.*

Proof. Otherwise the invariant axis is $\mathcal{I}^s = \cup_{i \in \mathbf{Z}} [x_i, y_i]$. Each $g \in \mathbf{A}$ acts in $\mathcal{I}^s$ and shifts the collection of intervals in $\mathcal{I}^s$. Therefore g induces an action in $\mathbf{Z}$ - the set of intervals. This produces an action of $\mathbf{A} = \mathbf{Z} \oplus \mathbf{Z}$ on $\mathbf{Z}$. There must be a non trivial $h \in \mathbf{A}$ which has a fixed point in $\mathbf{Z}$, which implies that $h([x_i, y_i]) = [x_i, y_i]$. This produces a fixed point of h in $[x_i, y_i]$, hence a fixed point of h in $\mathcal{H}^s$, contradiction. Therefore $\mathcal{I}^s$ is an open segment as desired. ∎

Lemma 6.2. *Let $g \in \mathbf{A}$. Then there are no almost invariant leaves under g.*

Proof. Suppose that g leaves L almost invariant, that is, L and $g(L)$ are non separated from each other. By theorem 4.7, the leaf L has a periodic orbit γ. Let $\mathcal{B}_L$ be the set of leaves of $\widetilde{\mathcal{F}}^s$ which are non separated from both L and $g(L)$. Let U be the component of $\widetilde{M} - L$ containing $g(L)$. If $g(U) \subset U$, then as seen in case I.1 of the proof of theorem 3.8, the leaves $L, g(L) \in \mathcal{A}(g)$ and by proposition 3.10, the axis $\mathcal{A}(g) = \mathcal{I}^s$ is an infinite union of closed, disjoint segments. This is disallowed by the previous lemma.

Therefore $g(U) \not\subset U$ and the analysis of case I.2 of theorem 3.8 implies the following: if Z is the set of stable leaves intersected by $\widetilde{W}^u(\gamma) \cap U$, then $g(Z) = Z$. Hence Z produces an invariant segment in $\mathcal{H}^s$ under g, so as seen before $Z = \mathcal{I}^s$. But then any $f \in \mathbf{A}$ has $f(\mathcal{I}^s) = \mathcal{I}^s$, preserving orientation, which implies that $f(\mathcal{B}_L) = \mathcal{B}_L$. By theorem 4.7, the set $\mathcal{B}_L$ is order isomorphic to either $\mathbf{Z}$ or $\{1, ..., n\}$ for some $n \in \mathbf{N}$. Since $\mathbf{A} \cong \mathbf{Z} \oplus \mathbf{Z}$ acts on $\mathcal{B}_L$ there would be a non trivial element in $\mathbf{A}$ with a fixed point in $\mathcal{B}_L$. This would leave a leaf of $\mathcal{B}_L$ invariant, contradiction to hypothesis of free action of $\mathbf{A}$ in this case. This contradiction shows that there cannot be any almost invariant leaves under g and finishes the proof of the lemma. □

Now we know that any $g \in \mathbf{A}$ acts freely in $\mathcal{H}^s$ and separates points. Similarly for the action of $\mathbf{A}$ in $\mathcal{H}^u$ producing an invariant axis $\mathcal{I}^u$. Recall the projections $\pi_s : \widetilde{M} \to \mathcal{H}^s$ and $\pi_u : \widetilde{M} \to \mathcal{H}^u$. Let $\mathcal{R}^s = \pi_s^{-1}(\mathcal{I}^s) \subset \widetilde{M}$ which is a union of leaves of $\widetilde{\mathcal{F}}^s$ and let $\mathcal{R}^u = \pi_u^{-1}(\mathcal{I}^u) \subset \widetilde{M}$. Notice $\mathcal{R}^s$ is connected, but usually not open because of singular leaves in $\mathcal{R}^s$. Then $\mathcal{I}^s, \mathcal{I}^u$ are homeomorphic to $\mathbf{R}$.

Lemma 6.3. *If G is any leaf in $\mathcal{I}^u$ then $G \cap \mathcal{R}^s \neq \emptyset$. In particular if J is any infinite interval in $\mathcal{I}^u \cong \mathbf{R}$, then $\pi_u^{-1}(J) \cap \mathcal{R}^s \neq \emptyset$. Similarly any $S \in \mathcal{I}^s$ intersects $\mathcal{R}^u$.*

Proof. Suppose not and there is $G \in \mathcal{I}^u$ with $G \cap \mathcal{R}^s = \emptyset$. Because $\mathcal{I}^s$ is an interval, then $\mathcal{R}^s$ is connected, hence it is contained in a component of $\mathcal{R}^u - G$. There is a component of $\mathcal{R}^u - G$ which cannot intersect the set $\mathcal{R}^s$, so there is an infinite interval J of $\mathcal{I}^u$ with $\pi_u^{-1}(J) \cap \mathcal{R}^s = \emptyset$. We will show this is impossible.

Let $g \in \mathbf{A}$, with g non trivial. Looking at the action of g in $\mathcal{I}^u$, we may assume by taking inverse if necessary, that $U \subset g(J)$ and so $\cup_{n>0} g^n(J) = \mathcal{I}^u$ - because g has no fixed points in $\mathcal{I}^u$. Using that $g(\mathcal{R}^s) = \mathcal{R}^s$ and $\pi_u^{-1}(g^n(J)) = g^n(\pi_u^{-1}(J))$, then

$$\mathcal{R}^s \cap \mathcal{R}^u \;=\; \mathcal{R}^s \cap \pi_u^{-1}(\mathcal{I}^s) \;=\; \mathcal{R}^s \cap \pi_u^{-1}\left(\bigcup_{n>0} g^n(J)\right)$$

$$= \bigcup_{n>0} \left(\mathcal{R}^s \cap g^n(\pi_u^{-1}(J))\right) \;=\; \bigcup_{n>0} g^n\left(\mathcal{R}^s \cap \pi_u^{-1}(J)\right) \;=\; \emptyset.$$

We now show that $\mathcal{R}^s \cap \mathcal{R}^u = \emptyset$ is impossible. Since $\mathcal{R}^s$ is connected and $\mathcal{R}^u \cap \mathcal{R}^s = \emptyset$, there is a unique leaf $F \in \widetilde{\mathcal{F}}^s$ so that $F \subset \partial \mathcal{R}^s$ and a line leaf of F separates $\mathcal{R}^s$ from $\mathcal{R}^u$. For any $g \in \mathbf{A}$, $g(\mathcal{R}^s) = \mathcal{R}^s, g(\mathcal{R}^u) = \mathcal{R}^u$ implies that $g(F) = F$. But $\mathbf{A}$ acts freely, contradiction. This contradiction finishes the proof of the lemma. □

7 The finite case

The proof is now divided in two cases. In this section we suppose that for any $G \in \mathcal{I}^u$, G intersects the leaves of $\mathcal{I}^s$ in at most a bounded subsegment of $\mathcal{I}^s$ - hence the name finite case. The goal is to show that this case cannot occur.

Fix an order $<_s$ in $\mathcal{I}^s$ and parametrize is as

$$\mathcal{I}^s \;=\; \{F_t \mid t \in \mathbf{R}\},$$

with $F_t <_s F_s$ if $t < s$. This in fact determines an order in $\mathcal{I}^u$ as follows:

Definition 7.1. *(Order in $\mathcal{I}^u$) Put a linear order in $\mathcal{I}^u$ by: if $G \in \mathcal{I}^u$ with $G \cap \mathcal{R}^s \neq \emptyset$ let $L \in \mathcal{I}^s$ with $G \cap L \neq \emptyset$. Let $E \in \mathcal{I}^s$ with $E >_s L$ and $E \cap G = \emptyset$ (here we use the running hypothesis in this*

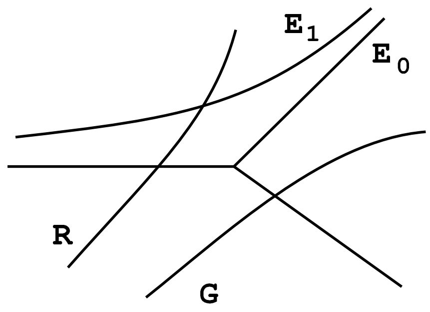

Figure 12: *Singularities can produce interesting behavior. Here $R >_u G$. The leaf E_0 is a singular leaf. For any E_1 arbitrarily near E_0, then $E_1 \cap G = \emptyset$, even though $E_0 \cap G \neq \emptyset$.*

case). By lemma 6.3 the leaf E has to intersect $\mathcal{R}^u$. Since $E \cap G = \emptyset$. There is only one component of $\mathcal{R}^u - G$ which can intersect E. Let this component of $\mathcal{R}^u - G$ define the positive direction from G in $\mathcal{I}^u$, or the elements bigger than G in $\mathcal{I}^u$ in this order. This defines a linear order in $\mathcal{I}^u$.

One fundamental fact for all the analysis in this case is:

Lemma 7.2. *The above linear order in $\mathcal{I}^u$ is independent of the choice of E or G.*

Proof. First notice that if $G \cap F_a \neq \emptyset$ and $G \cap F_b \neq \emptyset$, then $G \cap F_t \neq \emptyset$ for any $a < t < b$, because $\mathcal{I}^s$ is a segment, so it is connected and any leaf of $\mathcal{I}^s$ separates $\mathcal{I}^s$ into two components. Let now $E_0 \in \mathcal{I}^s$, so that $E_0 >_s F$ and E_0 is the smallest (in $<_s$) not intersecting G or $E_0 \cap G \neq \emptyset$, but for for any $E_1 \in \mathcal{I}^s$, $E_1 >_s E$ then $E_1 \cap G = \emptyset$, see fig. 12. Notice that a priori $G \cap E_0 \neq \emptyset$ does not imply $G \cap E_1 \neq \emptyset$ for any E_1 near E_0 as is the case for non singular foliations. This is because of the singularities, G may intersect a prong of E_0 which is not near E_1 even though E_1 is near E_0, see fig. 12.

The set $V_G = \{x \in L \in \mathcal{I}^s \mid L >_s E_0\} \subset \widetilde{M}$ is connected and does not intersect G. Hence V_G is contained in a single component of $\widetilde{M} - G$, so it can intersect only a fixed component of $\mathcal{R}^u - G$ no matter which E is chosen in $\mathcal{I}^s$ with $E >_s E_0$. The same component occurs if $E_0 \cap G = \emptyset$. Hence this component of $\widetilde{M} - G$ is independent of $E \geq_s E_0$. This shows that the linear order in $\mathcal{I}^u$ depends only in $G \in \mathcal{I}^u$ - denote this order by $<_G$.

If G' is another leaf of $\mathcal{I}^u$ let $<_{G'}$ be the linear order associated to G'. We want to show that both orders are the same. Assume first that $G' <_G G$. Let $F' \in \mathcal{I}^s$ with $F' \cap G' \neq \emptyset$. Choose

$$E \in \mathcal{I}^s, \quad E >_s F, \ E >_s F' \quad \text{and} \quad E \cap G = \emptyset, \ E \cap G' = \emptyset.$$

.

But $E \cap \mathcal{R}^u \neq \emptyset$, so E is in the component of $\mathcal{R}^u - G$ defining elements bigger than G in $<_G$. Since $G' <_G G$, then G' is not in this component of $\mathcal{R}^u - G$. Hence G separates G' from E. Also by definition of $<_{G'}$ using E, it follows that E is in the component of $\mathcal{R}^u - G'$ defining bigger elements that G' in $<_{G'}$. But as G separates G' from E, then G is also contained in this component of $\mathcal{R}^u - G'$. It follows that G is bigger than G' in the $<_{G'}$ order or $G <_{G'} G'$. If $G <_G G'$, do the same argument switching the roles of G and G' to obtain $G <_{G'} G'$. Hence $G <_G G'$ if and only if $G <_{G'} G'$. But the orders $<_G, <_{G'}$ are linear orders on $\mathcal{I}^u$, so the relative position of any two elements under the order determines the whole order, that is,

$$\forall \ H, R \in \mathcal{I}^u, \quad H <_G R \iff H <_{G'} R.$$

Since G, G' are arbitrary in $\mathcal{I}^s$, this shows the order is independent of G and finishes the proof. $\square$

Remark - The proof shows that for any $G \in \mathcal{I}^u$, there is $t_0 \in \mathbf{R}$ so that for any $t > t_0$, the leaf F_t is contained in the component of $\widetilde{M} - G$ containing those $G' \in \mathcal{I}^u$ with $G' >_u G$.

Let $<_u$ denote the unique linear order in $\mathcal{I}^u$ defined above. The uniqueness of the linear order $<_u$ is a key fact which implies a fundamental result for us:

Lemma 7.3. *Let $f \in \mathbf{A}$. Then f induces an increasing homeomorphism in $\mathcal{I}^s$ if and only if f induces an increasing homeomorphism of $\mathcal{I}^u$.*

Proof. Suppose f is increasing in $>_s$. Let $G \in \mathcal{I}^u$ and $F \in \mathcal{I}^s$ with $G \cap F \neq \emptyset$. Choose $E >_s F$ with $E \cap G = \emptyset$. Choose $n > 0$ so that $f^n(F) >_s E$. The component of $\mathcal{R}^s - G$ intersecting $f^n(F)$ defines elements $>_u G$ in $\mathcal{I}^u$. But $f^n(F)$ intersects $f^n(G)$, hence $f^n(G) >_u G$. So f^n acts as an increasing homeomorphism in $\mathcal{I}^u$ with the order $<_u$. As a result f also induces an increasing homemorphism in $\mathcal{I}^u$. This finishes the proof of the lemma. $\square$

This is the key lemma in the proof. It is at this point that things are different if some $G \in \mathcal{I}^u$ intersects a non bounded interval of leaves in $\mathcal{I}^s$ or vice versa.

Since $\mathbf{A}$ is abelian, Plante [Pl2] showed that the action of $\mathbf{A}$ over $\mathcal{I}^s \cong \mathbf{R}$ is semi conjugate to a linear action of $\mathbf{A}$ in $\mathbf{R}$: there is a surjective monotone map:

$$\varphi_s : \mathcal{I}^s \to \mathbf{R}, \quad \text{and} \quad \rho_s : \mathbf{A} \to \mathbf{R},$$

with ρ_s a homomorphism which satisfies the following condition:

$$\forall F \in \mathcal{I}^s, \quad \forall g \in \mathbf{A}, \quad \varphi_s(g(F)) \;=\; \varphi_s(F) + \rho_s(g).$$

Since $\mathbf{A}$ acts freely in $\mathcal{I}^s$ then ρ_s is injective [Pl2]. In addition since $\mathbf{A}$ has rank two, the action of $\mathbf{A}$ on $\mathcal{I}^s$ admits a unique minimal closed invariant set $\mu_s \subset \mathcal{I}^s$. This set is a perfect set and $\mathcal{I}^s - \mu_s$ (which could be empty) is exactly the set where φ_s is constant. Notice that $\rho_s(g) > 0$ if and only if g acts as an increasing homeomorphism of $\mathcal{I}^s$. The same happens for $\mathcal{I}^u$ producing φ_u, ρ_u and μ_u. Since μ_s is a perfect set, there are $F \in \mu_s$ which are not isolated on both sides.

The previous lemma shows that for any $g \in \mathbf{A}$, $\rho_s(g) > 0$ if and only if $\rho_u(g) > 0$. This easily implies that for any $f, g \in \mathbf{A}$:

$$\rho_s(g) \; > \; \rho_s(f) \quad \Longleftrightarrow \quad \rho_u(g) \; > \; \rho_u(f).$$

Lemma 7.4. *There is $F \in \mu_s$ not isolated on both sides in μ_s and F intersecting $G \in \mathcal{I}^u$ which is in μ_u.*

Proof. Let $F \in \mu_s$ not isolated on both sides. If F satisfies the condition of the lemma we are done. Otherwise consider the set $B \subset \mathcal{I}^s$ all $G' \in \mu_u$ intersecting only leaves in $\mathcal{I}^s$ which are smaller than F in $<_s$. The definition of $<_u$ using any G_0 with $G_0 \cap F \neq \emptyset$ implies that any such $G' \in B$ satisfies $G' <_u G_0$. Hence the set B is bounded above in $\mathcal{I}^u$. Let G be the supremum in $<_u$ of the set B. Use the fact that μ_u is closed in $\mathcal{I}^u$, hence $G \in \mu_u$. Notice that by assumption $G \cap F = \emptyset$.

By lemma 6.3 for any $G \in \mathcal{I}^u$, then $G \cap \mathcal{R}^s \neq \emptyset$. Let $E \in \mathcal{I}^s$ with $G \cap E \neq \emptyset$. If F is the smallest in $\mathcal{I}^s$ not intersecting G and with $F >_s E$, choose $F_1 \in \mu_s$ near enough F with $E <_s F_1 <_s F$ and $F_1 \cap G \neq \emptyset$ with F_1 not isolated on both sides. This uses the fact that F is not isolated on both sides. This finishes the proof in this case.

Otherwise choose $F' \in \mu_s$ with $E <_s F' <_s F$ and $F' \cap G = \emptyset$. This uses F is not isolated on the negative side. Since the orbit of F' under $\mathbf{A}$ is dense in μ_s and F is not separated on both sides in μ_s choose $g \in \mathbf{A}$ with $F' <_s g(F') <_s F$. Then $\rho_s(g) > 0$, so by the previous lemma $\rho_u(g) > 0$

and $g(G) >_u G$ in $\mathcal{I}^u$. But $g(G) \cap g(F') = \emptyset$. Recall that by definition, G can only intersect those L in $\mathcal{I}^s$ with $L <_s F'$. Hence $g(G)$ only intersects $L \in \mathcal{I}^s$ with $L <_s g(F') <_s F$. This implies that $g(G) \cap F = \emptyset$. This is contradiction to the definition of G, because $g(G) >_u G$, $g(G) \in \mathcal{I}^u$. Therefore this case cannot happen and this finishes the proof of the lemma. □

Lemma 7.5. *If $g_i \in \mathbf{A}$ with $\rho_s(g_i) \to 0$, then $\rho_u(g_i) \to 0$.*

Proof. We may assume $\rho_s(g_i) > 0$ and $\rho_s(g_i)$ decreasing with i increasing. Hence $\rho_u(g_i) > 0$ and $\rho_u(g_i)$ decreasing with i increasing. If $\rho_u(g_i) \not\to 0$ then $\rho_u(g_i) \to a > 0$. Since $\rho_u(\mathbf{A})$ is a rank two subgroup of $\mathbf{R}$, it is dense in $\mathbf{R}$. Choose $f \in \mathbf{A}$ with $0 < \rho_u(f) < a$. Since $a < \rho_u(g_i)$, then $0 < \rho_s(f) < \rho_s(g_i)$ for all i, contradicting $\rho_s(g_i) \to 0$. □

We now produce a contradiction to the hypothesis of the finite case.

Proposition 7.6. *There is $h \in \mathbf{A}$ not acting freely in $\mathcal{O}$.*

Proof. Using lemma 7.4, choose $F \in \mu_s$ not isolated on both sides and intersecting $G \in \mu_u$. Without loss of generality suppose that G is not isolated on the positive side of G in $\mathcal{I}^u$. Let $\gamma = F \cap G$ which is a single orbit of $\widetilde{\Phi}$. Choose $h_i \in \mathbf{A}$ with $\rho_s(h_i) > 0$ and $\rho_s(h_i) \to 0$ decreasing. Then $\rho_u(h_i) > 0$, $\rho_u(h_i) \to 0$. Also $h_i(F) \in \mu_s \subset \mathcal{I}^s$ and since F is not isolated in μ_s then $h_i(F) \to F$ in $\mathcal{I}^s$ and hence in $\mathcal{H}^s$. Similarly $h_i(G) \to G$ in $\mathcal{H}^u$. Because of singularities we may not have a product picture of the foliations near γ - so $h_i(F) \to F$ and $h_i(G) \to G$ does not a priori imply that $h_i(\gamma) \to \gamma$. For instance if γ is singular, the fact that $L_i \to F$ and $S_i \to G$, does not imply at all that $L_i \cap S_i \to F \cap G$. It could well happen that $F_i \cap S_i = \emptyset$. However, because $h_i(F)$ and $h_i(G)$ intersect, that is, $\emptyset \neq h_i(F) \cap h_i(G) = h_i(\gamma)$, it now follows that $h_i(\gamma) \to \gamma$ (even if γ is a singular orbit).

Since $h_i(\gamma) \to \gamma$, but not equal to γ, it follows in particular that $\alpha = \pi(\gamma)$ is not a closed orbit of Φ. Therefore α is not a singular orbit also. Choose $x \in \alpha$. Since the flow line α keeps returning arbitrarily near x and x is not singular, a long segment of α is shadowed by a closed orbit of Φ, see shadow lemma for pseudo-Anosov flows [Ha, Ma, Mos2]. This closed orbit of Φ corresponds to a covering translation f taking γ to $h_i(\gamma)$ (or vice versa). Hence $f^{-1}(h_i(\gamma)) = \gamma$ and since the stabilizer of γ is trivial (as γ is not periodic), it follows that $f = h_i$. But then h_i is associated to a closed orbit of Φ and therefore h_i does not act freely in $\mathcal{O}$. Let $h = h_i$. This finishes the proof of the proposition. □

The proposition shows that the running hypothesis in this section is contradictory with free action of $\mathbf{A}$.

8 The infinite case

We now prove the main theorem. The case that $\mathbf{A} \cong \mathbf{Z} \oplus \mathbf{Z}$ does not act freely in $\mathcal{O}$ was dealt with in section 5. Here we finish the analysis of the free action case:

Theorem 8.1. *Let Φ be a pseudo-Anosov flow in M^3 closed. If there is $\mathbf{A} = \mathbf{Z} \oplus \mathbf{Z}$ acting freely in $\widetilde{M}/\widetilde{\Phi} \cong \mathcal{O}$ then Φ is topologically conjugate to a suspension Anosov flow.*

Proof. The previous section shows that there is some leaf of $\mathcal{I}^u$ intersecting an infinite interval in $\mathcal{I}^s$. Recall that $\mathcal{I}^s = \{F_t \mid t \in \mathbf{R}\}$. By reversing this parametrization of $\mathcal{I}^s$ if necessary, we may assume there is $G \in \mathcal{I}^u$ with $G \cap F_t \neq \emptyset$ for any $t \geq a$, where a is a real number. For simplicity choose a so that F_a is a non singular leaf of $\widetilde{\mathcal{F}}^s$. This a is fixed from now on. Let $\gamma = G \cap F_a$, where a is a real number.

Lemma 8.2. *The F_t escapes in $\widetilde{M}$ as $t \to +\infty$.*

Proof. Otherwise since F_t is nested with increasing t, there is $S \in \widetilde{\mathcal{F}}^s$ with $F_t \to S$ when $t \to +\infty$. Given $g \in \mathbf{A}$ then $g(F_t)$ converges to $g(S)$ as $t \to +\infty$. But also $g(F_t) = F_{t'}$ and if $t \to +\infty$ then $t' \to +\infty$ because g acts by orientation preserving homeomorphisms of $\mathcal{I}^s$. Therefore $g(F_t) = F_{t'}$ also converges to S. Hence S and $g(S)$ are not separated from each other and g does not separate leaves. By lemma 6.2 the group $\mathbf{A}$ does not act freely in $\mathcal{O}$, contradiction. This finishes the proof of the lemma. $\square$

Let now $g \in \mathbf{A}$ with $\rho_s(g) < 0$. Hence $g(F_a) <_s F_a$ in $\mathcal{I}^s$ and by construction $g(G)$ also intersects all of $F_t \in \mathcal{I}^s$ with $F_t \geq_s g(F_a)$. Therefore for any $t > a$, $F_t \cap g(G) \neq \emptyset$. It follows that for any $t > a$ the leaves $F_a, F_t, G, g(G)$ form a rectangle and there are no singularities of $\widetilde{\Phi}$ in the rectangle. Let D be the flow band of F_a between G and $g(G)$. Notice that $g(G) \neq G$, because of free action. Let H be any leaf of $\widetilde{\mathcal{F}}^u$ separating G from $g(G)$, that is, any leaf $H \in \widetilde{\mathcal{F}}^s$ which intersects D. Then $H \cap F_t \neq \emptyset$ for any $t > a$ because of the rectangle condition [Fe6]. In addition since F_t escapes in $\widetilde{\mathcal{F}}^s$ as $t \to +\infty$ these are all the leaves intersected by the half leaf of H defined by $H \cap F_a$. It follows that for any $L \in \widetilde{\mathcal{F}}^s$, then L intersects H if and only if $L = F_t$ for some $t > a$. By definition of product regions, this proves that D is the base flow band of an unstable product region in $\widetilde{M}$. Theorem 4.8 implies that Φ is topologically conjugate to a suspension Anosov flow, which is what we wanted to prove. This finishes the proof of theorem 8.1. $\square$

Some consequences of this result are:

Corollary 8.3. *Let Φ be a pseudo-Anosov flow in M^3 and let T be an incompressible torus in M. Suppose that no loop in T is freely homotopic to a closed orbit of Φ. Then Φ is topologically conjugate to a suspension Anosov flow. If in addition T is embedded then T is isotopic to a torus transverse to Φ.*

Proof. The first statement follows directly from the previous theorem. Since M fibers over the circle with torus fiber and Anosov monodromy, there is only one incompressible torus in M up to isotopy [Bo-La, He], so the second assertion follows. $\square$

Corollary 8.4. *Let Φ be a pseudo-Anosov flow in M^3. Let F be an incompressible torus transverse to the stable foliation and let $\mathcal{F}^s_T$ be the induced foliation by $\mathcal{F}^s$ in T. Then $\mathcal{F}^s_T$ has closed leaves unless Φ is topologically conjugate to a suspension Anosov flow.*

Proof. This is the same result as theorem 8.1 of [Fe4] for Anosov flows. We need the following modifications: first, if necessary lift to a double cover and assume M is orientable. Also in general $\mathcal{H}^s, \mathcal{H}^u$ are not orientable. Rather than talking about the "front" or "back" of $F \in \widetilde{\mathcal{F}}^s$ (or $G \in \widetilde{\mathcal{F}}^u$) we consider the appropriate components of $\widetilde{M} - F$. Given these changes, the proof goes exactly as in [Fe4] to which we refer the reader. The goal is to show that $\pi_1(T)$ acts freely in $\mathcal{H}^s$ - then one can apply the main theorem here.

Notice that incompressibility is necessary and does not follow from the fact that T is transverse to $\mathcal{F}^s$, as opposed to T being transverse to Φ. For example take a small tubular neighborhood of a non singular closed orbit: one obtains a torus T transverse to $\mathcal{F}^s$ and having $\mathcal{F}^s_T$ with two closed leaves and 2 Reeb annuli. Clearly T is compressible. $\square$

We remark here that theorem 4.1 of [Fe4] is incorrect as stated and we provide a correct result.

Proposition 8.5. *Let Φ be an Anosov flow in M^3 orientable. Let T be an embedded incompressible torus in M. Suppose there is $g \in \pi_1(T)$ without invariant leaves in $\widetilde{\mathcal{F}}^s$ but having an almost invariant leaf $F \in \widetilde{\mathcal{F}}^s$ (or $\widetilde{\mathcal{F}}^u$). Then T is isotopic to a torus transverse to Φ.*

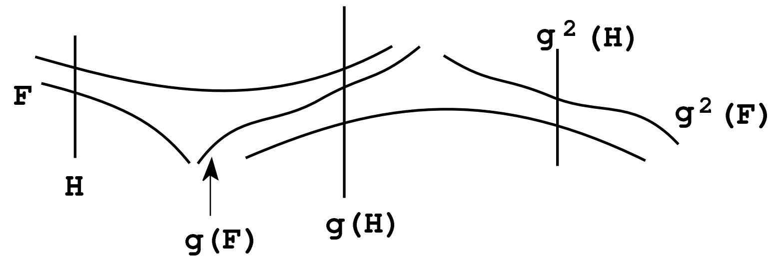

Figure 13: *Non separated leaves and reversal of orientation.*

Proof. Since Φ is an Anosov flow, then $\mathcal{H}^s$ is a simply connected 1-manifold and hence orientable and so is $\mathcal{H}^u$. Fix orientations in $\mathcal{H}^s, \mathcal{H}^u$. Then g can fix or reverse this orientation of $\mathcal{H}^s$. In [Fe4] we used the incorrect fact that if g reverses orientation of $\mathcal{H}^s$ then g has a fixed point in $\mathcal{H}^s$. We can have a situation where $g(F)$ is non separated from $F \in \widetilde{\mathcal{F}}^s$ but g reverses the orientation of $\mathcal{H}^s$, see fig. 13. This is exactly what happens in case I.1 of the analysis in the proof of theorem 3.8.

Hence the component of $\widetilde{M} - F$ containing $g(F)$ is taken to the component of $\widetilde{M} - g(F)$ not containing F. The axis of g is an infinite sequence of leaves $g^i(F)$ so that $g^i(F)$ is not separated from $g^{i+1}(F)$ for any $i \in \mathbf{Z}$.

Suppose this is the case, that is, g reverses the orientation of $\mathcal{H}^s$. As in the proof of lemma 3.11, any f commuting with g preserves $\mathcal{A}(g)$. Hence $\pi_1(T)$ has to act in the axis $\mathcal{A}(g) = \{g^i(F)\}, i \in \mathbf{Z}$ preserving the order. Therefore some $h \in \pi_1(T)$ fixes all elements in the chain. Therefore $\pi_1(T)$ does not act freely in $\mathcal{O}$ and this is the case that T can be put in a form of a free homotopy from a closed orbit of Φ to itself. In this case the torus is isotopic to one which is only quasi-transverse to Φ, see Barbot's work in [Ba3] for details. One does not obtain an isotopic torus transverse to Φ. So clearly this fails theorem 4.1 of [Fe4].

In order to rectify theorem 4.1 of [Fe4], the <u>additional</u> hypothesis here is that M is orientable. We now explain how that helps us. Suppose there is g as above with g reversing the orientation of $\mathcal{H}^s$. Consider the action of g in $\mathcal{H}^u$. Let γ be the periodic orbit in F and let $H = \widetilde{W}^u(\gamma)$. As $F, g(F)$ are not separated from each other, then $g(H)$ and H intersect a common stable leaf, see fig. 13. If g reverses the orientation of $\mathcal{H}^u$, then the components C of $\widetilde{M} - H$ not containing $g(H)$ is taken to the component of $\widetilde{M} - g(H)$ not containing H. Then $g^2(F) \subset g(C)$, but also $g^2(H) \cap g(C) = \emptyset$. Hence $g^2(H)$ cannot intersect $g^2(F)$ contradiction. It follows that g preserves the orientation of $\mathcal{H}^u$. The invariant axis for the action of g on $\mathcal{H}^u$ is a segment, which contains $H, g(H), g^2(H)$, ... Since g reverses the orientation to $\mathcal{H}^s$ but preserves the orientation of $\mathcal{H}^u$, it follows that g reverses the orientation in $\mathcal{O}$ and therefore reverses the orientation of $\widetilde{M}$. Hence M is not orientable. This contradicts the hypothesis of M orientable.

The conclusion is that g must preserve the orientation of $\mathcal{H}^s$. As in the analysis of case I.2 of the proof of theorem 3.8, it follows that the axis of g is a segment in $\mathcal{H}^s$. If g reverses the orientation of $\mathcal{H}^u$, then again M would be non orientable, contradiction. Hence g preserves both orientations. It follows that all $g^n(F), n \in \mathbf{Z}$ are non separated from each other. Also if $g^n(F) = F$ for some $n \neq 0$, then g^n would have to reverse the orientation to $\widetilde{\mathcal{F}}^s$ or $\widetilde{\mathcal{F}}^u$, again a contradiction. Therefore the $g^n(F), n \in \mathbf{Z}$ are all distinct. Let $\mathcal{B}_F$ be the set of leaves non separated from F and $g(F)$. By the above, this set is infinite, so by theorem 4.7 it is order isomorphic to $\mathbf{Z}$. Given any $f \in \pi_1(T)$, then f leaves invariant $\mathcal{A}(g)$ and commutes with g, so f preserves the orientation of $\mathcal{H}^s$. Since M is orientable f also leaves invariant the orientation of $\mathcal{H}^u$. From this point on we can proceed exactly as in the proof of theorem 4.1 in [Fe4]. The end result is that T is isotopic to a torus transverse to Φ. ∎

9 Pseudo-Anosov flows transverse to foliations

There are many situations where a codimension one foliation $\mathcal{G}$ in a 3-manifold M admits a transverse pseudo-Anosov flow [Th2, Th3, Th4, Mos2, Fe8]. Our results help to understand the atoroidal property in this setting. Recall that $\mathcal{G}$ is $\mathbf{R}$-covered if the lifted foliation $\widetilde{\mathcal{G}}$ to $\widetilde{M}$ has leaf space homeomorphic to the real numbers $\mathbf{R}$ [Fe1]. Also an Anosov flow is $\mathbf{R}$-covered if its stable foliation $\mathcal{F}^s$ (equivalently unstable foliation $\mathcal{F}^u$ [Fe1, Ba2]) is $\mathbf{R}$-covered.

Theorem 9.1. *Let $\mathcal{G}$ be an $\mathbf{R}$-covered foliation in M^3 closed so that $\mathcal{G}$ is transverse to a pseudo-Anosov flow Φ. Suppose that Φ is not an $\mathbf{R}$-covered Anosov flow. Then M is homotopically atoroidal, that is, there are no $\mathbf{Z} \oplus \mathbf{Z}$ subgroups of $\pi_1(M)$.*

Proof. One says that that Φ is *regulating* for $\mathcal{G}$ if any orbit of $\widetilde{\Phi}$ intersects all leaves of $\widetilde{\mathcal{G}}$ [Fe7]. The main result of [Fe7] states that if $\mathcal{G}$ is $\mathbf{R}$-covered and transverse to Φ pseudo-Anosov, then Φ is regulating for $\mathcal{G}$ unless Φ is an $\mathbf{R}$-covered Anosov flow. Since we are assuming that the last option does not occur, then Φ is regulating for $\mathcal{G}$. Therefore every flow line intersects all leaves of $\widetilde{\mathcal{G}}$ and vice versa. Fix an identification of the leaf space $\mathcal{H}(\widetilde{\mathcal{G}})$ of $\widetilde{\mathcal{G}}$ with $\mathbf{R}$ so that positive movement in flow lines of $\widetilde{\Phi}$ corresponds to increasing the parameter in $\mathbf{R}$.

Suppose there is $\mathbf{A} \cong \mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$. If $\mathbf{A}$ acts freely in $\mathcal{O}$, the the main theorem of this article shows that Φ is topologically conjugate to a suspension Anosov flow. But a suspension Anosov flow is $\mathbf{R}$-covered, so this is disallowed by hypothesis.

The remaining option is that there is $f \in \mathbf{A} - \{id\}$ and γ orbit of $\widetilde{\Phi}$ with $f(\gamma) = \gamma$. By lemma 5.1, we may assume that f is indivisible in $\mathbf{A}$. Let $g \in \mathbf{A}$ so that $\{f, g\}$ form a basis of $\mathbf{A}$. Then $g(\gamma) \neq \gamma$ and γ, $g(\gamma)$ are connected by a finite chain of lozenges $\mathcal{C}_i, 1 \leq i \leq n$, with γ a corner of $\mathcal{C}_1$. Let α be the other corner of $\mathcal{C}_1$ so $f(\alpha) = \alpha$. The action of f on $\mathcal{C}_1$ shows that the closed orbit $\pi(\gamma)$ of Φ is freely homotopic to the closed orbit $(\pi(\alpha))^{-1}$ [Fe2, Fe6]. Notice that γ and α intersect all leaves of $\widetilde{\mathcal{G}}$. Assume that f translates γ in the positive flow direction, hence increasing the parameter in $\mathcal{H}(\widetilde{\mathcal{G}})$. But $f(\alpha) = \alpha$ and since $\pi(\gamma) \cong (\pi(\alpha))^{-1}$, it follows that f acts as a translation in α in the negative flow direction, therefore decreasing the parameter in $\mathcal{H}(\widetilde{\mathcal{G}})$. These two facts contradict each other. This shows that the second option cannot happen either.

We conclude that there is no $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$, so M is homotopically atoroidal. This finishes the proof of the theorem. $\square$

Remark - Clearly the hypothesis of Φ is not an $\mathbf{R}$-covered Anosov flow is necessary. If Φ is for instance a suspension Anosov flow, one can perturb the unstable foliation of Φ to produce a new $\mathbf{R}$-covered foliation transverse to Φ, see details of this construction in [Fe7]. But clearly M is toroidal - the fiber produces an incompressible torus. The same can be done for a geodesic flow in the unit tangent bundle of a closed hyperbolic surface [Fe7] - which is also an $\mathbf{R}$-covered Anosov flow [Fe1]. Again the underlying manifold is toroidal.

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