Foliations, topology and geometry of 3-manifolds: **R**-covered foliations and transverse pseudo-Anosov flows

Sérgio R. Fenley *

December 1, 2001

1 Introduction

In this article we analyse the topological and geometrical consequences that foliations induce on 3-manifolds. More specifically we study the transverse structure of an **R**-covered foliation in a 3-manifold, where **R**-covered means that in the universal cover the leaf space of the foliation is Hausdorff. When the manifold is aspherical we prove that either there is a region in the leaves where the geometry does not change much transversely, yielding an incompressible torus in the manifold; or there is a transverse pseudo-Anosov flow which captures the directions of maximal stretch/contraction transverse to the foliation. Hence if the manifold is atoroidal and aspherical there is always a transverse pseudo-Anosov flow. As a consequence manifolds with **R**-covered foliations satisfy the weak hyperbolization conjecture.

The goal of this article is to understand the geometrical/topological implications of the existence of a foliation in a 3-manifold. First we review some basic facts of foliation theory. The main villain in 3-manifold foliation theory is the Reeb component: a foliation of the solid torus where the boundary is a leaf and the interior leaves are topological planes spiralling towards the boundary leaf. Reebless foliations, that is those without Reeb components, are extremely useful in understanding the topology of 3-manifolds: fundamental work of Novikov, and later Rosenberg, Palmeira showed that leaves inject in the fundamental group level (incompressible leaves) [No], the manifold is irreducible (that is every embedded sphere bounds a ball) [Ro] and the universal cover is homeomorphic to $\mathbf{R}^3$ [Pa]. Such foliations have excellent properties and they reflect the topology of the manifold. On the other hand Gabai constructed Reebless foliations in any irreducible, oriented, compact 3-manifold with non trivial second homology and derived fundamental results in 3-manifold theory, such as property R and many other results [Ga1, Ga2, Ga3]. Roberts also constructed many Reebless foliations in large classes of 3-manifolds which are not Haken [Rob] and jointly with Delman used this to prove property P for alternating knots [De-Ro]. Notice that the Reebless property is crucial here, since any closed 3-manifold admits a codimension one foliation [Li, Wo], most of which are not useful for topology - for instance $\mathbf{S}^3$ has many foliations (with Reeb components).

Our focus will be on the transverse geometric structure of a Reebless foliation. Thurston [Th8, Th9, Th10] recently showed that foliations are much better behaved in the transverse direction than was previously expected: nearby leaves stay nearby forever in many directions of the leaf. This gives a tremendous boost in understanding the global structure of foliations and it aids the understanding of the geometry of the foliation and the manifold in connection with the geometrization conjecture [Th4].

There is a natural breakup into two cases here: the lifted foliation in the universal cover is a foliation by planes (or spheres) and the leaf space of this lifted foliation is a 1-manifold which may be Hausdorff or not. In a lot of situations the question of Hausdorff/non Hausdorff turns out to play an important role and have strong consequences [Ve, Ba1, Ba2, Ba3, Fe2, Fe3, Fe4]. If the leaf space is Hausdorff then it is homeomorphic to the real numbers **R** and the foliation is said to be **R**-*covered* [Pl2, Fe2].

*Reseach partially supported by NSF grants DMS 9612317 and DMS 0071683.

In this article we analyse $\mathbf{R}$-covered foliations in 3-manifolds - the simplest case in studying the global structure of foliations in the universal cover. Examples of this large class of foliations are: 1) fibrations over the circle; 2) foliations defined by non singular closed 1-forms; 3) stable/unstable foliations of large classes of Anosov flows in 3-manifolds [Fe2]; 4) slitherings over the circle as defined by Thurston [Th7] - roughly a slithering is a map from the universal cover of the manifold to the circle $\mathbf{S}^1$ which is a fibration equivariant under covering translations, inducing a foliation in the manifold; 5) Uniform foliations: a foliation is uniform if any two leaves in the universal cover are a bounded distance from each other (the bound depends on the pair of leaves) - they are closely related to slitherings [Th7]; 6) Many examples $\mathbf{R}$-covered foliations not induced by slitherings [Cal2]. On the other hand, Reebless finite depth foliations [Ga1, Ga3] are not $\mathbf{R}$-covered unless the compact leaf is a fiber of M over the circle [Go-Sh].

The case of fibrations is very illuminating and is a precursor of the whole idea of analysing the transverse geometry of foliations. In a seminal work Thurston proved that in the aspherical case either there is an incompressible torus transverse to the fibration or there is a suspension flow which is a pseudo-Anosov flow producing singular stable/unstable foliations [Th2, Bl-Ca]. He went on to prove that the pseudo-Anosov case yields hyperbolic manifolds, establishing a deep relationship with geometry [Th3, Th4, Th5]. We concentrate on the first step. Thurston's result can be summarized as follows from the foliations point of view: any transversal flow to the fibration produces homeomorphisms between leaves. There may be a region in the fiber whose geometry stays bounded under the transversal flow – this produces an invariant curve and a transverse incompressible torus. The second option is that transversely there will be unbounded distortion of the geometry everywhere and this produces a transverse flow which is (pseudo) hyperbolic - a pseudo-Anosov flow.

The goal of this article is to extend this result to general $\mathbf{R}$-covered foliations:

Main theorem - Let $\mathcal{F}$ be a transversely oriented, $\mathbf{R}$-covered foliation in M^3 closed, aspherical. Then either there is a $\mathbf{Z} \oplus \mathbf{Z}$ in $\pi_1(M)$ or there is a (singular) pseudo-Anosov Φ transverse to $\mathcal{F}$.

Calegari [Cal1] has independently also proved the main theorem. Many of the tools used by Calegari are similar to those used in this article and the strategy for the proof of a preliminary result follows general ideas of Thurston [Th9, Th10], which have never been written up. On the other hand this article is more complete than [Cal1] and contains full details. This work was done independently of [Cal1].

The aspherical condition is only used to rule out manifolds finitely covered by $\mathbf{S}^2 \times \mathbf{S}^1$, see below. As in the fibering case this shows that either there is a region where the geometry varies boundedly in the transverse direction or there are directions of maximal stretch/contraction everywhere.

Thurston produced a transverse pseudo-Anosov flow in the case that the foliation is associated to a slithering, which implies that the foliation is uniform [Th7]. General $\mathbf{R}$-covered foliations need not be uniform: an easy example is the stable foliation of an Anosov flow which is the suspension of an Anosov diffeomorphism of the torus. Thurston asked whether any $\mathbf{R}$-covered foliation in atoroidal manifolds had to be uniform. This is not true in general: recently Calegari [Cal2] has produced many examples of $\mathbf{R}$-covered, non uniform foliations in hyperbolic 3-manifolds. In the uniform situation Thurston used the existence of projectively invariant measures in the appropriate setting to the produce transversal laminations to the foliation. The proof in the general case is completely different and is more topological.

We now explain the basic ideas in the proof. First of all the intrinsic geometry of the leaves plays a fundamental role in our analysis. Two manifolds can be uniformized to be spherical, euclidean or hyperbolic and to a great extent the same is true for 2-dimensional foliations. This study started with Reeb's result [Re] on stability of compact leaves. Then there was the seminal work of Plante [Pl1] on holonomy invariant transverse measures, which was extended by Sullivan [Sul] and put in the context of spaces which are negatively curved in the large by Gromov [Gr]. As a result there is a fundamental trichotomy for general foliations of 3-manifolds:

1) There is a sphere or projective plane leaf,

2) There is a holonomy invariant transverse measure of 0 Euler characteristic, approximated by a torus (either transverse or in a leaf);

3) Leaves are uniformly Gromov negatively curved in the large.

In case 1) Reeb showed that M is finitely covered by $\mathbf{S}^2 \times \mathbf{S}^1$ with the product foliation [Re]. In case 2) if the foliation is Reebless, then the torus in question is incompressible and the manifold is toroidal. As spherical and toroidal manifolds are in some sense rare, this implies that 3) is the generic case if $\mathcal{F}$ is Reebless. In addition if $\mathcal{F}$ is Reebless then in cases 1) and 2) the manifold M can be decomposed into geometric pieces [Th3, Th4, Th5] and is well understood. More recently Candel [Can] showed that in case 3) there is a metric in the manifold which makes all leaves hyperbolic (constant Gaussian curvature equal to -1). Therefore case 3) is the remaining case to be analysed in the proof of the main result.

A fibration over the circle is very nice because any transverse flow which induces homeomorphisms between leaves (in M or in the universal cover $\widetilde{M}$). This homeomorphism was used to analyse the transversal distortion of the geometry of leaves. General foliations have holonomy, so it only makes sense to look for homeomorphisms between leaves in the universal cover. This is not possible for non $\mathbf{R}$-covered foliations, so the $\mathbf{R}$-covered property is necessary here. One of the biggest difficulties in general is the lack of a transversal flow which gives homeomorphisms between leaves in the universal cover. Any transversal flow gives local homeomorphism between subsets of leaves in $\widetilde{M}$ but it is far from clear they should give global homeomorphisms. In fact there are many natural counterexamples: for instance let Ψ be a geodesic flow on the unit tangent bundle of a closed hyperbolic surface R, so Ψ is an Anosov flow [An, An-Si]. Let $\mathcal{F}$ be the (weak) stable foliation of Ψ which is $\mathbf{R}$-covered and choose the transversal flow to be generated by the strong unstable foliation of Ψ. This transversal flow produces local homeomorphisms between leaves in $\widetilde{M}$ which definitely are not global homeomorphisms [Ba1, Fe2].

But all is not lost. For foliations with hyperbolic leaves one useful strategy is to first analyse the variation of distance between leaves of $\widetilde{\mathcal{F}}$ in $\widetilde{M}$ to obtain relationships between ideal boundaries of leaves. Each leaf of $\widetilde{\mathcal{F}}$ has a circle at infinity. Thurston [Th9, Th10] explained how to use contracting directions between different leaves to locally and then globally collate these individual circles at infinity into a single universal circle which encodes all circles at infinity. For $\mathbf{R}$-covered foliations it turns out the local stitching between different circles at infinity is in fact a homeomorphism of ideal boundaries. There are two cases: if leaves are a bounded distance from each other in $\widetilde{M}$ (uniform case), this yields a quasi-isometry between leaves and hence a homeomorphism between ideal boundaries. If leaves are not a bounded distance from each other, this forces an arbitrary pair of leaves of $\widetilde{\mathcal{F}}$ to contract together in a dense set of directions also producing a homeomorphism between circles at infinity of leaves of $\widetilde{\mathcal{F}}$. These boundary identifications are group equivariant. The common identified circle is called the *universal circle* of the foliation in this setting [Th9, Th10]. Universal circles for foliations with hyperbolic leaves were introduced by Thurston recently [Th9, Th10]. In general these ideal maps between circles at infinity come from maps defined only between strict subsets of leaves of $\widetilde{\mathcal{F}}$ even for $\mathbf{R}$-covered foliations.

This identification of circles at infinity can be used to produce natural maps between the entire leaves in the universal cover. Given any two leaves F, E of $\widetilde{\mathcal{F}}$ there is a homeomorphism β between the circles at infinity of F and E. If β can be continuously extended to an isometry between F and E, one calls β a *Möebius* map. In general one can quantify how far β is from a Möebius map: one way is to look for the best possible extension of β to a map from F to E - one such tighest extension was called an *earthquake* map by Thurston [Th6]. Another way is to use the universal circle and check the distortion on the geometry of various ideal quadrilaterals in leaves of the foliation. In either case one possibility is that the distortion in geometry (measured via earthquakes or ideal quadrilaterals) is in some sense globally bounded. This corresponds to the notion that the geometry does not change very much in the transversal direction and yields an incompressible torus in the manifold. The other option is that the analysis of the distortion produces either a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of the fundamental group $\pi_1(M)$ of M or a transverse lamination to $\mathcal{F}$ - this result was announced by Thurston in 1997 [Th9, Th10].

We analyse the second option in much more detail here. First we show that if the homeomorphisms between circles at infinity are not uniformly bounded then there is always a transversal lamination to $\mathcal{F}$, that is, in this case even if there is $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$ there will be one transverse lamination which encodes regions of maximal distortion. Thurston had obtained either one or the other conclusion. We then analyse the atoroidal case in much more detail: we show there are in fact two distinct transverse laminations which have the behavior of stable and unstable laminations. These laminations are transverse to each other and fill M. They intersect in an orientable 1-dimensional foliation producing a flow in the intersection of the laminations. Collapse the complementary regions of the union of the laminations to produce a flow Φ in M. The transverse laminations blow down to singular foliations $\mathcal{F}^s, \mathcal{F}^u$, which are shown to have "hyperbolic" behavior, so Φ is pseudo-Anosov. In the first option of bounded distortion of geometry we prove a rigidity result: up to topological conjugation the foliation $\mathcal{F}$ admits a transverse foliation which is a local isometry between leaves - a transversely hyperbolic foliation [Ep, Th3].

The laminations constructed here are genuine essential laminations [Ga-Ka]. Using results of Gabai and Kazez [Ga-Ka] it immediately implies the following result, also proved by Calegari [Cal1]:

Corollary – Suppose that M aspherical supports an $\mathbf{R}$-covered foliation $\mathcal{F}$. Then M satisfies the weak hyperbolization conjecture: either there is $\mathbf{Z} \oplus \mathbf{Z} < \pi_1(M)$ or $\pi_1(M)$ is Gromov negatively curved.

We mention a potential but extremely important possible use of the results here. In the atoroidal case above the geometrization conjecture predicts that the manifold is hyperbolic [Th4]. The pseudo-Anosov flow can be used to compare the geometries of leaves of $\widetilde{\mathcal{F}}$ and this can possibly be used as a starting point to geometrize M. A similar approach was successfull in the case of fibrations [Th5].

The flow Φ constructed here is is regulating for $\mathcal{F}$. This means that every orbit of $\widetilde{\Phi}$ intersects every leaf of $\widetilde{\mathcal{F}}$ and vice versa: there is a (topological) product picture in the universal cover. Hence $\widetilde{\Phi}$ produces global homeomorphisms between leaves of $\widetilde{\mathcal{F}}$ - as was desired in the initial analysis of the $\mathbf{R}$-covered case. In [Fe5] we analyse when a transverse pseudo-Anosov flow to an $\mathbf{R}$-covered foliation can fail to be regulating. It turns out that this can only occur if Φ itself has an $\mathbf{R}$-covered stable foliation of a flow.

In [Fe7] we use the results of this article and of [Fe5, Fe6] to prove that except in the case of $\mathbf{R}$-covered Anosov foliations, then up to topological conjugacy there is only one transverse pseudo-Anosov flow transverse to the $\mathbf{R}$-covered foliation. Hence our construction is in fact canonical.

We thank Bill Thurston for sharing with us his wonderful results, in particular his construction of the universal circle for foliations and transverse laminations. We thank Danny Calegari for conversations and ideas concerning $\mathbf{R}$-covered foliations. We also thank the referee for an extremely careful reading with inumerous corrections and suggestions which greatly improved the presentation of the paper. Most of the research was done while the author visited Princeton University and we thank this institution for its hospitality.

2 Uniform foliations, compact leaves and minimality

Throughout the article $\mathcal{F}$ will denote a 2-dimensional foliation of a closed 3-manifold M. The universal cover of M will be denoted by $\widetilde{M}$ and $\widetilde{\mathcal{F}}$ is the lifted foliation to $\widetilde{M}$. The map

$$\pi : \widetilde{M} \to M$$

will always denote the universal covering map. The fundamental group $\pi_1(M)$ is identified with the group of covering translations of $\widetilde{M}$. For any subset B of $\widetilde{M}$, let $\mathcal{I}(B)$ denote the isotropy group of B, which is the subgroup of $\pi_1(M)$ leaving B invariant. Finally $\mathcal{H}$ is the leaf space of $\widetilde{\mathcal{F}}$, which is a topological space.

Definition 2.1. *([Fe2]) $\mathcal{F}$ is $\mathbf{R}$-covered if $\mathcal{H}$ is homeomorphic to the real numbers $\mathbf{R}$.*

A weaker property is equivalent to $\mathbf{R}$-covered:

Lemma 2.2. *$\mathcal{F}$ is $\mathbf{R}$-covered if and only if $\mathcal{H}$ is Hausdorff.*

Proof. Assume first that $\mathcal{H}$ is Hausdorff. Suppose there is a closed curve γ in $\widetilde{M}$ which is transverse to $\widetilde{\mathcal{F}}$. Then γ bounds an immersed disk D which can be put into general position with respect to $\widetilde{\mathcal{F}}$ [Ha, No, So]. An analysis of the induced (singular) foliation in D shows there are leaves α_1, α_2 so that α_1 is a closed curve and α_2 spirals towards α_1. Let F_i be the leaf of $\widetilde{\mathcal{F}}$ containing α_i. Then F_2 limits on F_1 so any neighborhood of F_1 contains F_2 and hence $\mathcal{H}$ is not Hausdorff contradiction. Hence any transversal curve to $\widetilde{\mathcal{F}}$ projects injectively to $\mathcal{H}$, so $\mathcal{H}$ is a 1-manifold with a countable base. It is also Hausdorff so it can only be the circle or the real line. In the circle case, construct a closed transversal to $\widetilde{\mathcal{F}}$, contradiction. Hence $\mathcal{F}$ is $\mathbf{R}$-covered. The converse implication is immediate. ☐

Here is how Reeb components interact with the $\mathbf{R}$-covered property: if $\mathcal{F}$ is $\mathbf{R}$-covered then $\mathcal{F}$ is Reebless unless M is finitely covered by $\mathbf{S}^2 \times \mathbf{S}^1$ [Go-Sh]. The restriction is necessary: glue two Reeb components along their boundaries to produce $\mathbf{S}^2 \times \mathbf{S}^1$ with an $\mathbf{R}$-covered foliation.

Most of the time we assume that $\mathcal{F}$ is transversely orientable. The non orientable case usually follows from some additional considerations.

Definition 2.3. *(foliated I-bundle) A foliated I-bundle (in dimension 3) is a pair $(N, \mathcal{G})$, with N homeomorphic to a product $R \times I$, where R is a surface, which may be compact or not, and may have boundary or not; and I is the unit interval. In addition $\mathcal{G}$ is a foliation in N so that:*

- *$R \times \{0\}$ and $R \times \{1\}$ are leaves of $\mathcal{G}$;*
- *$\mathcal{G}$ is transverse to the I-fibers in N (including $\partial R \times [0,1]$).*

Up to topological conjugacy, the foliation $\mathcal{G}$ is completely determined by the monodromy, which is a map from $\pi_1(R)$ to the group of homeomorphisms of I. Sometimes we abuse the notation and say that N or $\mathcal{G}$ is a foliated I-bundle.

A fundamental concept here is that of a *uniform* foliation:

Definition 2.4. *([Th7]) A foliation $\mathcal{F}$ is uniform if given any two leaves E, F of $\widetilde{\mathcal{F}}$, there is a positive constant b, so that the Hausdorff distance between E, F is smaller than b. Explicitly, for any point x of E there is y in F so that $d(x,y)$ is less than b and conversely. The bound depends on the pair E, F.*

Obviously it is not at all important that leaves be hyperbolic to define uniform foliations. In the case of $\mathbf{R}$-covered foliations, the existence of compact leaves implies that the foliation is uniform:

Lemma 2.5. *(compact leaves) Let $\mathcal{F}$ be an $\mathbf{R}$-covered foliation in M^3, closed and not finitely covered by $\mathbf{S}^2 \times \mathbf{S}^1$. Then $\mathcal{F}$ is taut. In addition if $\mathcal{F}$ has a compact leaf R then $\mathcal{F}$ is uniform.*

Proof. If necessary lift to a finite cover and assume that $\mathcal{F}$ is transversely orientable and M is orientable. If the lifted foliation is taut then so it the original one. The uniform condition concerns objects in the universal cover so the same holds concerning this property. Since $\mathcal{F}$ is $\mathbf{R}$-covered and M is not finitely covered by $\mathbf{S}^2 \times \mathbf{S}^1$ then $\mathcal{F}$ is Reebless as shown by Goodman and Shields [Go-Sh].

With the orientation conditions, if $\mathcal{F}$ is not taut then that there is a codimension 0 submanifold V bounded by a union $T_1, ..., T_n$ of tori so the transversal flow is say outgoing along the boundary [Go]. Hence there are no transversals connecting distinct lifts of the T_i to $\widetilde{M}$. As $\mathcal{F}$ is $\mathbf{R}$-covered any two leaves of $\widetilde{\mathcal{F}}$ are connected by a transversal. The only possibility is that $n = 1$ and there is only one lift of T_1 to $\widetilde{M}$. Hence $\pi_1(T_1)$ surjects in $\pi_1(V)$. But if T_1 is incompressible this is impossible by theorem 10.5 of [He]. But $\mathcal{F}$ is Reebless so T_1 in incompressible. This contradiction shows that $\mathcal{F}$ is taut.

If in addition $\mathcal{F}$ has a compact leaf R, then Goodman and Shields proved that R is the fiber of a fibration of M over the circle [Go-Sh] (this uses $\mathcal{F}$ transversely orientable). Cut M along R to produce a manifold M_1 homeomorphic to $R \times I$ with an induced foliation $\mathcal{F}_1$. We want to show that $\mathcal{F}_1$ is a foliated I-bundle. Let A be an annulus in M_1 of the form $\gamma \times I$ where γ is a simple closed curve in R. Isotope A to be in general position with respect to $\mathcal{F}_1$. By Euler characteristic arguments A has no singularities. Since $\mathcal{F}$ is taut, Gabai [Ga5] showed that A can be made transverse to $\mathcal{F}_1$. Cut along A to produce a new manifold with a new foliation transverse to the vertical boundary. Continue cutting along transverse annuli and disks until obtaining a union of manifolds homeomorphic to $D^2 \times I$, where D^2 is the closed disk. The foliation in the boundary of these balls has a tangential part $D^2 \times \partial I$ and a transverse part $\partial D^2 \times I$. The transverse part has no holonomy because this is a ball. Therefore (up to topological conjugacy) this is a foliation by horizontal disks. Conclusion: we can isotope $\mathcal{F}_1$ so that it is transverse to the I-fibers in M_1 and hence $\mathcal{F}_1$ is a foliated I-bundle. Glue back along R and lift to $\widetilde{M}$. As M_1 is a foliated I-bundle, it now follows that $\mathcal{F}$ is uniform. ☐

When there are no compact leaves we can reduce to the minimal case:

Proposition 2.6. *(minimal case) Suppose that $\mathcal{F}$ is $\mathbf{R}$-covered, does not have compact leaves and its not minimal. Then $\mathcal{F}$ can be collapsed to a minimal foliation: there is a foliation $\mathcal{F}'$ which is minimal and which is obtained from $\mathcal{F}$ by collapsing at most countably many foliated I-bundles of $\mathcal{F}$ to single leaves.*

Proof. Suppose first that $\mathcal{F}$ is transversely orientable.

Let Z be a minimal set of $\mathcal{F}$. By hypothesis Z is not a compact leaf and not all of M. This implies that Z intersects any transversal curve to $\mathcal{F}$ in a Cantor set.

Let U be a component of the complement of Z in M and $\widehat{U}$ the metric completion of U. The interior of $\widehat{U}$ embeds in M and there is an induced map in the boundary, but the boundary may be a double cover of a leaf B_0 of $\mathcal{F}$. In the last case the leaf B_0 will be isolated on both sides. Since B_0 is in a minimal set, it follows that B_0 is a compact leaf. This contradicts the hypothesis. Hence $\widehat{U}$ embedds in M.

We claim that $\widehat{U}$ is a foliated I-bundle. By the general theory of codimension one foliations [Di] the set $\widehat{U}$ is equal to a union of two subsets K and A, intersecting only in their boundaries and satisfying:

- K is a compact, connected, codimension 0 submanifold of $\widehat{U}$, which is called the *core* of $\widehat{U}$,
- A is a union of manifolds Q_i which are homeomorphic to products $R_i \times I$, where each R_i is a non compact, connected surface with compact, connected boundary and $\mathcal{F}$ restricted to Q_i is a foliated I-bundle (see also [Ca-Co]).

In addition in the induced metric the I-fibers in Q_i are very short, so we can choose them to be the transversal flow segments of Φ.

Let $\widetilde{U}$ be a component of $\pi^{-1}(\widehat{U})$. Since $\widehat{U}$ embedds in M and is not all of M, then $\widetilde{U}$ is not all of $\widetilde{M}$. Therefore it has boundary components. Then $\widetilde{U}$ is closed, connected, has non empty interior and is not all of $\widetilde{M}$. Since the leaf space of $\widetilde{\mathcal{F}}$ is $\mathbf{R}$, then $\widetilde{U}$ has leaf space which is a closed interval J. Here it is fundamental that $\mathcal{F}$ is $\mathbf{R}$-covered. Let L_1, L_2 be the boundary leaves of $\widetilde{U}$. If an element of the isotropy group of $\widetilde{U}$ switched L_1 and L_2 then $\mathcal{F}$ would not be transversely oriented, contradiction. It follows that the isotropy group of $\widetilde{U}$ is the same as the isotropy group of L_1 or L_2.

Let now

– $\widetilde{K}$ be a component of $\pi^{-1}(K)$ contained in $\widetilde{U}$;

– E be the intersection of $\widetilde{K}$ and L_1

– B_1 be a component of E. Let $C_1 = \pi(B_1)$ which is a subset of ∂K.

If there is another component B_2 of E, then there is a curve α contained in ∂B_1 separating B_1 from B_2 in L_1. But α projects to a closed curve β in ∂K. Recall that ∂K is equal to ∂A (as subsets of $\widehat{U}$) and then β is contained in an annulus $\beta \times I$ in ∂K. The annulus $\beta \times I$ separates K from a component of A in $\widehat{U}$. The component C_2 of $\pi^{-1}(\beta \times I)$ containing α separates $\widetilde{K}$ from the component of $\widetilde{U} - C_2$ containing B_2. Therefore the lift $\widetilde{K}$ of K does not intersect B_2. This is a contradiction. We conclude that E is connected and therefore equal to B_1. As any covering translation preserving $\widetilde{K}$, also preserves E then it also preserves B_1. It follows that the isotropy groups $\mathcal{I}(B_1)$ and $\mathcal{I}(\widetilde{K})$ are equal. Therefore $\pi_1(C_1)$ surjects in $\pi_1(K)$.

But K is compact and irreducible, so theorem 10.5 of Hempel [He] implies that K is homeomorphic to $C_1 \times I$. As in the previous lemma isotope the foliation to be transverse to the I-fibers in K. This proves the claim that $\widehat{U}$ is a foliated I-bundle.

Notice that this discussion shows that there is a unique minimal set Z. This is because we just proved that the complement of Z is a union of foliated I-bundles with non compact bases. Any leaf in the interior of the I-bundles will limit in points that the boundary leaves also limit on, that is they will have limit points in Z. But Z is a minimal set so the additional leaf is not part of a minimal set. It is fundamental in all of this discussion that $\mathcal{F}$ is $\mathbf{R}$-covered - clearly these results do not work in more generality.

So we can consider the at most countably many components of $M - Z$. There is a positive number ϵ so that any two points in M which are less than ϵ apart, then their local leaves are connected by a very small transversal arc. At most finitely components of $M - Z$ many have thickness bigger than ϵ, hence in the other ones the transversal flow already produces an I-bundle structure. For the finitely many other ones change the original transversal flow in the core part (which is an I-bundle) to consist of the I-fibers in the particular component. This is done only in finitely many compact pieces - so we may assume the flow is smooth. Finally blow down all the complementary regions of Z using the new transversal flow to produce a foliation $\mathcal{F}'$. Because the minimal set Z intersects transversals to the original $\mathcal{F}$ in a Cantor set, the collapsed object is still M with a foliation $\mathcal{F}'$. Also $\mathcal{F}'$ is minimal - if there is a non trivial minimal set of $\mathcal{F}'$, it would generate a complementary component of Z which was not collapsed. This finishes the proof in the transversely orientable case.

Suppose now that $\mathcal{F}$ is not transversely orientable. In any case choose a transverse line field to $\mathcal{F}$. There is a double cover M_2 and a lift $\mathcal{F}_2$ of $\mathcal{F}$ which is transversely orientable. The cover is normal and there is an involution f of M_2 so that M is the quotient of M_2 by f. Certainly $\mathcal{F}_2$ is $\mathbf{R}$-covered and the results above work for $\mathcal{F}_2$.

Let Z be a minimal set of $\mathcal{F}$ and Z' be its inverse image in M_2. Let Z_2 be a minimal set contained in Z'. Then Z_2 projects to a set in M which is closed and contained in Z, hence the projection is Z. Furthermore $f(Z_2)$ is also a minimal set of $\mathcal{F}_2$, hence by the above discussion $f(Z_2)$ equals Z_2. This now implies that Z_2 equals Z' and then Z is the unique minimal set of $\mathcal{F}$.

Let U be a complementary region of Z which is an I-bundle. One option is that it projects homeomorphically to M and as above we can use I-fibers to collapse this to a single leaf. The other option is that it double covers a set in M, which may have one or two boundary components. Isotope the I-fibration in $\widehat{U}$ so that it is invariant under f – one only needs to do this in the compact pieces of the I-bundle. Then one can collapse the resulting I-fibration in M. If the region has only one boundary component, then it collapses to a leaf which is not transversely orientable. The resulting foliation is minimal as above. $\square$

Notice $\mathcal{F}$ is a blow up of at most countably many leaves of $\mathcal{F}'$. To prove the main theorem, if we find a subgroup $\mathbf{Z} \oplus \mathbf{Z}$ in $\pi_1(M)$ we are done. If we find a pseudo-Anosov flow transverse to $\mathcal{F}'$, then it pulls back by the blow up operation to a pseudo-Anosov flow transverse to $\mathcal{F}$. Hence from now on assume that $\mathcal{F}$ is minimal if it does not have compact leaves.

3 Ideal geometry and the universal circle

In order to prove the main theorem, the remaining case is when the leaves are Gromov hyperbolic. Using Candel's theorem [Can] we assume each leaf of $\mathcal{F}$ is hyperbolic – notice the metric may vary only continuously in the transversal direction [Can]. Thurston explained how to locally stitch the circles at infinity of different leaves and then to globalize the local stitching to produce a universal circle which encodes all circles at infinity [Th9, Th10]. In the case of $\mathbf{R}$-covered foliations we show how to do the identifications so that the local maps are homeomorphisms between the circles at infinity.

Remark. In this section and the next there are no orientability conditions.

If E, F are leaves of $\widetilde{\mathcal{F}}$, let (E, F) denote the set of leaves of $\widetilde{\mathcal{F}}$ separating E from F. As $\mathcal{H}$ is homeomorphic to $\mathbf{R}$, then if E, F are distinct the set (E, F) is homeomorphic to an interval. Let $[E, F]$ be the union of (E, F) and the two leaves E, F. Each leaf of $\widetilde{\mathcal{F}}$ is isometric to the hyperbolic plane $\mathbf{H}^2$ and has an ideal circle at infinity $S^1_\infty(F)$. We now come to a key object of our study:

Definition 3.1. *(cylinder at infinity) Let $\mathcal{F}$ be an $\mathbf{R}$-covered foliation with hyperbolic leaves. Let*

$$\mathcal{A} \quad = \quad \bigcup_{F \in \widetilde{\mathcal{F}}} S^1_\infty(F).$$

which is the cylinder at infinity of $\widetilde{\mathcal{F}}$ - the union of all ideal circles of leaves of $\widetilde{\mathcal{F}}$.

Since $\mathcal{F}$ is $\mathbf{R}$-covered, then set wise $\mathcal{A}$ is an infinite cylinder $\mathbf{S}^1 \times \mathbf{R}$. First of all we put a topology in $\mathcal{A}$ so that it is also homeomorphic to a cylinder.

Notation: To be used throughout the article: given $x \in \widetilde{M}$ let $F(x)$ denote the leaf of $\widetilde{\mathcal{F}}$ containing x. The same holds for x in M.

Each geodesic ray in $F(x)$ starting at x defines a unique ideal point in $S^1_\infty(F(x))$ giving a homeomorphism between the unit tangent bundle of $F(x)$ at x and $S^1_\infty(F(x))$. Let $T_1\widetilde{\mathcal{F}}$ be the unit tangent bundle of $\widetilde{\mathcal{F}}$. Given any $B \subset \widetilde{M}$, let $\widetilde{\mathcal{F}}_B$ be the union of leaves of $\widetilde{\mathcal{F}}$ which intersect B and

$$\mathcal{A}_B = \bigcup_{F \in \widetilde{\mathcal{F}}_B} S^1_\infty(F).$$

This is particularly useful if $B = \mu$ is a transversal arc to $\widetilde{\mathcal{F}}$. In addition if F is a leaf of $\widetilde{\mathcal{F}}$, then d_F denotes the path distance in F. The term "open transversal" will be used for a transversal to $\mathcal{F}$ or $\widetilde{\mathcal{F}}$ which is homeomorphic to an open interval $(0,1)$.

Lemma 3.2. *(topology of $\mathcal{A}$) Let μ be an open transversal to $\widetilde{\mathcal{F}}$. Then $T_1\widetilde{\mathcal{F}}$ restricted to μ is homeomorphic to an open cylinder $\mathbf{S}^1 \times (0,1)$. This provides an identification of $\mathcal{A}_\mu$ with an open cylinder $\mathbf{S}^1 \times (0,1)$ and defines a topology in $\mathcal{A}$ making it homeomorphic to a cylinder. The union $\widetilde{M} \cup \mathcal{A}$ has a natural topology making it homeomorphic to $D^2 \times \mathbf{R}$, where D^2 is the closed disk and $D^2 \times \{t\}$ correspond to the union $F \cup S^1_\infty(F)$ for F a leaf of $\widetilde{\mathcal{F}}$.*

Proof. If μ, μ' are two transversals to $\widetilde{\mathcal{F}}$ so that $\widetilde{\mathcal{F}}_\mu, \widetilde{\mathcal{F}}_{\mu'}$ intersect, we need to show that the topology $\mathcal{T}_\mu$ induced by μ in the intersection of $\mathcal{A}_\mu$ and $\mathcal{A}_{\mu'}$ is the same as the topology $\mathcal{T}_{\mu'}$ induced by μ' in this intersection. By restricting to their intersection we can assume that the sets $\mathcal{A}_\mu, \mathcal{A}_{\mu'}$ are equal.

Since both topologies induced in $\mathcal{A}_\mu$ are first countable it suffices to consider the behavior of sequences. Consider a sequence y_i, i in $\mathbf{N}$ converging to y_0 in $\mathcal{T}_\mu$. Then y_i are in $S^1_\infty(F_i)$, for uniquely defined leaves F_i which are in $\widetilde{\mathcal{F}}_\mu$ (equal to $\widetilde{\mathcal{F}}_{\mu'}$). Let

$$x_i = F_i \cap \mu, \qquad z_i = F_i \cap \mu'.$$

Then the sequence F_i converges to F_0 in $\mathcal{H}$ with x_0 in F_0 and F_0 a leaf of $\widetilde{\mathcal{F}}_\mu$. Also

$$x_i \to x_0 \ \text{ in } \ \mu \quad \text{and} \quad z_i \to z_0 \ \text{ in } \ \mu'.$$

For each i let l_i, s_i geodesic rays in F_i from x_i, z_i respectively with ideal point y_i in $S^1_\infty(F_i)$. These come from identifications of the unit tangent bundle to $\widetilde{\mathcal{F}}$ at x_i, z_i with $S^1_\infty(F_i)$ respectively. Since the sequence y_i converges to y_0 in $\mathcal{T}_\mu$, then the directions of l_i at x_i converge to the direction of l_0 at x_0.

Notice y_i converges to y_0 in $\mathcal{T}_{\mu'}$ if and only if the directions of s_i in F_i converge to the direction of s_0 in F_0. We use a couple properties of the hyperbolic metric. Since x_i converges to x_0 and z_i converges to z_0 then $d_{F_i}(x_i, z_i)$ is bounded above for all i. In addition the rays l_i, s_i, for $i \geq 1$ define the same ideal point y_i in $S^1_\infty(F_i)$. Hence l_i and s_i are asymptotic in F_i. These two facts imply that given any positive ϵ there is a positive $a(\epsilon)$ so that except for initial length $a(\epsilon)$, the remainder of the rays l_i, s_i are within ϵ of each other in F_i for each $i \geq 1$. Notice that the constants are independent of i – this only uses the fact that all leaves are hyperbolic and the distance between x_i and z_i in F_i is bounded above.

Consider any subsequence $s_{i(k)}$ so that $s_{i(k)}$ converges to a ray v_0 in F_0 or equivalently that the directions of $s_{i(k)}$ at $z_{i(k)}$ converge to the direction of v_0 at z_0. For notational simplicity assume this is the original sequence s_i. As l_i converges to l_0 and s_i converges to v_0, the above property implies that except for initial segments of length smaller than $a(\epsilon)$, the remainder of the rays l_0, v_0 are within ϵ of each other in F_0. Explicitly, if w is a point in l_0 which is more than $a(\epsilon)$ away from x_0 in F_0, then

$$w = \lim_{i \to \infty} w_i \quad \text{with} \quad w_i \in l_i \quad \text{and} \quad d_{F_i}(x_i, w_i) > a(\epsilon).$$

Also $d_{F_i}(x_i, w_i)$ is bounded above. By the property above there are u_i in s_i with $d_{F_i}(w_i, u_i)$ smaller than ϵ and up to subsequence again we may assume u_i converges. As $d_{F_i}(x_i, w_i)$ is bounded above, then so is $d_{F_i}(u_i, v_i)$. Therefore u_i has to converge to a point u in v_0. Then $d_{F_0}(w, u)$ is bounded above by ϵ and conversely. This implies that l_0, v_0 have subrays which are at most ϵ distant from each other, so it again follows from hyperbolic geometry that they are asymptotic in F_0. That means that v_0 defines the ideal point y_0 in $S^1_\infty(F_0)$. Therefore the rays v_0, s_0 are equal. This is equivalent to the sequence s_i converging to s_0: the directions of s_i converge to that of s_0. But notice that this is in fact a subsequence of the original sequence! This proves that any sequence y_i converging to y_0 in $\mathcal{T}_\mu$ has a subsequence which converges to y_0 in $\mathcal{T}_{\mu'}$. This then implies that the original sequence converges to y_0 in $\mathcal{T}_{\mu'}$ as we wanted to prove. This shows that the topology in $\mathcal{A}$ is well defined. Clearly

$$\mathcal{A} \ = \ \bigcup_{i \in \mathbf{N}} \ \mathcal{A}_{\mu_i},$$

with μ_i transversals intersecting more and more of the leaf space of $\widetilde{\mathcal{F}}$. Each $\mathcal{A}_{\mu_i}$ is homeomorphic to a cylinder hence $\mathcal{A}$ is homeomorphic to $\mathbf{S}^1 \times \mathbf{R}$.

Similar arguments show that there is a natural topology on

$$\bigcup_{x \in \mu} \ (F(x) \cup S^1_\infty(F(x))),$$

making it homeomorphic to $D^2 \times (0,1)$, where each leaf with its ideal circle corresponds to $D^2 \times \{t\}$. It follows that $\widetilde{M} \cup \mathcal{A}$ is naturally homeomorphic to $D^2 \times \mathbf{R}$. $\square$

If g is a covering translation of $\widetilde{M}$ and L a leaf of $\widetilde{\mathcal{F}}$, then g maps L to $g(L)$ by an isometry which extends to a homeomorphism g^L_∞ between their circles at infinity. This produces a bijection g_∞ from $\mathcal{A}$ to itself. Similar arguments as in the lemma above show that g_∞ is a homeomorphism of $\mathcal{A}$, which will be called a *covering homeomorphism*. Many times we will abuse notation and write g instead of g_∞ for this "ideal" map. In this way $\pi_1(M)$ acts in the cylinder at infinity.

Clearly $\mathcal{A} \ \cong \ \mathbf{S}^1 \times \mathbf{R}$ has a natural foliation by circles, which comes from the circles at infinity of leaves. This is what we call the "horizontal" foliation of $\mathcal{A}$. Natural means that this foliation is left invariant by the action of $\pi_1(M)$. In general the action of $\pi_1(M)$ on $\mathcal{A} \cong \mathbf{S}^1 \times \mathbf{R}$ does not respect the vertical foliation by $\{x\} \times \mathbf{R}$. The main goal of this section is to produce a natural "vertical" foliation of $\mathcal{A}$ which also is associated to the geometry of the foliation. This will creat the universal circle of the foliation $\mathcal{F}$ (or $\widetilde{\mathcal{F}}$). First recall the definition of quasi-isometries:

Definition 3.3. *(quasi-isometry)[Th3] A quasi-isometry is a map $\varphi : (M_1, d_1) \to (M_2, d_2)$ between metric spaces so that there is positive k satisfying: for any x, y in M_1 then*

$$\frac{1}{k} d_1(x,y) - k \quad < \quad d_2(\varphi(x), \varphi(y)) \quad < \quad k d_1(x,y) + k$$

and in addition there is a positive k' so that for any point z of M_2 there is x of M_1 with $d_2(z, \varphi(x))$ smaller than k'. If the constant is important we say that φ is a k-quasi-isometry.

First we produce the natural vertical foliation in the uniform case:

Proposition 3.4. *(vertical foliation - uniform case) Let $\mathcal{F}$ be an uniform $\mathbf{R}$-covered foliation with hyperbolic leaves. Then given any two leaves E, F of $\widetilde{\mathcal{F}}$, there is a canonical homeomorphism between $S^1_\infty(E)$ and $S^1_\infty(F)$. This yields a universal circle which is naturally homeomorphic to any circle at infinity. There is a "vertical" foliation in $\mathcal{A}$ which is transverse to the horizontal foliation and is group invariant. The homeomorphisms between $S^1_\infty(E)$ and $S^1_\infty(F)$ are given by the holonomy of this vertical foliation.*

Proof. There is a brief proof of this result in [Th7] − for completeness we provide the details here.

Fix E, F in $\widetilde{\mathcal{F}}$ and positive b_1 so that their Hausdorff distance is less than b_1. Define a map $\varphi : E \to F$:

$$\varphi(x) = y \text{ for some } y \text{ in } F \text{ with } d(x,y) < b_1.$$

The map φ is not well defined, but it is coarsely defined. This follows from a fundamental property of $\mathbf{R}$-covered foliations: If $\mathcal{F}$ is $\mathbf{R}$-covered, then for any positive b_2, there is positive $b_3 = f(b_2)$ satisfying:

$$\forall\, z, w \in \widetilde{M} \text{ with } w \in F(z), \text{ then } d(z,w) < b_2 \Rightarrow d_{F(z)}(z,w) < b_3,$$

see [Fe1]. The important thing is that b_3 depends only on b_2 and not on individual leaves or points. This property is in fact equivalent to the $\mathbf{R}$-covered property (for Reebless foliations) and does not hold in general. Hence there is positive b_4 so that if x in E and y, z in F with

$$d(x,y) < b_1, \quad d(x,z) < b_1, \quad \text{then } d_F(y,z) < b_4 = f(2b_1).$$

We conclude that $\varphi(x)$ is well defined up to a set of diameter b_4 in F. This is what we mean by coarsely defined. We want to show that φ is a quasi-isometry from E to F.

For any $x, y \in E$ choose a geodesic arc from x to y in E, having length a_1 and let n be the integer $\lfloor a_1 \rfloor$ where $\lfloor\ \rfloor$ is the greatest integer function. Then $d_E(x,y)$ is a number in the interval $[n, n+1)$. Split γ to produce points $x_0 = x, x_1, ..., x_n, x_{n+1} = y$ with $d_E(x_{i-1}, x_i)$ equal to 1 for any i smaller than n and $d_E(x_n, x_{n+1})$ less than 1. Then

$$\begin{aligned} d(\varphi(x_{i-1}, \varphi(x_i))) &\leq d(\varphi(x_{i-1}), x_i) + d(x_{i-1}, x_i) + d(x_i, \varphi(x_i)) \\ &\leq b_1 + 1 + b_1 = (2b_1 + 1). \end{aligned}$$

Let $b_5 = f(2b_1 + 1)$ so if w, z are in the same leaf of $\widetilde{\mathcal{F}}$ and $d(w,z)$ is smaller than $(2b_1+1)$ then $d_{F(z)}(z,w)$ is smaller than b_5. It follows that

$$d_F(\varphi(x), \varphi(y)) \leq (n+1)b_5 < (d_E(x,y) + 1)b_5 = b_5 d_E(x,y) + b_5.$$

This shows one side of the required inequalies for quasi-isometries. In the same way there is a map ξ from F to E with $d(w, \xi(w))$ smaller than b_1 for all w in F. Hence

$$d(w, \varphi\xi(w)) \leq 2b_1 \quad \text{and so} \quad d_F(w, \varphi\xi(w)) \leq b_4 = f(2b_1)$$

for all w in F. This shows that φ is almost onto as required in the definition of quasi-isometry. Similarly $d_E(x, \xi\varphi(x))$ is smaller than b_4 for all x in E. Given x, y in E, let $z = \varphi(x),\ w = \varphi(y)$. An argument as above implies that

$$d_E(\xi(z), \xi(w)) \leq b_5 d_F(z,w) + b_5.$$

So

$$\begin{aligned} d_E(x,y) &\leq d_E(x, \xi\varphi(x)) + d_E(\xi\varphi(x), \xi\varphi(y)) + d_E(\xi\varphi(y), y) \\ &\leq 2b_4 + b_5 d_F(\varphi(x), \varphi(y)) + b_5, \end{aligned}$$

or

$$\frac{1}{b_5} d_E(x,y) - \left(\frac{2b_4}{b_5} + 1\right) \leq d_F(\varphi(x), \varphi(y)).$$

We conclude that $\varphi : E \to F$ is a quasi-isometry. Therefore it extends to a homeomorphism ϕ between $S^1_\infty(E)$ and $S^1_\infty(F)$ [Gr, Th2]. This works for any pair of leaves L, G of $\widetilde{\mathcal{F}}$, producing corresponding maps: φ^G_L from L to G – a quasi-isometry; and ϕ^G_L homeomorphism between $S^1_\infty(L)$ and $S^1_\infty(G)$.

We now produce a natural "vertical" foliation in $\mathcal{A}$. Fix E in $\widetilde{\mathcal{F}}$. For any y in $S^1_\infty(E)$ and any F in $\widetilde{\mathcal{F}}$, then $\phi^F_E(y)$ is a point in $S^1_\infty(F)$. Let

$$\alpha_y \;=\; \bigcup_{F \in \widetilde{\mathcal{F}}} \phi^F_E(y).$$

By the above α_y intersects every circle $S^1_\infty(F)$ in a single point. We claim that α_y is a continuous curve in $\mathcal{A}$. Let μ be a transversal to $\widetilde{\mathcal{F}}$ and x_i a sequence in μ converging to x_0. Let $F_i = F(x_i)$. We want to show that $y_i = \varphi^{F_i}_E(y)$ produces a sequence converging to $y_0 = \varphi^{F_0}_E(y)$.

Consider l_i geodesic rays in F_i starting in x_i and with ideal point y_i. For simplicity assume that all F_i, with i bigger than 1 are in the interval (F_0, F_1). Such F_i separate F_0 from F_1 in $\widetilde{M}$. The Hausdorff distance is monotone increasing: if $[F, G]$ is a subset of $[L, H]$ in $\mathcal{H}$, then $d_H(F, G)$ is smaller than $d_H(L, H)$. Therefore

$$d_H(F_0, F_i) \quad \text{is bounded above by} \quad d_H(F_0, F_1)$$

for all i. Using the arguments above and this uniform bound on $d_H(F_0, F_i)$ this implies that that all $\varphi^{F_i}_{F_0}$ are uniform quasi-isometries – they are all k-quasi-isometries for some fixed k. The images $\varphi^{F_i}_{F_0}(l_0)$ are uniform quasigeodesics in F_i with ideal point y_i. Hence they are a bounded distance from a geodesic ray in F_i starting in z_i and with ideal point y_i. Since they are uniform quasigeodesics starting in z_i which is a uniformly bounded distance from x_i then the images $\varphi^{F_i}_{F_0}$ are a uniform bounded distance from l_i in F_i. If the sequence l_i does not converge to l_0, up to subsequence suppose that l_i converges to v_0 not equal to l_0. But $d_H(l_i, l_0)$ is bounded above by a_0 for some globably defined a_0, where this Hausdorff distance is computed in $\widetilde{M}$. Hence $d_H(v_0, l_0)$ is bounded above by a_0 as well. The $\mathbf{R}$-covered property implies that $d_{F_0}(v_0, l_0)$ is bounded, contradicting the fact that v_0 and l_0 diverge exponentially in F_0. Therefore l_i converges to l_0.

Hence α_y is a continous curve in $\mathcal{A}$. Consider the collection $\{\alpha_y\}$ where y is arbitrary in $S^1_\infty(E)$. For any point z of $\mathcal{A}$, z is in $S^1_\infty(F)$ for some F of $\widetilde{\mathcal{F}}$ and $z = \phi^F_E(y)$ for a unique y in $S^1_\infty(E)$. Equivalently

$$z \;=\; \alpha_y \;\cap\; S^1_\infty(F(z)) \quad \text{and hence} \quad \mathcal{A} \;=\; \bigcup_{y \in \mathbf{S}^1_\infty(E)} \alpha_y.$$

Furthermore the sets $\{\alpha_y\}$ with y in $S^1_\infty(E)$ are disjoint for distinct y. Since they are continuous curves, this collection produces a vertical triviliazation of $\mathcal{A} \;\cong\; \mathbf{S}^1 \;\times\; \mathbf{R}$. Since covering translations preserve distances and relations between distances, it is very easy to check that this foliation of $\mathcal{A}$ is invariant under covering translations, producing a "natural" vertical foliation in $\mathcal{A}$. This will be used to analyse how the geometry changes transverse to $\widetilde{\mathcal{F}}$. This finishes the proof of proposition 3.4. $\square$

Before we analyse the non uniform situation we introduce contracting directions and markers.

Definition 3.5. *(contracting direction) Let x be a point in a leaf L of $\widetilde{\mathcal{F}}$ and let $\{\gamma(t),\ t \text{ in } [0, +\infty)\}$ be a geodesic ray in L starting in x and with tangent vector v at x. Let p in $S^1_\infty(L)$ be the ideal point of γ. Then γ (or v) is a contracting direction if the following happens: there is a transversal μ to $\widetilde{\mathcal{F}}$ containing x (maybe as an endpoint or maybe in the interior) so that for any leaf E of $\widetilde{\mathcal{F}}$ which intersects μ the distance $d(E, \gamma(t))$ converges to 0 as $t \to \infty$. In other words holonomy along γ (or in the v direction) contracts a neighborhood of leaves towards L. Similarly define contracting directions in $\mathcal{F}$.*

Remarks: 1) Contracting directions can be defined for any foliation: it just means that nearby leaves get contracted together in that direction. Using harmonic measures Thurston [Th8] showed that contracting directions are quite common in codimension one foliations in closed manifolds (any dimension).

2) In our setting the contracting direction is really a property of the ideal point p in $S^1_\infty(L)$ and is independent of the initial point x or the geodesic ray defining p. This is because all such rays are asymptotic, so a packet of leaves gets contracted together irrespective of the initial point or ray.

Lemma 3.6. *Let x in L with a contracting direction given by a geodesic ray $\{\gamma(t)\}$ and μ a transversal to $\widetilde{\mathcal{F}}$ contracted in the γ direction. For any E of $\widetilde{\mathcal{F}}$ intersecting μ, the contracting direction γ defines an ideal point $\zeta(E)$ of E and any geodesic ray of E with ideal point $\zeta(E)$ is contracted to L. In addition for any F in $\widetilde{\mathcal{F}}$ there is at most one direction in F which gets contracted towards γ.*

Proof. Fix a transverse line field to $\mathcal{F}$ and lift to $\widetilde{\mathcal{F}}$. Let E in $\widetilde{\mathcal{F}}$ intersecting μ. For any positive a_0 there is positive t_0 so that $d(\gamma(t), E)$ is smaller than a_0 for t bigger than t_0. If a_0 is small, the translate of $\gamma(t)$ for t bigger than t_0 to L along the transverse foliation is defined for all time (as they are very close) and is a curve with arbitrarily small geodesic curvature in E. Hence the translate is a quasigeodesic in E [Th3] and it defines an ideal point in $S^1_\infty(E)$ which is denoted by $\zeta(E)$. Also for smaller and smaller a_0 the translates have smaller and smaller geodesic curvature and become more and more geodesic. Hence a geodesic ray in E with ideal point $\zeta(E)$ is asymptotic with the initial ray γ in E_0.

Finally suppose there is F in $\widetilde{\mathcal{F}}$ and there are geodesic rays r_1, r_2 in F which are asymptotic to γ. Therefore they are asymptotic to each other. But $\mathcal{F}$ is Reebless, so there are no closed transversals to $\widetilde{\mathcal{F}}$. This implies when r_1 and r_2 are close in $\widetilde{M}$, they have to be in the same local sheet of $\widetilde{\mathcal{F}}$. This in turn implies that r_1, r_2 are also asymptotic in F. Therefore they define the same direction in F. ☐

Contracting directions in $\widetilde{\mathcal{F}}$ in turn produce markers in the cylinder at infinity:

Definition 3.7. *(marker) Let L in $\widetilde{\mathcal{F}}$ with a contracting direction given by the geodesic ray γ which contracts a transversal segment μ. For any E intersecting μ let $\zeta(E)$ be the unique ideal point of E defined in the previous lemma. The set of $\{\zeta(E)\}$ with E intersecting μ is a subset of $\mathcal{A}$ which defines a marker in $\mathcal{A}$ associated to the pair (γ, μ). For any E intersecting μ we say there is a marker between $S^1_\infty(L)$ and $S^1_\infty(E)$ or equivalently a contracting direction between L and E. Sometimes we abuse notation and say that this produces a marker between L and E. Let ζ denote the marker.*

Remark: If $\mathcal{F}$ is a non $\mathbf{R}$-covered foliation with hyperbolic leaves there is not a global cylinder at infinity. However the union of the circles at infinity associated to a transversal to $\widetilde{\mathcal{F}}$ still is a cylinder and one can define markers associated to intervals of leaves in the leaf space.

Some needed properties of markers are now established. If ζ is a marker and E in $\widetilde{\mathcal{F}}$, let $\zeta(E)$ be the intersection of ζ and $S^1_\infty(E)$ which is at most one point.

Lemma 3.8. *If α, β are markers in $\mathcal{A}$ which intersect each other, then they do not intersect transversely, that is: For any E in $\widetilde{\mathcal{F}}$ with $\alpha(E), \beta(E)$ not empty, then $\alpha(E) = \beta(E)$.*

Proof. Let α, β be markers which intersect in a point p and let and E in $\widetilde{\mathcal{F}}$ with $\alpha(E), \beta(E)$ both non empty. There is L in $\widetilde{\mathcal{F}}$ with p equal to $\alpha(L)$ and $\beta(L)$. Let r be a geodesic ray in L with ideal point p. Let

$$r_1, r_2 \quad \text{geodesic rays in} \quad E \quad \text{with ideal points} \quad \alpha(E), \beta(E) \quad \text{respectively.}$$

As $p, \alpha(E)$ are in α then r and r_1 are asymptotic (in $\widetilde{M}$). Similarly r and r_2 are asymptotic so r_1 and r_2 are asymptotic. As in lemma 3.6 this implies that r_1 and r_2 are asymptotic in E. In other words $\alpha(E), \beta(E)$ are equal. ☐

Lemma 3.9. *Markers are continuous curves in $\mathcal{A}$.*

Proof. Consider a contracting direction in a leaf F of $\widetilde{\mathcal{F}}$ defined by the geodesic ray $\gamma = \{\gamma(t),\ t$ in $[0,+\infty)\}$ and ideal point p in $S^1_\infty(F)$. There is a packet of leaves near F which contracts to F in the γ direction. For any positive ϵ the whole packet is ϵ near $\gamma(t)$ for any t bigger than t_0 for some $t_0 > 0$, depending only on ϵ. Since the remainder is a compact initial segment $\gamma([0,t_0])$, if the packet is reduced the whole ray γ is ϵ near any leaf in the (smaller) packet. Hence one can move γ to curves in nearby leaves using the transversal foliation. These curves have arbitrarily small geodesic curvature, which goes to 0 as ϵ goes to 0. Therefore the curves are closer and closer to being geodesics and their ideal points are better and better determined by the initial directions. But the directions of the initial segments converge to the direction of γ at $\gamma(0)$ and so the ideal points of the lifted curves in the nearby leaves converge to the ideal point p in the topology of $\mathcal{A}$. This shows continuity of the marker at p and as p is arbitrary this completes the proof. The proof shows that a marker defined by a transversal arc μ is a homeomorphic image of μ in $\mathcal{A}$ which is transverse to the horizontal foliation in $\mathcal{A}$. $\square$

We will now consider the case of $\mathbf{R}$-covered non uniform foliations. Again the goal is to produce a natural vertical foliation in $\mathcal{A}$. Lemma 2.5 shows that $\mathcal{F}$ has no compact leaves and by proposition 2.6 we may assume that that $\mathcal{F}$ is minimal. Hence:

Running hypothesis for the rest of the section: $\mathcal{F}$ is a minimal, $\mathbf{R}$-covered, non uniform foliation with hyperbolic leaves.

In the uniform case if one of the leaves has a point sufficiently far from the other, then the leaves can never get too close to each other at all (for ideas on this see [Th7]). The non uniform case is completely different: any pair of leaves of $\widetilde{\mathcal{F}}$ has many directions where they are arbitrarily close.

We learned some ideas in this section from Danny Calegari in 1998 – at that point he was studying foliations with "confined regions". This means there are F and L in $\widetilde{\mathcal{F}}$ and a half plane of F which is asymptotic to L. His goal was to prove that if in addition $\mathcal{F}$ is minimal then it is conjugate to the stable foliation of a suspension Anosov flow. We realized that some ideas of the confined case can be used to treat the general case. The article [Cal1] has a similar treatment of the general non uniform situation.

Fix an orientation in $\mathcal{H}$. We first prove several needed properties of non uniform $\mathbf{R}$-covered foliations.

Lemma 3.10. *No two leaves of $\mathcal{F}$ are a bounded distance from each other.*

Proof. Suppose there are leaves E, F of $\widetilde{\mathcal{F}}$ which are a bounded distance from each other. Let $\mathcal{J}$ be the interval $[E,F]$ of $\mathcal{H}$ and consider the union of $\mathcal{J}$ with all its translates under covering translations. Take the component C containing $\mathcal{J}$. Assume first that C is a bounded interval in $\mathcal{H}$. Then translates of C are either C itself or disjoint from C. It follows that the closure $\hat{C}$ of C in $\mathcal{H}$ is precisely invariant. But then the leaves of $\widetilde{\mathcal{F}}$ corresponding to the translates of ∂C project to a non trivial closed set of $\mathcal{F}$ in M. This is not possible by hypothesis. If C is unbounded below in $\mathcal{H}$ then it has to be invariant under all of $\pi_1(M)$ and so it must be unbounded above, that is C is equal to $\mathcal{H}$. But as the Hausdorff distance is monotone increasing, that implies that any two leaves in $\widetilde{\mathcal{F}}$ are a bounded distance from each other or that $\mathcal{F}$ is uniform, contrary to assumption. This finishes the proof. $\square$

The following proposition is crucial for our results. It states that anything bounded can be put in between two arbitrary leaves and then uses that to produce contracting directions between the leaves.

Proposition 3.11. *(compression of the universal cover and contracting directions) Given arbitrary distinct leaves E, F of $\widetilde{\mathcal{F}}$ and B a bounded set, there is a covering translate of B contained between E and F. As a consequence there is at least one contracting direction between E and F.*

Proof. By hypothesis $d_H(E,F)$ is infinite. For simplicity we may assume without loss of generality that $\mathcal{F}$ is transversely orientable, F is in front of E and choose

$$p_i \in E \quad \text{with} \quad d(p_i, F) \quad \text{converging to infinity.}$$

Let B be a bounded set in $\widetilde{M}$. We are looking for a translate $h(B)$ of B so that $h(B)$ is in the front of E and in the back of F, that is, between E and F. Choose covering translations g_i with $g_i(p_i)$ converging to p_0 and so $g_i(E)$ converge to E_0 containing p_0. Let L leaf of $\widetilde{\mathcal{F}}$ very near p_0 and in front of E_0.

We claim that we can choose a covering translate $h(B)$ contained in the front of L. This seemingly obvious fact is not true in general, even for Reebless foliations! For example, start with a Reeb foliated annulus A and consider $A \times \mathbf{S}^1$ with the product foliation. Then glue the two boundary tori to produce a non taut, but Reebless foliation $\mathcal{F}$. A noncompact leaf in A produces an annulus leaf of $\mathcal{F}$. Lifting to Z in $\widetilde{\mathcal{F}}$, one of the complementary regions of Z in $\widetilde{M}$ has every point a bounded distance from Z, and there are sets of big diameter which cannot be mapped into that component. This example also shows that for general $\mathcal{F}$, given arbitrary leaves G, H of $\widetilde{\mathcal{F}}$, the fact that G is in a bounded neighborhood of H does not imply that H is in a bounded neighborhood of G - this relation is not symmetric.

To prove the claim we use that $\mathcal{F}$ is taut. Suppose there is a finite supremum a_0 of $d(z, L)$ for z in front of L. Let z in front of L with $d(z, L)$ very near a_0. Any geodesic arc from z to L with length very close to a_0 is almost perpendicular to L. There is positive ϵ so we can choose foliated box neighborhoods of these points in L with $d(z, y)$ bigger than $a_0 + \epsilon$ for any y in the other side of L from these foliated boxes. As $\mathcal{F}$ is taut there is a transversal from L to a translate $f(L)$ in the back of L and not intersecting those neighborhoods. Then z is in front of $f(L)$ and $d(z, f(L))$ is greater than $a_0 + \epsilon$. Hence $f^{-1}(z)$ is in front of L and $d(f^{-1}(z), L)$ is bigger than a_0 contradiction to assumption. This proves the claim.

As $g_i(E)$ converges to E_0 then for i big enough L is in front of $g_i(E)$ and so is $h(B)$. Then

$$d(g_i(p_i), g_i(F)) \ \rightarrow \ +\infty, \quad \text{but} \ \ d(g_i(p_i), h(B)) \quad \text{is bounded.}$$

It follows that $g_i(F)$ does not intersect $h(B)$ and does not separate it from $g_i(E)$ for i big enough. Since $\mathcal{F}$ is $\mathbf{R}$-covered this implies that $h(B)$ is in the front of $g_i(E)$ and in the back of $g_i(F)$, that is between $g_i(E)$ and $g_i(F)$. Hence $g_i^{-1} h(B)$ is between E and F, see fig. **??**. This proves the first statement, compression of the universal cover.

Now take B intersecting E and F and choose $g(B)$ to be between E and F, see fig. **??**, a. This implies that $g(E), g(F)$ are between E and F and so

$$g([E, F]) \ \subset \ (E, F).$$

As $[E, F]$ is an interval in $\mathcal{H}$, there is F_0 in (E, F) with $g(F_0)$ equal to F_0 and $g^i(F)$ converging to F_0 when i converges to infinity. Hence there is a geodesic γ in F_0 with $g(\gamma) = \gamma$ and which has contracting holonomy in one side, that is g contracts the interval $[F_0, F]$ to F_0 as i converges to infinity, see fig. **??**, b.

By the same arguments as above then for any two leaves U, V of $\widetilde{\mathcal{F}}$ one can map the interval $[U, V]$ of $\mathcal{H}$ inside $[F_0, F]$. Since F_0 has a contracting direction with F it produces a contracting direction between any two leaves in $[F_0, F]$. Pulling back by a covering translation, this produces a contracting direction between U and V. This finishes the proof of the proposition. $\square$

The goal is to use these contracting directions to produce identifications of the circles at infinity.

Lemma 3.12. *There are at least two contracting directions between any two leaves $E, F \in \widetilde{\mathcal{F}}$.*

Proof. Otherwise there are E_0, F_0 in $\widetilde{\mathcal{F}}$ with only one contracting direction between them. For any E, F in $\widetilde{\mathcal{F}}$ we can map $[E_0, F_0]$ inside $[E, F]$, so there is only one contracting direction between $[E, F]$ as well. Choosing leaves E_i, F_i of $\widetilde{\mathcal{F}}$ with E_i, F_i escaping to opposite ends of $\mathcal{H}$ and the packets $[E_i, F_i]$ increasing, it follows that there is a unique "vertical" curve α in $\mathcal{A}$, which contains all markers. For any leaf F let α_F be the intersection of α and F. The action of $\pi_1(M)$ on $\mathcal{A}$ sends markers to markers, therefore any covering translation acts in $\mathcal{A}$ sending the unique vertical marker α to itself.

Given a covering translation g not acting freely in $\mathcal{H}$, there is a leaf F with $g(F) = F$, so there is a geodesic axis l in F invariant under g. Since $g(\alpha_F) = \alpha_F$, then α_F is one of the ideal points of l. Let now f be any other covering translation with $f(F)$ and F distinct. Then

$$(fgf^{-1})(f(F)) \;=\; f(F) \quad \text{and} \quad (fgf^{-1})(f(l)) \;=\; f(l)$$

so $f(l)$ has an ideal point in $\alpha_{f(F)}$. This implies that l and $f(l)$ are asymptotic in $\widetilde{M}$, which is impossible since $\pi(l)$ is a closed curve in M and there is a minimum distance between any two distinct lifts to $\widetilde{M}$. We conclude that there are at least two markers connecting any two leaves. □

Proposition 3.13. *(local density of markers) Given F in $\widetilde{\mathcal{F}}$, Y any open set in $S^1_\infty(F)$, and A any component of $\mathcal{A} - S^1_\infty(F)$, there is a marker with an endpoint in Y and contained in the closure of A.*

Proof. This shows that there are markers on "both" sides of Y in $\mathcal{A}$. The key property is that $\mathcal{F}$ is minimal. Suppose the result is not true. Let

– F be a leaf of $\widetilde{\mathcal{F}}$, Y an open set in $S^1_\infty(F)$;

– A component of $\mathcal{A} - S^1_\infty(F)$, so that there is no marker in $\mathcal{A}$ with an endpoint in Y and contained in the closure of A.

Choose points p_i in F with p_i converging to p in Y. Fix a small transversal μ to $\widetilde{\mathcal{F}}$, let E, L be the leaves of $\widetilde{\mathcal{F}}$ through the endpoints of μ. Since $\mathcal{F}$ is minimal, there is a positive constant a_0 so that any point in a leaf of $\mathcal{F}$ is within a_0 (in its leaf) of another point very near the center of $\pi(\mu)$. Lifting to $\widetilde{M}$, there are q_i in F with $d_F(q_i, p_i)$ smaller than a_0 and covering translations g_i with $g_i(q_i)$ in μ. Up to subsequence

$$g_i(q_i) \quad \text{converges to} \quad q_0 \;\in\; \mu, \; q_0 \;\in\; F_0 \;\in\; \widetilde{\mathcal{F}}.$$

Notice that in $F \cup S^1_\infty(F)$, q_i also converges to p in Y, hence the visual measure of Y (in $S^1_\infty(F)$) as measured from q_i is θ_i with θ_i converging to 2π. Hence from $g_i(q_i)$, the visual measure of $g_i(Y)$ in $S^1_\infty(g_i(F))$ is θ_i also.

By lemma 3.12 there are at least two markers

$$\zeta_1, \zeta_2 \quad \text{from} \quad S^1_\infty(E) \text{ to } S^1_\infty(L).$$

Use the parametrization of the circles at infinity between E and L given by $T_1\widetilde{\mathcal{F}}|\mu$. The markers ζ_1, ζ_2 intersect $S^1_\infty(F_0)$ in angles δ_1, δ_2 as measured by this identification. Since the markers are continuous in $\mathcal{A}$ and disjoint, there is positive a_1 so that for any G in $\widetilde{\mathcal{F}}$ intersecting μ the markers ζ_1, ζ_2 define directions in G which are at least a_1 angles apart – as measured in $T_1\widetilde{\mathcal{F}} \mid (G \cap \mu)$. But $g_i(F)$ converges to F_0 as i converges to infinity and the markers in one side of $S^1_\infty(g_i(F))$ are restricted to have an endpoint in $S^1_\infty(g_i(F)) - g_i(Y)$. This set has visual measure smaller than $2\pi - \theta_i$ which converges to zero with i. These two facts contradict each other. This shows the local density of markers. □

Markers were introduced by Thurston in [Th7]: he showed that markers are (locally) dense in $\mathcal{A}$ (also in the non **R**-covered case). We will show a much stronger fact in our setting: there is a dense set of contracting directions between any two leaves of $\widetilde{\mathcal{F}}$. The markers will be the skeleton of the vertical foliation in $\mathcal{A}$. It is fundamental for all the analysis that markers are continuous curves in $\mathcal{A}$. The continuity of markers can be strengthened to a property that says markers are not too horizontal:

Lemma 3.14. *Let L a leaf of $\widetilde{\mathcal{F}}$ and Z a closed subset of $S^1_\infty(L)$. For any open neighborhood N of Z in $\mathcal{A}$, there are neighborhoods V of L in $\mathcal{H}$ defined by transversal μ to $\widetilde{\mathcal{F}}$ ($\widetilde{\mathcal{F}}_\mu = V$) and W of Z in $\mathcal{A}$, so that any marker ζ which intersects W then its intersection with $\mathcal{A}_\mu$ is contained in N.*

Proof. If there were a horizontal marker - that is contained in some $S^1_\infty(F)$ - it would clearly fail the lemma. This is because no matter how small a neighborhood of $S^1_\infty(F)$ in $\mathcal{A}$, this curve still moves a fixed amount in the horizontal direction. Still this is a continuous curve. The aim is to show that markers cannot even get too close to horizontal arcs.

If the lemma is not true there are 1) a leaf L in $\mathcal{F}$, 2) a closed subset Z of $S^1_\infty(L)$, and 3) an open neighborhood N of Z in $\mathcal{A}$ satisfying: there are shrinking neighborhoods V_i of L in $\mathcal{H}$ (that is $\cap V_i = L$) defined by transversals μ_i (that is $\widetilde{\mathcal{F}}_{\mu_i} = V_i$), there are shrinking open neighborhoods W_i of Z in $\mathcal{A}$ (that is $\cap W_i = Z$) and markers ζ_i with

$$\zeta_i \cap W_i \neq \emptyset \quad \text{but} \quad \zeta_i \cap \mathcal{A}_{\mu_i} \not\subset N.$$

Choose points x_i in the intersection of W_i and ζ_i. As the W_i shrink to Z assume up to subsequence that x_i converges to x_0 with x_0 in Z. There are

$$y_i \in \zeta_i \quad \text{with} \quad y_i \in \mathcal{A}_{\mu_i} \quad \text{but not in } N.$$

Since $\mathcal{A}_{\mu_i}$ shrinks to $S^1_\infty(L)$, we can choose another subsequence so that y_i converges to y_0 a point in $S^1_\infty(L)$. But y_i is not in N, so y_0 is not in Z, hence y_0, x_0 are different points. For simplicity assume y_i are points in $S^1_\infty(F_i)$, with F_i above L and only consider the part of the markers on the corresponding side of $S^1_\infty(L)$ in $\mathcal{A}$. Since the markers are continuous curves in $\mathcal{A}$, then up to another subsequence the markers ζ_i have to limit in at least one of the segments in $S^1_\infty(L)$ defined by x_0 and y_0. Let B be this segment. If B has a marker ζ on that side of $S^1_\infty(L)$ in $\mathcal{A}$, then because the ζ_i limit on B, it follows that ζ_i will intersect ζ for i big enough. Lemma 3.8 shows that for each such i, ζ and ζ_i are subpieces of a possibly bigger marker ζ'. Hence for i big enough the intersection of ζ' and $\mathcal{A}_{\mu_i}$ is equal to the intersection of ζ_i and $\mathcal{A}_{\mu_i}$. The marker ζ' is a continuous curve in $\mathcal{A}$, transverse to the horizontal foliation, so for i big enough i,

$$\zeta' \cap \mathcal{A}_{\mu_i} \subset N \quad \text{which implies} \quad \zeta_i \cap \mathcal{A}_{\mu_i} \subset N,$$

contradiction to assumption. We conclude that this is impossible.

The remaining option is that there are no markers on that side of $S^1_\infty(L)$ with endpoint in B. This is disallowed by the previous proposition. The proof is complete. □

The following lemma says that if a sequence of markers converges to a point in a marker ζ then the whole markers also converge to ζ. It is needed later for the analysis of global density of markers.

Lemma 3.15. *Let S, S' leaves of $\widetilde{\mathcal{F}}$ and $\{\zeta_i\}$ with $i \geq 0$ a sequence of markers from $S^1_\infty(S)$ to $S^1_\infty(S')$. If the intersection a_i of ζ_i and $S^1_\infty(S)$ converges to a_0 with i, then the ζ_i converge to ζ_0 in $\mathcal{A}$, that is, for any Z in $[S, S']$, the intersection b_i of ζ_i and $S^1_\infty(Z)$ converges to b_0, the intersection of ζ_0 and $S^1_\infty(Z)$.*

Proof. Else there are $\{\zeta_i\}$, Z as above so that b_i does not converge to b_0. For simplicity suppose the sequence a_i for i bigger than 0 is nested (with i) in $S^1_\infty(S)$. The non transversal intersection of markers implies that the b_i are also nested in $S^1_\infty(Z)$. Let r be a geodesic ray in S with ideal point a_0 and let v be a geodesic ray in Z with ideal point b_0. Let p_j be a sequence in r converging to a_0. We can choose q_j in v with $d(q_j, p_j)$ converging to zero since a_0 defines a contracting direction from S to Z. For each positive i let $r_{j,i}$ (respectively $v_{j,i}$) be the ray in S starting in p_j with ideal point a_i (respectively in Z starting in q_j with ideal point b_i). For each j we can choose $i(j)$ big enough so that the directed angle in S at p_i between r and $r_{j,i(j)}$ is θ_j and θ_j converges to 0. Directed means it is measured from r to $r_{j,i(j)}$ in the side the $r_{j,i}$ accumulates on r (when i grows). Since the $b_{i(j)}$ do not converge to b_0, then as seen from q_0, the visual angle of the segment in $S^1_\infty(Z)$ from b_0 to $b_{i(j)}$ does not converge to 0. It follows that the directed angle β_j at q_j between the rays

$$v \quad \text{and} \quad v_{j,i(j)}$$

does not converge to zero (in fact it converges to π). Then choose covering translations f_j so that $f_j(p_j)$ converges to a point p_0, hence $f_j(q_j)$ converges to p_0 as well. At $f_j(p_j)$ the angle between

$$f_j(r) \quad \text{and} \quad f_j(r_{j,i(j)})$$

converges to 0, but at q_j the angle between $f_j(v)$ and $f_j(v_{j,i(j)})$ does not converge to zero. This shows that least one of the markers $f_j(\zeta_0)$ or $f_j(\zeta_{i(j)})$ moves a definite amount horizontally in arbitrarily small vertical displacement. For j big enough this contradicts lemma 3.14. This finishes the proof. □

Definition 3.16. *(invariant curves) An invariant curve in $\mathcal{A}$ is an embedded curve intersecting each circle at infinity exactly once and invariant under all covering homeomorphisms of $\mathcal{A}$. An invariant curve which is a limit of longer and longer markers is called a limit invariant curve.*

For instance if $\mathcal{F}$ is the stable foliation of a suspension Anosov flow, form the curve of all the positive ideal points of leaves of $\widetilde{\mathcal{F}}$. This is continuous in $\mathcal{A}$ and invariant. This foliation is $\mathbf{R}$-covered and not uniform. The analysis of $\mathbf{R}$-covered non uniform foliations will go roughly as follows:

If the set of contracting directions between a pair of leaves is not dense then one produces a limit invariant curve $\mathcal{L}$ in $\mathcal{A}$. One can show that the leaves are asymptotic away from the invariant curve - that is all directions but one are contracting. So in any case one obtains a dense set of contracting directions. The strategy here is to first analyse limit invariant curves in detail in lemmas 3.17 through 3.20 and proposition 3.21 and then use that to produce the vertical foliation in proposition 3.22.

Lemma 3.17. *Any limit invariant curve $\mathcal{L}$ has no points associated to contracting directions of $\mathcal{F}$.*

Proof. Suppose the limit invariant curve line $\mathcal{L}$ has a point q associated to a contracting direction. Then there is a marker ζ_0 through q. By hypothesis there are markers ζ_j which converge pointwise to $\mathcal{L}$. The previous lemma shows that these markers converge pointwise to ζ_0 in the circles at infinity that ζ_0 intersects. This shows that $\mathcal{L}$ contains the marker ζ_0 $-$ that is, $\mathcal{L}$ coincides with ζ_0 locally. We can map any interval $[U, U']$ of $\mathcal{H}$ inside this small segment, hence the whole curve $\mathcal{L}$ is a marker. But since $\mathcal{L}$ is $\pi_1(M)$ invariant, the argument of lemma 3.12 shows that this is impossible. This finishes the proof. □

We use the *transversal flow distance* between points and leaves: Fix a transversal line field to $\mathcal{F}$ generating a foliation τ with lift $\widetilde{\tau}$ to $\widetilde{M}$. Given G in $\widetilde{\mathcal{F}}$ and z in $\widetilde{M}$, consider the transversal flow line τ_z through z. As $\mathcal{F}$ is Reebless τ_z can intersect G at most once. If they do not intersect let $d_\tau(z, G)$ be infinity. Otherwise let $d_\tau(z, G)$ be the length of the segment of τ_z from z to the intersection with G. If $\mathcal{L}$ is an invariant curve in $\mathcal{A}$ and L a leaf of $\widetilde{\mathcal{F}}$ let $\mathcal{L}_L$ be the intersection of $S^1_\infty(L)$ and $\mathcal{L}$.

Lemma 3.18. *Let $\mathcal{L}$ be a limit invariant curve. Given L in $\widetilde{\mathcal{F}}$ and a side of L in $\widetilde{M}$ there is G of $\widetilde{\mathcal{F}}$ in that side so that: for any half plane H of L which does not limit on $\mathcal{L}_L$ and any escaping sequence of points z_i in H then the limsup of $d_\tau(z_i, G)$ is bounded above (depending only on H and G).*

Proof. This is stronger than limsup $d(z_i, G)$ being bounded, which can occur even if $d_\tau(z_i, G)$ is infinite for all i - for example if $\mathcal{F}$ is the stable foliation of an Anosov geodesic flow and τ is given by the strong unstable foliation. We do the proof for G above L, the same proof applies for G below L.

Roughly the proof goes as follows: if there is u in $S^1_\infty(L)$ distinct from $\mathcal{L}_L$ so that d_τ "blows up" near u, then one can map any transversal segment to one "near u". This produces covering translations with invariant leaves in $\widetilde{\mathcal{F}}$ and a contracting fixed point in $\mathcal{L}$ - contradicting the previous lemma.

Suppose the proposition is not true. Let G_i be a sequence in $\widetilde{\mathcal{F}}$ converging to L. Given i there is a sequence $z_{i,j}$ in H with $d_\tau(z_{i,j}, G_i)$ bigger than j and $z_{i,j}$ escapes in H (with j growing). Using subsequences find

$$z_i \ \in \ H \quad \text{with} \ \ d_\tau(z_i, G_i) \ > \ i \quad \text{and} \quad z_i \ \to \ u \in su(L), \ u \ \text{ distinct from } \ \mathcal{L}_L.$$

Fix v in L. Let α be τ_v and α_i the subsegments of α between L and G_i, whose lengths converge to 0.

Since $\mathcal{F}$ is minimal any leaf is dense. Given positive a_0 there is positive a_1 so that if β is a segment of the foliation τ of length bigger than a_0, w any point in M and W the leaf of $\mathcal{F}$ through w then the following happens: W intersects β in a point w' which is within $a_0/4$ of the midpoint of β (in the flow length of β) and w' is at most a_1 distant from w in the path distance of W.

Also there is positive a_2 sufficiently small, so that for any segment β' of τ of length smaller than a_2, then if it is moved by holonomy so that starting point moves a distance less than a_1 in its leaf (of $\mathcal{F}$) then the final segment of τ has length bounded above by $a_0/4$. Hence any segment of τ of length bounded by a_2 can be moved by holonomy, with initial point moved a distance less than a_1 within its leaf to have a point in the segment β within $a_0/4$ of the middle point of β. Since the length of the holonomy translate is less than $a_0/4$ the final holonomy translate is entirely contained in β.

By truncating finitely many terms assume length of α_i is bounded above by a_2. Let β_i be segments in leaves of $\widetilde{\tau}$ of length a_0 with an endpoint in z_i and contained in the positive side of L. The property of the z_i's implies that (at least for i big enough) all β_i are in the union of leaves S of $\widetilde{\mathcal{F}}$ contained in the interval $[L, G_i)$ of $\mathcal{H}$. Using the previous paragraph there are covering translations h_i so that $h_i(v)$ is in a leaf $h_i(L)$ of $\widetilde{\mathcal{F}}$ intersecting β_i within distance $a_0/4$ of the midpoint of β_i and path distance from $h_i(v)$ to β_i is less than a_1 in $h_i(L)$. By the previous paragraph the image of $h_i(\alpha_i)$ by holonomy will map into β_i. The endpoints of α_i are in L, G_i and the endpoints of β_i are in L and in another leaf between L and G_i. This implies that on the level of the leaf space h_i sends the interval $[L, G_i]$ of $\mathcal{H}$ into a subset of its interior (L, G_i) – so h_i has a fixed point in (L, G_i). Then $h_i^n(L)$ converges to a leaf L_i of $\widetilde{\mathcal{F}}$ when n converges to infinity (for each i!) and L_i is invariant under h_i. Notice that L_i converges to L as i converges to infinity, because L_i is in (L, G_i) and G_i converges to L in $\mathcal{H}$.

Since $h_i(L_i) = L_i$ then h_i acts as a hyperbolic isometry in L_i and has two fixed points in $S^1_\infty(L_i)$. Let

$$h_i^+ \quad = \quad \lim_{n \to +\infty} \ h_i^n(x)$$

for any point x of L_i. Let h_i^- be the other fixed point of h_i. The key fact needed here is the following:

Lemma 3.19. *h_i^+ converges to u in $\mathcal{A}$ when i converges to infinity.*

Proof. Let N be a neighborhood of $u \in \mathcal{A}$ in the top side of $S^1_\infty(L)$ (u defined at the beginning of the proof lemma 3.18). Identify N to a subset of $T_1\alpha$ using the ideal circles. Then N contains an open segment T in $S^1_\infty(L)$ with u in T. As markers are locally dense in $S^1_\infty(L)$, there are markers

$$\xi_1, \ \xi_2 \quad \text{from} \quad S^1_\infty(L) \quad \text{to} \quad S^1_\infty(S_1)$$

with S_1 above L, intersecting $S^1_\infty(L)$ in $\xi_1(L), \xi_2(L)$ respectively so that: the intersections with $S^1_\infty(L)$ are in T and define a small segment in $S^1_\infty(L)$ with u in the interior. Let r, r_1, r_2 be geodesic segments in L starting in v and with ideal points u, $\xi_1(L), \xi_2(L)$ respectively. Notice that r_1, r_2 are contracting directions between L and S_1. Let a_3 positive, very small. Since r_1, r_2 are contracting directions between L and S_1, there is S_2 in (L, S_1) so that any point in r_1, r_2 is within a_3 of S_2, and hence within a_3 of any S between L and S_2. For any such S we can move r_1 and r_2 to S using the transversal flow - if a_3 is sufficiently small. The geodesic curvature of the pushed curves in S is small tending to zero as a_3 tends to zero hence they are quasigeodesics in S and their initial directions give arbitrarily close estimates of the direction defined by the lifts of r_1, r_2 to S. Hence these directions are in N and are close to the direction of r in $T_1\alpha$ (if N is small). The markers ξ_1, ξ_2 and the circles $S^1_\infty(L)$, $S^1_\infty(S_2)$ define a small neighborhood N_1 of u in $\mathcal{A}$ in that side of $S^1_\infty(L)$ in $\mathcal{A}$. We can choose N_1 to be a subset of N.

Let v_i be the intersection of α and L_i. Then for i big z_i is in the wedge of L defined by r_1, r_2, so the intersection b_i of β_i and L_i is in the wedge defined by the images of r_1, r_2 in L_i. The direction of the geodesic segment in L_i from v_i to b_i is within this wedge and defines a point in N_1 and hence in N. As $h_i(v_i)$ is boundedly close to b_i in L_i, then the direction of the geodesic segment in L_i from v_i to $h_i(v_i)$ also defines a point in N for i big enough. The points v_i are in the fixed transversal α and very close to

v, hence they are in a compact subset of L_i. The points $h_i(v_i)$ are boundedly close to β_i hence also from z_i. As $d_L(v, z_i)$ converges to infinity then

$$d_{L_i}(v_i, h_i(v_i))$$

is also converging to infinity. Since h_i is a hyperbolic isometry of L_i, this now implies that $h_i(v_i)$ is close to h_i^+ in the compactification $L_i \cup S^1_\infty(L_i)$. Notice this argument does not give any information about h_i^-. This shows that the direction in L_i defined by h_i^+ is in N. As N is arbitrary this shows that h_i^+ converges to u. This finishes the proof. $\square$

Remarks: 1) These arguments in fact show: if there is positive c_0 and there are z_i in L converging to u, G_i in $\widetilde{\mathcal{F}}$ converging to L to that $d_\tau(z_i, G_i)$ bigger than c_0, then one obtains h_i in $\pi_1(M)$ with fixed points h_i^+ in $\mathcal{A}$ converging to u.

2) Similarly if u in $S^1_\infty(L)$ is a contracting direction on the positive side (of $\widetilde{\mathcal{F}}$), one switches the roles of α_i and β_i to get: let G_i in $\widetilde{\mathcal{F}}$ converging to L all in the domain of contraction of holonomy in the direction u. Fix geodesic ray r in L with ideal point u. Fix i and let c_0 be the length of α_i. As above there is positive but very small c_1 so that any segment of $\widetilde{\tau}$ of length smaller than c_1 can be transported by a bounded distance holonomy to be in the interior of a covering translate of α_i. As u is contracting direction choose z_i in r with $d_\tau(z_i, G_i)$ smaller than c_1. Let β_i defined as before, now with length less than c_1. This produces g_i in $\pi_1(M)$ with $g_i(\beta_i)$ contained in the interior of the set of leaves of $\widetilde{\mathcal{F}}$ intersected by α_i. The g_i^{-1} acting on $\mathcal{A}$ have (positive) fixed points c_i which converge to u in $\mathcal{A}$. This shows that arbitrarily near any contracting direction there are contracting fixed points of covering homeomorphisms.

Conclusion of the proof of lemma 3.18. Let γ_i be the geodesic in L_i which is the axis of h_i in L_i so $h_i(\gamma_i) = \gamma_i$. The ideal points of γ_i are h_i^+, h_i^-. Then

$$h_i \quad \text{sends} \quad [L, L_i] \quad \text{inside} \quad L, L_i]$$

and has no other invariant leaf in $(L, L_i]$. Hence h_i contracts the leaf space near L_i and therefore the direction of γ_i associated to h_i^+ is an expanding direction for $\widetilde{\mathcal{F}}$: nearby leaves of $\widetilde{\mathcal{F}}$ diverge from L_i in this direction.

This implies that the direction of γ_i associated to h_i^- is a contracting direction (or equivalently h_i^{-1} expands the leaf space near L_i). But $h_i(\mathcal{L}) = \mathcal{L}$, so one of the ideal points of γ is in $\mathcal{L}$. As h^+ converges to u and u is not $\mathcal{L}_L$ then for i big enough h_i^+ is not $\mathcal{L}_{L_i}$. So for i big enough, h_i^- is $\mathcal{L}_{L_i}$. But this would imply $\mathcal{L}$ has a point h_i^- associated to a contracting direction. This contradicts lemma 3.17 and finishes the proof of lemma 3.18. $\square$

With more work we can show $d_\tau(z_i, G)$ converges to 0:

Lemma 3.20. *Suppose there is a limit invariant curve $\mathcal{L}$. For any L in $\widetilde{\mathcal{F}}$ and a side of L in $\widetilde{M}$ there is G of $\widetilde{\mathcal{F}}$ in that side so that: for any u in $S^1_\infty(L)$ and distinct from $\mathcal{L}_L$ and any sequence z_i in L converging to u then $d_\tau(z_i, G)$ converges to 0. In particular u is a contracting direction between L and G.*

Proof. Given L and a side of it pick a G as given by lemma 3.18. Suppose the proposition is not true. Then find u in $S^1_\infty(L)$ distinct from $\mathcal{L}_L$ and sequence z_i with $d_\tau(z_i, G)$ not converging to 0. By lemma 3.18 the limsup of $d_\tau(z_i, G)$ is bounded above by a constant a_4 which depends only on L, G and u. Since $d_\tau(z_i, G)$ does not converge to 0, up to subsequence assume $d_\tau(z_i, G)$ converges to a_5 positive. Up to another subsequence choose f_i in $\pi_1(M)$ with $f_i(z_i)$ converging to a point z_0. Then $f_i(L)$ converges to L_0 containing z_0 and $f_i(G)$ converges to a leaf G_0 because $d_\tau(z_i, G)$ converges to a_5. Here G_0, L_0 are distinct leaves because a_5 is positive. For any w in L_0, $d_{L_0}(w, z_0)$ is finite, so w is the limit of w_i with w_i in $f_i(L)$ and $d_{f_i(L)}(w_i, f_i(z_i))$ bounded (the bound depends on $d_{L_0}(w, z_0)$). The points $f_i^{-1}(w_i)$ of L are a bounded distance from z_i and in particular

$$f_i^{-1}(w_i) \ \rightarrow \ u \ \in \ S^1_\infty(L) \ \text{ when } \ i \ \rightarrow \ +\infty.$$

Therefore the limsup of $d_\tau(f_i^{-1}(w_i), G)$ is less than a_4. Up to subsequence $d_\tau(f_i^{-1}(w_i), G)$ converges to a_6, which is not 0 because $d_L(f_i^{-1}(w_i), z_i)$ is bounded above and $d_\tau(z_i, G)$ is bounded below by a positive constant. There are y_i in G with $y_i, f^{-1}(w_i)$ in the same leaf of $\widetilde{\tau}$ and $d_\tau(y_i, f_i^{-1}(w_i))$ converging to a_6. Then $f_i(y_i)$ converges to a point y in τ_w and $d_\tau(w, y)$ is a_6. But $f_i(y_i)$ is in $f_i(G)$ and $f_i(G)$ converges only to G_0 hence y is in G_0. This produces a map φ from L_0 to G_0 given by $\varphi(w) = y$. Notice that for any w in L_0, the $w, \varphi(w)$ are in the same leaf of $\widetilde{\tau}$ and $d_\tau(w, \varphi(w))$ is less than a_4.

The map φ from L_0 to G_0 is injective because $\mathcal{F}$ is Reebless and hence it is a homeomorphism onto its image. If $\varphi(L_0)$ is not all of G_0 then there is b in G_0 with b in the boundary of $\varphi(L_0)$ (as a subset of G_0). Choose s_j in $\varphi(L_0)$ converging to b. Let x_j in L_0 with $\varphi(x_j) = s_j$. Then

$$d(x_j, s_j) \ \ \leq \ \ d_\tau(x_j, s_j) \ \ \leq \ \ a_4$$

hence $d(x_j, b)$ is bounded and so is $d(x_j, x_1)$. As $\mathcal{F}$ is $\mathbf{R}$-covered this implies that $d_{L_0}(x_j, x_1)$ is bounded too (this is a key point!). Up to subsequence assume that x_j converges to x_0. Then $s_j = \varphi(x_j)$ converges to $\varphi(x_0)$ - a point in G_0. But b is equal to $\varphi(x_0)$ and is in $\varphi(L_0)$ contradicting the hypothesis.

We conclude that φ is surjective and in fact for every point s in G_0, $d_\tau(s, L_0)$ is less than a_4. Using the fact that $\mathcal{F}$ is minimal this quickly shows that any two leaves of $\widetilde{\mathcal{F}}$ are a bounded distance from each other, contradicting the non uniform hypothesis. This finishes the proof of lemma 3.20. ☐

Since L in $\mathcal{H}$ and u in $S^1_\infty(L) - \mathcal{L}_L$ are arbitrary, lemma 3.20 shows that every point u of $\mathcal{A} - \mathcal{L}$ is an interior point of some marker ζ in $\mathcal{A}$. If two markers intersect then their union is a marker. This produces a 1-dimensional foliation $\mathcal{N}$ in $\mathcal{A} - \mathcal{L}$ consisting of the collection of all markers. The goal is to show that any leaf of $\mathcal{N}$ intersects all circles at infinity.

Proposition 3.21. *Suppose there is a limit invariant curve $\mathcal{L}$. For any E, F in $\widetilde{\mathcal{F}}$ and any v in $S^1_\infty(E)$ distinct from $\mathcal{L}_E$ then v is a contracting direction with F - that is every direction but those in $\mathcal{L}$ are contracting direction between arbitrary leaves.*

Proof. Let v in $\mathcal{A} - \mathcal{L}$ be a fixed point of a covering homeomorphism f; ζ the leaf of $\mathcal{N}$ through v and R_0 in $\widetilde{\mathcal{F}}$ with v in $S^1_\infty(R_0)$. For simplicity assume that ζ misses some $S^1_\infty(R)$ with R above R_0. Let R_1 above R_0 the smallest such that ζ misses - the set of R such that ζ intersects $S^1_\infty(R)$ is open in $\mathcal{H}$ because every point in $\mathcal{A} - \mathcal{L}$ is in the <u>interior</u> of a marker. Then

$$f(\zeta) \ = \ \zeta \quad \text{implies} \quad f(R_1) \ = \ R_1$$

because $\mathcal{F}$ is $\mathbf{R}$-covered. Also for any R in the interval (R_0, R_1) of $\mathcal{H}$ one has $f(R), R$ distinct, because all such R are asymptotic to R_0 in the v direction and cannot be left invariant by covering translations associated to that direction. If needed switch f, f^{-1} so that v is the atracting fixed point of f in $S^1_\infty(R_0)$. Then R_0 is an expanding point for the action of f in $[R_0, R_1]$, see fig. **??**, a. The action of f on the closed interval

$$B \ \ = \ \ \mathcal{L} \ \cap \ (\cup_{R \in [R_0, R_1]} S^1_\infty(R))$$

has an expanding fixed point in $\mathcal{L}_{R_0}$. As f has no invariant leaf in (R_0, R_1), the action of f on B has $\mathcal{L}_{R_1}$ as an attracting point, see fig. **??**, a. As $f(R_1) = R_1$ this shows that $\mathcal{L}_{R_1}$ corresponds to a contracting direction in R_1, contradicting lemma 3.17. We conclude that ζ intersects all circles at infinity.

If a leaf ν of $\mathcal{N}$ intersecting $S^1_\infty(S_0)$ does not intersect all circles at infinity; assume there is (say) a top limit $S^1_\infty(S_1)$. Then ν limits to $\mathcal{L}$ near $S^1_\infty(S_1)$, see fig. **??**, b. It follows that $\nu, S^1_\infty(S_0)$ and $\mathcal{L}$ bound a region $\mathcal{Z}$ which does not intersect $S^1_\infty(S_1)$, see fig. **??**, b. Any marker intersecting $\mathcal{Z}$ is bounded above. Now just choose u in $\mathcal{Z}$ which is a fixed point of some covering translation – we showed before that any contracting direction is the limit of fixed points of covering translations. Let ν' be the leaf of $\mathcal{N}$ through v'. As v' is in $\mathcal{Z}$ then ν' is bounded above, which was previously disallowed.

The conclusion is that for any marker ζ in $\mathcal{A}$, then ζ intersects all $S^1_\infty(R)$. In particular given E, F in $\widetilde{\mathcal{F}}$ and u in $S^1_\infty(E) - \mathcal{L}_E$ then the marker ζ through u intersects $S^1_\infty(F)$ - that is u is a contracting direction between E and F. This finishes the proof of proposition 3.21. $\square$

Using these results, we can now finish the analysis of the non uniform case:

Proposition 3.22. *(vertical foliation - non uniform case) Let $\mathcal{F}$ be a minimal, non-uniform* **R***-covered foliation with hyperbolic leaves. Given any F, E of $\widetilde{\mathcal{F}}$, there is a dense set of directions in F contracting towards E. The set of markers extends to a natural vertical foliation in $\mathcal{A}$ which is group invariant.*

Proof. The argument goes like this: If markers are not dense, zoom in towards an interior point of the markerless set. This pushes markers to the opposite end and defines an ideal point of the leaf - all markers between sufficiently spaced leaves of $\widetilde{\mathcal{F}}$ have to pass near this point. This collection of points at infinity produces a curve $\mathcal{L}$ in $\mathcal{A}$ which is a limit invariant curve. Then appeal to the previous proposition.

Suppose the proposition is not true. Then there are F in $\widetilde{\mathcal{F}}$ and E in $\widetilde{\mathcal{F}}$ (say above F) and not a dense set of contracting directions from F to E $-$ this is from the point of view of F! Hence there is an open interval J_0 in $S^1_\infty(F)$ so that no point in J_0 corresponds to a contracting direction from F to E. Let q_0 in J_0. Since q_0 is not a contracting direction between F and E, there is positive ϵ and p_i in F converging to q_0 along a geodesic ray l and so that $d(p_i, E)$ is bigger than ϵ.

In the leaf F, the visual measure of J_0 from the point of view of p_i is θ_i with θ_i converging to 2π as i converges to infinity. Up to a subsequence of p_i choose covering translations g_i with $g_i(p_i)$ converging to p_0 and $g_i(l)$ converging to a geodesic ray l_0. Let F_0 in $\widetilde{\mathcal{F}}$ containing p_0. Let

$$\mathcal{O} \quad = \quad \{\ G \in \widetilde{\mathcal{F}} \ \mid \ G = f(F_0), \ \text{for some } f \in \pi_1(M)\ \} \ \subset \ \mathcal{H}.$$

We will define a function η from $\mathcal{O}$ to $\mathcal{A}$ which picks out the "limit" marker direction and will produce a limit invariant curve. Since $d(g_i(E), g_i(p_i))$ is bigger than ϵ then $g_i(E)$ does not have any subsequence converging to F_0. A marker from $g_i(F)$ to $g_i(E)$ must start in the set

$$U_i \quad = \quad S^1_\infty(g_i(F)) \ - \ g_i(J_0).$$

From the point of view of $g_i(p_i)$ in $g_i(F)$, the visual measure of U_i is $2\pi - \theta_i$ which converges to 0. Also visually from $g_i(p_i)$ the set U_i is very close to the direction of the segment of $g_i(l)$ from $g_i(p_1)$ to $g_i(p_i)$. Because the directions of $g_i(l)$ converge to that of l_0 and the topology of $\mathcal{A}$ is given by the visual topology from transversals to $\widetilde{\mathcal{F}}$, it follows that the segments U_i converge to a unique point in $S^1_\infty(F_0)$.

Definition 3.23. *(function η) Define $\eta : \mathcal{O} \to \mathcal{A}$ by*

$$\eta(F_0) \ = \ \lim_{i\to\infty} U_i \ = \ \lim_{i\to\infty} \ (S^1_\infty(g_i(F) - g_i(J_0))$$

and for any covering translation f define $\eta(f(F_0)) = f(\eta(F_0))$.

The leaves F, E, F_0 of $\widetilde{\mathcal{F}}$ as well as the covering translations g_i will be fixed in this proof.

Lemma 3.24. *The function η from $\mathcal{O}$ to $\mathcal{A}$ extends to an embedding $\eta : \mathcal{H} \cong$* **R** *$\to \mathcal{A}$.*

Proof. Roughly the idea is: if the lemma is not true we produce spaced enough leaves A, B in $\widetilde{\mathcal{F}}$ with no markers between $S^1_\infty(A)$ and $S^1_\infty(B)$, contradiction.

Suppose the lemma is not true. There is L in $\widetilde{\mathcal{F}}$ and two sequences L_j, H_j converging to L with $\eta(L_j)$ converging to a, $\eta(H_j)$ converging to b, with a, b distinct points in $S^1_\infty(L)$. The L_j, H_j are covering translates of F_0:

$$L_j \ = \ f_j(F_0), \ \ H_j \ = \ h_j(F_0) \quad \text{for chosen } \ f_j, h_j \ \in \ \pi_1(M).$$

Choose small disjoint open neighborhoods V_a, V_b of a, b respectively in $\mathcal{A}$. Lemma 3.14 shows that there are small disjoint open neighborhoods V'_a, V'_b of a, b respectively in $\mathcal{A}$, and a small neighborhood Y of L in $\mathcal{H}$ defined by transversal μ, so that any marker through V'_a and contained in $\mathcal{A}_\mu$ is contained in V_a and similarly any marker intersecting V'_b is contained in V_b. In particular the two sets of markers contained in $\mathcal{A}_\mu$ through V'_a and V'_b are disjoint from each other. Lemma 3.8 implies that any marker in $\mathcal{A}$ through V'_a does not intersect V'_b.

Fix j big enough so that $\eta(L_j)$ is in V'_a and $\eta(H_j)$ is in V'_b. As $L_j = f_j(F_0)$ then

$$\eta(L_j) \;=\; f_j(\eta(F_0)) \;=\; f_j(\lim_{i\to\infty} U_i) = \lim_{i\to\infty} f_j(U_i) \;\in\; V'_a.$$

Similarly $\eta(H_j)$ is the limit of $h_j(U_i)$ with i converging to infinity. Now fix i big enough so that $f_j(U_i)$ is contained in V'_a and $h_j(U_i)$ is contained in V'_b. By the property of V'_a and V'_b, this implies that any marker through $f_j(U_i)$ is disjoint from a marker through $h_j(U_i)$, see fig. **??**.

Choose A in $\widetilde{\mathcal{F}}$ with A less than L in the linear ordering of $\mathcal{H}$. By taking j, i big enough we can assume that L_j, H_j are bigger than A and so are

$$f_j(g_i(F)), \quad h_j(g_i(F)).$$

Also choose B in $\widetilde{\mathcal{F}}$ with B bigger than both $f_j(g_i(E))$ and $h_j(g_i(E))$ in $\mathcal{H}$. A marker from $S^1_\infty(A)$ to $S^1_\infty(B)$ has to pass through $S^1_\infty(f_j(g_i(F)))$ and through $S^1_\infty(f_j(g_i(E)))$, since the leaves $f_j(g_i(F))$ and $f_j(g_i(E))$ separate A from B. By the property of U_i, it follows that the marker has to pass through $f_j(U_i)$ contained in V'_a. Similarly any such marker has to pass through $S^1_\infty(h_j(g_i(F)))$ and $S^1_\infty(h_j(g_i(E)))$ hence it has to pass through $h_j(U_i)$ contained in V'_b. But we just showed no marker can pass through both V'_a and V'_b. This would imply there is no marker from $S^1_\infty(A)$ to $S^1_\infty(B)$ which contradicts proposition 3.11. These arguments show that η can be extended to a continuous function from the closure of $\mathcal{O}$ to $\mathcal{A}$. But $\mathcal{F}$ is a minimal foliation so $\mathcal{O}$ is dense in $\mathcal{H}$ so there is a continuous extension $\eta : \mathcal{H} \cong \mathbf{R} \to \mathcal{A}$. The image is a curve $\mathcal{L}$ which is transverse to the horizontal foliation and intersects every circle at infinity. This finishes the proof of the lemma. □

Conclusion of the proof of proposition 3.22.

The set

$$\{\eta(f(F_0))\} \;\; f \;\in\; \pi_1(M)$$

is an equivariant subset of $\mathcal{A}$. By the previous lemma the curve $\mathcal{L}$ is left invariant by every covering homeomorphism - it is an invariant curve. Also given any covering translation g with an invariant leaf L $(g(L) = L)$, then one of the fixed points of g in $S^1_\infty(L)$ is in $\mathcal{L}$.

Let L in $\mathcal{O}$, $L = g(F_0)$. Then L is the limit of $g(g_i(F))$ as i converges to infinity. Any marker from

$$S^1_\infty(g(g_i(F)) \quad \text{to} \quad S^1_\infty(g(g_i(E))$$

has to start in in $g(U_i)$. Recall that $g(U_i)$ converges to $\eta(L)$ as i converges to infinity. Choose a collection of leaves G_k, R_k in $\widetilde{\mathcal{F}}$, escaping to opposite ends of $\mathcal{H}$ and G_k always smaller than R_k in $\mathcal{H}$. For each k choose a marker ζ_k from $S^1_\infty(G_k)$ to $S^1_\infty(R_k)$. Fix a neighborhood N of $\eta(L)$ in $\mathcal{A}$. Choose i big enough so that $g(U_i)$ is contained in N. For k big enough the leaves G_k, $g(g_i(F))$, $g(g_i(E))$ and R_k are linearly ordered in increasing order in $\mathcal{H}$, hence ζ_k has to pass through some point z_k in $g(U_i)$, so z_k is in N. Therefore

$$\text{for all } \; L \;\text{ in }\; \mathcal{O}, \quad \eta(L) \;=\; \lim_{k\to\infty} \; (\zeta_k \cap S^1_\infty(L)) \quad (*).$$

As $\mathcal{O}$ is dense in $\mathcal{H}$, and η is continuous in $\mathcal{H}$, lemma 3.14 implies that equation $(*)$ holds for any G in $\widetilde{\mathcal{F}}$. We conclude that $\mathcal{L}$ is the limit of the sequence of longer and longer markers ζ_k and $\mathcal{L}$ is a limit invariant curve.. But in that case proposition 3.22 implies that given any G, H leaves in $\widetilde{\mathcal{F}}$ and any u in $S^1_\infty(G) - \mathcal{L}_G$ then u is a contracting direction between G and H. This now contradicts the assumption in the proof of proposition 3.22 that there is not a dense set of contracting directions from F to E. So in any case for any G, H in $\widetilde{\mathcal{F}}$ there a dense set of contracting directions between G and H.

We now finish the proof of proposition 3.22. Given arbitrary G, H in $\widetilde{\mathcal{F}}$, the dense set of markers between $S^1_\infty(G)$ and $S^1_\infty(H)$, extends uniquely to a vertical foliation of the region of $\mathcal{A}$ between $S^1_\infty(G)$ and $S^1_\infty(H)$. This is because it is dense from the point of view of both G and H! In addition if G', G, H, H' are linearly ordered in $\mathcal{H}$ and one does the same operation using G', H', the resulting foliation is an extension of the foliation between $S^1_\infty(H)$ and $S^1_\infty(G)$. This is because markers from $S^1_\infty(G')$ to $S^1_\infty(H')$ produce markers from $S^1_\infty(G)$ to $S^1_\infty(H)$ and there is a unique extension of the foliation to the bigger annulus. Consequently there is a well defined vertical foliation in $\mathcal{A}$. Since the collection of markers in $\mathcal{A}$ is invariant under covering homeomorphisms, the vertical foliation also is and is a natural foliation associated to $\mathcal{F}$. This finishes the construction of the vertical foliation in the non uniform case. $\square$

4 The uniformly quasisymmetric case

As shown in the previous section, if $\mathcal{F}$ is $\mathbf{R}$-covered, with hyperbolic leaves, then both in the uniform or non uniform cases there is a vertical foliation in $\mathcal{A}$ which is equivariant. No need of transverse orientability for these results. The leaf space of the vertical foliation is a circle, which is the *universal circle* of the foliation as defined by Thurston [Th9, Th10] and is denoted by $\mathcal{U}$. For the arguments in this and the following sections, fix once and for all a leaf $F^* \in \widetilde{\mathcal{F}}$ and identify $\mathcal{U}$ to the circle at infinity $S^1_\infty(F^*)$ – for each point in $\mathcal{U}$ associate the intersection of the corresponding vertical leaf with $S^1_\infty(F^*)$. The leaf F^* is isometric to the hyperbolic plane $\mathbf{H}^2$ and we use the model of F^* as the unit disk in the plane, hence $S^1_\infty(F^*)$ homeomorphic to $\mathcal{U}$ is the unit circle $\mathbf{S}^1$ in the complex plane $\mathbf{C}$.

Notation: If g is in $\pi_1(M)$, let $\theta(g)$ denote the induced homeomorphism of $\mathcal{U} \cong \mathbf{S}^1$.

The transverse change in geometry of leaves of $\widetilde{\mathcal{F}}$ is encoded by how the hyperbolic metrics vary from leaf to leaf. The distortion can also be measured in the ideal circles in the following way: We say that a homeomorphism of $\mathcal{U} \cong \mathbf{S}^1$ is *Möebius* if it continuously extends to an isometry of $\mathbf{H}^2$. Since hyperbolic isometries act freely and transitively on triples of distinct points in $\mathbf{S}^1$, one cannot verify directly whether f in $Homeo(\mathbf{S}^1)$ is Möebius by looking at the action on triples of points. However one can do that by considering the action on quadruples of points. Given 4 distinct points $Z = \{z_1, z_2, z_3, z_4\}$ in $\mathbf{S}^1$, that follow each other in the positive counterclockwise direction, recall that the *cross ratio* of the set is

$$C(Z) \quad = \quad \frac{z_4 - z_1}{z_4 - z_2} . \frac{z_3 - z_2}{z_3 - z_1}.$$

Then $C(Z)$ is always real and in $(0,1)$. The 4 points in Z define an unique *ideal quadrilateral* in $\mathbf{H}^2$ with ideal points in Z, which is regular if and only if $C(Z)$ is equal to $1/2$. For any homeomorphism f of $\mathcal{U}$ let $fZ = \{f(z_1), f(z_2), f(z_3), f(z_4)\}$. Let K bigger than 1. Then f is said to be K*-quasisymmetric* if

$$(2K)^{-1} \ \leq \ C(fZ) \ \leq \ 1 - (2K)^{-1},$$

whenever $C(Z) = 1/2$ [Hi1, Hi2]. This means regular quadrilaterals do not get too distorted. The notation is f is K-qs [Hi1]. It is easy to see that f is Möebius if and only if $C(Z) = C(fZ)$ for all sets of 4 distinct points – equivalently f is 1-qs. This definition is the analogue in dimension 1 of the concept of a quasiconformal map in a complex domain of dimension ≥ 2.

There is a rich theory of quasisymmetric maps [Hi1, Hi2, Le]. A group Γ acting on $\mathbf{S}^1$ is *uniformly quasisymmetric* if there is K so that for any $f \in \Gamma$, then f is a K-quasisymmetric homeomorphism of $\mathbf{S}^1$. We denote this by Γ is K-qs [Hi1]. In this section we deal with following the situation:

Case 1 – $\pi_1(M)$ acts on $\mathcal{U}$ as a uniformly quasisymmetric group.

This is the rigid case and it implies that the action is always topologically conjugate to a Möebius action in $\mathbf{S}^1$: there is f in $Homeo(S^1)$ so that for every g in $\pi_1(M)$, $f \circ \theta(g) \circ f^{-1}$ is Möebius on $\mathbf{S}^1$. This has already been done in the literature using works of various authors. We just outline the possibilities.

Suppose first that $\pi_1(M)$ acts a non discrete group on $\mathcal{U}$. Given that $\pi_1(M)$ acts a K-qs group, it was proved by Hinkkanen in chapter 9 of [Hi1] that the action of $\pi_1(M)$ is conjugate to a Möebius group.

Suppose now that $\pi_1(M)$ acts as a discrete group of homeomorphisms of $\mathcal{U}$. This means that given a sequence $g_n, n \in \mathbf{N}$ in $\pi_1(M)$ with $f_n = \theta(g_n)$ and f_n converging to the identity then f_n is the identity for n big enough. The idea is to first prove that $\pi_1(M)$ is a *convergence group*: that is, $\pi_1(M)$ acts discreetly in the triple space which is the set of triples $\{a, b, c\}$ in $\mathbf{S}^1 \times \mathbf{S}^1 \times \mathbf{S}^1$ with a, b, c distinct [Ge-Ma, Ga4]. In general the convergence group property is stronger than discreteness of the group, but not in the case of uniformly quasisymmetric groups. This is because there is a normal property associated to K-qs groups: Let g_n in $\pi_1(M)$, with $f_n = \theta(g_n)$ and suppose there are a, b, c distinct in $\mathcal{U}$ so that

$$f_n(a), \ f_n(b), \ f_n(c)$$

converge to 3 distinct points. It follows that there is a subsequence which converges uniformly to a K-qs homeomorphism. Gehring and Martin [Ge-Ma] do this is detail for uniformly quasiconformal maps in higher dimension and the result for K-qs groups acting in the circle is mentioned by Hinkkanen in [Hi1], page 62, even though the proof is not written down there. The key ideas are well known, for instance: choose z in $\mathcal{U}$ so that $Z = \{a, b, c, z\}$ form the vertices of an ideal quadrilateral in F^*. The key fact is that $C(f_n(Z))$ is bounded away from 0 and 1 so that the quadrilaterals associated to these points are never too thin (meaning that two opposite sides of the quadrilateral do not have points very close to each other). Hence there is a subsequence f_{n_i} with $f_{n_i}(z)$ converging to w different from the limits of $f_n(a), f_n(b), f_n(c)$. Starting with the initial ideal triangle in F^* with ideal points a, b, c we can tesselate $F^* \cong \mathbf{H}^2$ with ideal triangles so that any two adjacent ones form a regular ideal quadrilateral. Then as above there is a subsequence f_{n_i} which converges in all the ideal points of the triangles, hence in a dense set of the circle $\mathcal{U}$. Again using the K-qs property of the action of $\pi_1(M)$ one shows that the limit map extends to a continuous map h from $\mathcal{U}$ to $\mathbf{S}^1$, which is a homeomorphism and the convergence is in fact uniform. Also the inverses converge to h^{-1}. This implies that the compositions $f^{-1}_{n_{i+1}} \circ f_{n_i}$ converge uniformly to the identity. As the group is discrete f_{n_i} are all equal for i sufficiently big. This shows that $\pi_1(M)$ acts discretly in the triple space and is a convergence group. Fundamental work of Tukia [Tu], Gabai [Ga4] and Casson-Jungreis [Ca-Ju] then implies that $\pi_1(M)$ is conjugate to a Möebius group.

Let f in $Homeo(\mathbf{S}^1)$ be the conjugating homeomorphism. We prove a rigidity result. First produce a transversely hyperbolic [Th3, Ep] 1-dimensional foliation in a manifold M' as follows:

Identify the leaf space $\mathcal{H}$ to $\mathbf{R}$ and parametrize it as $\{F_t\}$ with t a real number and $F_0 = F^*$. Given g in $\pi_1(M)$, let g^* be the induced homeomorphism of $\mathcal{H}$. Let $\widetilde{M}'$ be the product $\mathbf{H}^2 \times \mathbf{R}$. Let $\pi_1(M)$ act on $\widetilde{M}'$ as follows. By hypothesis for any g in $\pi_1(M)$, the homeomorphism $f \circ \theta(g) \circ f^{-1}$ is Möebius and it extends to an isometry of $\mathbf{H}^2$, still denoted by $f \circ \theta(g) \circ f^{-1}$. Define the action on $\widetilde{M}'$ by

$$\overline{g}(u, t) \quad = \quad (f \circ \theta(g) \circ f^{-1}(u), \ g^*(t)), \qquad u \text{ in } \mathbf{H}^2, \ t \text{ in } \mathbf{R}.$$

We analyse properties of this action to prove the rigidity result.

<u>Claim 1</u> $-$ The action is free.

Suppose there is g in $\pi_1(M)$ and (u, t) in $\widetilde{M}'$ with $\overline{g}$ fixing (u, t). Then g^* fixes t so g leaves F_t invariant. If g is not the identity in F_t, then g is a non trivial isometry in F_t which must be of hyperbolic type. Hence g acts on $S^1_\infty(F_t)$ with two fixed points, one contracting one expanding and the same is true for the action of $\theta(g)$ on $\mathcal{U}$ and the action of $f \circ \theta(g) \circ f^{-1}$ in $\mathbf{S}^1$. As the extension of $f \circ \theta(g) \circ f^{-1}$ to $\mathbf{H}^2$ is an isometry it has to have hyperbolic type and has no fixed points in $\mathbf{H}^2$. This contradiction shows that g acts as the identity in F_t and hence g is the identity. This proves claim 1.

At this point we need the following simple but extremely useful continuity property which relates curves in $\mathcal{A}$ with geodesics in leaves of $\widetilde{\mathcal{F}}$. We establish notation which will be used often: if a, b are two ideal points of a leaf L of $\widetilde{\mathcal{F}}$, let $\overline{ab}$ be the geodesic in L defined by the ideal points a, b if they are different and let this be the emptyset if a, b are equal. We show a basic continuity property of geodesics:

Lemma 4.1. *Let L_i, i in $\mathbf{N}$ be a sequence of leaves of $\widetilde{\mathcal{F}}$ converging to L_0. Let p_i, q_i distinct points in $S^1_\infty(L_i)$ with p_i converging to p_0, q_i converging to q_0 in $\mathcal{A}$. Then $\overline{p_iq_i}$ converges in $\widetilde{M}$ to $\overline{p_0q_0}$.*

Proof. First suppose that p_0, q_0 are equal. Let α be a transversal to $\widetilde{\mathcal{F}}$ through x_0 in L_0, which intersects L_i in a point x_i. Identify the unit tanget bundle to the leaves $T^1_{\widetilde{\mathcal{F}}}|\alpha$ to the union of circles at infinity near $S^1_\infty(L_0)$. From the point of view of α, the visual angle seen by $\overline{p_iq_i}$ is converging to 0, because p_i, q_i are both converging to p_0. Hence $d_{L_i}(x_i, \overline{p_iq_i})$ converges to infinity. As $\mathcal{F}$ is $\mathbf{R}$-covered $d(x_i, \overline{p_iq_i})$ also converges to infinity. Therefore the geodesics $\overline{p_iq_i}$ escape in $\widetilde{M}$ and have no limit point in L_0.

Suppose now that p_0, q_0 are distinct points. With the notation as in option 1, the visual angle of $\overline{p_iq_i}$ as seen from x_i converges to the visual angle of $\overline{p_0q_0}$ as seen from x_0 – this last one is not 0, so we may assume all of them are bounded away from 0. Hence the geodesics $\overline{p_iq_i}$ have points y_i a bounded distance from x_i. There is a subsequence y_{i_n} converging to y_0 which is in L_0 and so that directions at y_{i_n} also converge. Choose a transversal α' to $\widetilde{\mathcal{F}}$ through y_0 and containing y_{i_n}. From the new point of view still p_{i_n} converges to p_0 and q_{i_n} converges to q_0 in $\mathcal{A}$. Hence the two rays of $\overline{p_{i_n}q_{i_n}}$ defined by y_{i_n} converge to two rays in L_0 starting in y_0 and with ideal points p_0, q_0. In addition the angle between the rays of $\overline{p_{i_n}q_{i_n}}$ is always equal to π, hence so is the angle between the two rays in L_0 starting at y_0. This means that the union of these two rays in L_0 is a geodesic, which is none other than $\overline{p_0q_0}$. So $\overline{p_{i_n}q_{i_n}}$ converges to $\overline{p_0q_0}$. Since any such sequence has a convergent subsequence to $\overline{p_0q_0}$ this proves the lemma. ☐

It follows that if β, γ are continuous curves in $\mathcal{A}$ which are transverse to the horizontal foliation, then the geodesics in L defined by the intersections of β and γ with $S^1_\infty(L)$ vary continuously in $\widetilde{M}$ as L varies in $\widetilde{\mathcal{F}}$. In particular this occurs if β, γ are contained in leaves of the vertical foliation of $\mathcal{A}$.

Claim 2 – The action of $\pi_1(M)$ on $\widetilde{M}'$ is properly discontinuous.

Let C compact in $\widetilde{M}'$. Let g_i in $\pi_1(M)$ so that there are x_i in C with x_i also in $(\overline{g}_i)^{-1}(C)$. Let y_i be $\overline{g}_i(x_i)$. Up to subsequence x_i converges to x_0 and y_i converges to $y_0 = (u', t')$. Let

$$x_i \;=\; (u_i, t_i) \quad \text{with} \quad u_i \,\in\, \mathbf{H}^2, \;\; t_i \,\in \mathbf{R}, \;\; u_i \,\to\, u_0, t_i \,\to\, t_0.$$

Choose triples of points (z_i, w_i, v_i) in $\partial \mathbf{H}^2 = \mathbf{S}^1$ with u_i the barycenter of the ideal triangle in $\mathbf{H}^2$ defined by these 3 points. Assume that z_i converges to z_0 which implies w_i, v_i also converge to distinct points w_0, v_0. Up to another subsequence $f \circ \theta(g_i) \circ f^{-1}(z_i)$ converges to z'_0, so also $f \circ \theta(g_i) \circ f^{-1}(w_i)$ converges to w'_0 and $f \circ \theta(g_i) \circ f^{-1}(v_i)$ converges to v'_0, distinct in $\mathbf{S}^1$. Hence y_i is equal to (s_i, r_i) with s_i converging to the barycenter of the triangle defined by z'_0, w'_0, v'_0. Using the conjugacy by f, it follows that in $\mathcal{U}$ the sequences

$$f^{-1}(z_i), \;\; f^{-1}(w_i), \;\; f^{-1}(v_i) \quad \text{converge to distinct points} \quad f^{-1}(z_0), \;\; f^{-1}(w_0), \;\; f^{-1}(v_0)$$

respectively. Let a_i in F_{t_i} which are the barycenters of the ideal triangles in F_{t_i} defined by the points b^1_i, b^2_i, b^3_i in $S^1_\infty(F_{t_i})$ associated to $f^{-1}(w_i), f^{-1}(z_i), f^{-1}(v_i)$ of $\mathcal{U}$. As these points converge to 3 distict points in $\mathcal{A}$, the lemma above implies that $\overline{b^1_i b^2_i}$, $\overline{b^1_i b^3_i}$, $\overline{b^2_i b^3_i}$ converge to geodesics in the limit leaf. The associated barycenters a_i also converge to the barycenter a_0 in F_{t_0} of the limit ideal triangle. But

$$\theta(g_i)(f^{-1}(z_i)) \;=\; f^{-1}(f \circ \theta(g_i) \circ f^{-1}(z_i)) \;\to\; f^{-1}(z'_0) \;\text{ in }\; \mathcal{U},$$

similarly for w_i, v_i. The $g_i(a_i)$ are barycenters of ideal triangles in $F_{g^*_i(t_i)}$ with ideal points associated to

$$\theta(g_i)(f^{-1}(z_i)), \;\; \theta(g_i)(f^{-1}(w_i)), \;\; \theta(g_i)(f^{-1}(v_i))$$

and they converge to the barycenter c_0 of the ideal triangle in F_{t_0} defined by $f^{-1}(z'_0)$, $f^{-1}(w'_0)$, $f^{-1}(v'_0)$. That means $g_i(a_i)$ converges to c_0. As the action of $\pi_1(M)$ on $\widetilde{M}$ is properly discontinuous there are only finitely many distinct g_i. This proves claim 2.

Let Γ be the action of $\pi_1(M)$ induced in $\widetilde{M}'$. Claims 1) and 2) imply that $M' = \widetilde{M}'/\Gamma$ is a manifold.

Claim 3 – M' is a compact manifold.

Let x_i in M' and lift to $\overline{x}_i$ in $\widetilde{M}'$. Similarly to arguments in claim 2, find associated points a_i in $\widetilde{M}$. Up to subsequence there are covering translates $g_i(a_i)$ converging in $\widetilde{M}$. Again similarly to claim 2 show that $\overline{g}_i(\overline{x}_i)$ is in a compact subset of $\widetilde{M}'$, implying compactness of M'. We leave the details to the reader.

Notice that M is homotopy equivalent to M'.

There are two product foliations in $\widetilde{M}'$ one by leaves $\mathbf{H}^2 \times \{t\}$ and another by vertical lines $\{x\} \times \mathbf{R}$. Both of these foliations are invariant by the action of Γ producing two transverse foliations in M'. The two dimensional foliation implies that M' is irreducible. The foliation by vertical lines $\{x\} \times \mathbf{R}$ induces a 1-dim $\mathcal{V}$ foliation in M' which is transversely hyperbolic: there is a transversal $\mathbf{H}^2$ structure which is preserved by holonomy [Th3, Ep]. Under these circumstances Thurston [Th3, Ep] showed that either

1) M' is a Seifert fibered manifold with $\mathcal{V}$ a Seifert fibration, or

2) M' is a torus bundle over $\mathbf{S}^1$ with Anosov monodromy and $\mathcal{V}$ is (say) the strong unstable foliation of the corresponding suspension Anosov flow.

In case 2) M' is Haken and as M is homotopy equivalent to M', then M is in fact homeomorphic to M' [He, Wa]. In case 1) M' is homotopy equivalent to a Seifert fibered space and since M is irreducible, Scott [Sc] proved that M is homeomorphic to M'.

Using averaging techniques one can show that $\mathcal{F}$ is topologically conjugate to $\mathcal{F}'$. These techniques have been used for instance by Ghys [Gh] and others in the one dimensional setup. The two dimensional case is more involved and for brevity we only do the following: In situation 1) M is a Seifert fibered space, and work of Brittenham [Br] (see also [Th1]), implies that $\mathcal{F}$ is either vertical (a union of circle fibers in the Seifert fibration) or horizontal (transverse to the circle fibers). The first option cannot occur because the leaves of $\mathcal{F}$ are hyperbolic. Hence $\mathcal{F}$ can be put transverse to the Seifert fibration and by careful choices, the transversal flow in M lifts to a flow in $\widetilde{M}$ which produces global homeomorphims between leaves of $\widetilde{\mathcal{F}}$ which are isometries of the hyperbolic metric. No change in geometry! In any case it is easy to see that in this case there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$.

In case 2), M fibers over $\mathbf{S}^1$ with Anosov monodromy. One can put the incompressible torus T transverse to $\mathcal{F}$ [Rou, Th1, Ga5] and hence there is an induced foliation in T. This foliation is invariant by the monodromy of the fibration and hence has to be the stable or unstable foliation of the monodromy. Hence $\mathcal{F}$ is conjugate to (say) the (weak) stable foliation of the associated Anosov flow and the transversal flow can be chosen to be the strong unstable foliation of this flow. As in case 1) no transversal change of the leaf metrics. In any case there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$.

This finishes the proof of the main theorem in the case $\pi_1(M)$ acts by uniformly quasisymmetric homemorphisms of $\mathcal{U}$.

5 The non uniformly quasisymmetric case

For the rest of the article we we analyse the following situation:

Case 2 – The action of $\pi_1(M)$ on $\mathcal{U}$ is not uniformly quasisymmetric.

Theorem 5.1. *If the action of $\pi_1(M)$ on $\mathcal{U}$ is not uniformly quasisymmetric and $\mathcal{F}$ is transversely orientable, then there is a lamination $\mathcal{G}$ transverse to $\mathcal{F}$ intersecting leaves of $\mathcal{F}$ in geodesics.*

Proof. The goal of this section is to prove this theorem. Thurston [Th9, Th10] announced a very similar result with an additional possibility in the conclusion: a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$. He explained to us the main steps of his proof [Th9]. We extend Thurston's result by always obtaining a transverse lamination. The detailed constructions in this section are essential for the results in the next section.

Roughly the proof goes like this: using the hypothesis on the action we produce ideal quadrilaterals in leaves of $\widetilde{\mathcal{F}}$ which get arbitrarily distorted. They shrink to geodesics in leaves of the foliation. Using the universal circle one can sweep these geodesics across the foliation to produce an immersed lamination transverse to $\mathcal{F}$. The difficult part will be to show they are embedded.

We use the same notation as in the previous section: the universal circle $\mathcal{U}$ is identified to a circle $\mathbf{S}^1$ and also to the circle at infinity $S^1_\infty(F^*)$ of a fixed leaf F^* of $\widetilde{\mathcal{F}}$, where F^* is identified to $\mathbf{H}^2$. Given a covering translation g, then $\theta(g)$ denotes the induced homeomorphism in $\mathcal{U}$ (or in $S^1_\infty(F^*)$). The set $\mathcal{H}$ (the leaf space of $\widetilde{\mathcal{F}}$) is parametrized as $\{F_t\}$ with t in $\mathbf{R}$. The proof is divided into several steps.

<u>**Step 1**</u> - Constructing ideal quadrilaterals which get stretched in opposite directions.

By hypothesis there are quadruples Z_i of points in $\mathcal{U}$ with $C(Z_i) = 1/2$ and g_i in $\pi_1(M)$ with the cross ratio $C(\theta(g_i)(Z_i))$ arbitrarily close to 0 or 1. There is a way to produce the transverse laminations to $\mathcal{F}$ using earthquake maps on the hyperbolic plane [Th6] as explained by Thurston [Th9]. Here we use simple properties of the cross ratio to obtain the laminations.

If $Z = \{z_1, z_2, z_3, z_4\}$ is a positively oriented quadruple of points in $\mathbf{S}^1$, then $C(Z)$ is very near 0 or 1 if and only if the ideal quadrilateral in $\mathbf{H}^2$ associated to it is very thin: there are two opposite sides of the quadrilateral which are very close to each other in the hyperbolic metric. This obviously implies that the other two opposite sides are very far from each other and is equivalent to it. Given an ideal quadrilateral we define the *waist* to be the minimum distance between opposite sides. Using the formula for $C(Z)$ it is very easy to verify that $C(Z)$ is very near 0 if and only if the geodesic $\overline{z_1 z_2}$ of $\mathbf{H}^2$ defined by z_1, z_2 has a point very near $\overline{z_3 z_4}$ and that $C(Z)$ is very near 1 if and only if $\overline{z_2 z_3}$ has a point very near $\overline{z_4 z_1}$.

Definition 5.2. *Given a quadruple U of points in a circle at infinity $S^1_\infty(L)$ of a leaf L of $\widetilde{\mathcal{F}}$, let $\mathcal{W}(U)$ denote the ideal quadrilateral in L with endpoints in U.*

Let Z be a quadruple in $S^1_\infty(F^*)$. For g in $\pi_1(M)$ the map $\theta(g)$ acts on $\mathcal{U}$ (identified to $\mathbf{S}^1$). This defines an action on geodesics and ideal quadrilaterals of F^*, for simplicity of notation also denoted by $\theta(g)$. In particular

$$\theta(g)(\mathcal{W}(Z)) \ = \ \mathcal{W}(\theta(g)(Z)).$$

We stress that

$$\theta(g)(\mathcal{W}(Z)) \ \subset \ F^*, \ \ \text{usually not isometric to} \ \ \mathcal{W}(Z),$$

$$\text{whereas} \quad g(\mathcal{W}(Z)) \ \subset \ g(F^*) \quad \text{is always isometric to} \ \ \mathcal{W}(Z).$$

First check the action in F^*. Let Z_i be a sequence of quadruples in $\mathcal{U}$ and g_i in $\pi_1(M)$ with $C(Z_i) = 1/2$ (that is, the $\mathcal{W}(Z_i)$ are regular ideal quadrilaterals in $F^* = \mathbf{H}^2$) but the cross ratios $C(\theta(g_i)(Z_i))$ converge to either 0 or 1. Circularly rename the points in Z_i so that these cross ratios converge to 0.

Lemma 5.3. *There are ideal quadrilaterals $\mathcal{C}_i$ in $\mathbf{H}^2$ defined by quadruples Y_i in $\mathcal{U}$ and covering translations g_i with $C(Y_i)$ converging to 0, but $C(\theta(g_i))(Y_i))$ converging to 1.*

Proof. This means that with the ordering in the quadruples, the map $\theta(h_i)$ sends quadrilaterals very thin in one direction ($C(Z)$ near 0) to quadrilaterals very thin in the other direction ($C(Z)$ near 1). Given n in $\mathbf{N}$ and any waist size b_0 sufficiently small, there is a waist size b_1 much smaller than b_0 so that any quadrilateral of waist b_1 can be covered by n quadrilaterals

$$\{\mathcal{E}_m\}, \quad 1 \leq m \leq n,$$

of waist smaller than b_0 (the associated cross ratio is very close to 0) satisfying: the interiors of the $\mathcal{E}_m$ are disjoint from each other, consecutive quadrilaterals are adjacent, see fig. ??, a.

Now if $\mathcal{W}(\theta(g_i)(Z_i))$ has waist size less than b_1, cover it by quadrilaterals $\{\mathcal{E}_m\}$ with m in $[1,n]$ all of waist size smaller than b_0 as above, see fig. ??, a. Let $\theta(g_i^{-1})$ act on this. The union of

$$\theta(g_i)^{-1}(\mathcal{E}_m)$$

will cover the regular quadrilateral $\mathcal{W}(Z_i)$. Since the quadrilaterals $\theta(g_i)^{-1}(\mathcal{E}_m)$ are restricted in one direction to be inside the regular quadrilateral $\mathcal{W}(Z_i)$ (see fig. ??, b) then: If n is sufficiently big, at least one of the quadrilaterals $\theta(g_i)^{-1}(\mathcal{E}_m)$ is very thin in the other direction, that is the associated cross ratio is very close to 1. Let $\mathcal{C}_i$ be one such quadrilateral $\mathcal{E}_m$. This finishes the proof of lemma 5.3. $\square$

Definition 5.4. *If l is a geodesic in a leaf a leaf F of $\widetilde{\mathcal{F}}$, it has two ideal points in $S^1_\infty(F)$ and therefore two distinct points in $\mathcal{U}$. The set $l \times \mathbf{R}$ consists of the union of the geodesics in leaves of $\widetilde{\mathcal{F}}$ associated to the same points in $\mathcal{U}$ defined by l. The curves in $\mathcal{A}$ defined by each point in $\mathcal{U}$ are continuous hence the set $l \times \mathbf{R}$ is a topological plane which is properbly embedded in $\widetilde{M}$. In the same way if V is a convex set in a leaf of $\widetilde{\mathcal{F}}$ bounded by geodesics s_i, one forms $s_i \times \mathbf{R}$ and jointly they bound the set $V \times \mathbf{R}$.*

Step 2 - The distortion parallelepipeds.

We will use the thin quadrilaterals

$$\mathcal{W}(Y_i), \quad \mathcal{W}(\theta(g_i)(Y_i))$$

of the previous step to produce immersed transverse laminations to $\mathcal{F}$. First construct a distorted ideal parallelepiped in $\widetilde{M}$ as follows. For simplicity suppose that F^* is in the back of $g_i^{-1}(F^*)$. Notice that $\theta(g_i)(Y_i)$ defines an ideal quadrilateral in F^* with cross ratio very close to 1. Since g_i acts as isometries between leaves of $\widetilde{\mathcal{F}}$, then

$$\mathcal{Z}_i \;=\; g_i^{-1}(\mathcal{W}(\theta(g_i)(Y_i))) \quad \subset \quad g_i^{-1}(F^*)$$

is isometric to $\mathcal{W}(\theta(g_i)(Y_i))$ and has cross ratio very close to 1. The ideal points of

$$g_i^{-1}(\mathcal{W}(\theta(g_i)(Y_i))) \quad \text{in} \quad S^1_\infty(g_i^{-1}(F^*))$$

define the same points b_1, b_2, b_3, b_4 in $\mathcal{U}$ that Y_i does. For each F in $\mathcal{H}$ between F^* and $g_i^{-1}(F^*)$ form the ideal quadrilateral with the ideal points corresponding to b_1, b_2, b_3, b_4 in $\mathcal{U}$. The 4 curves in $\mathcal{A}$ defined by these points in $\mathcal{U}$ are continuous curves in $\mathcal{A}$, hence lemma 4.1 implies that the sides of the ideal quadrilaterals in F vary continuously with the leaf F. The union of these ideal quadrilaterals between F^* and $g_i^{-1}(F^*)$ is a *parallepiped* $\mathcal{P}_i$ in $\widetilde{M}$, see fig. ??. That is

$$\mathcal{P}_i \;=\; \bigcup \;\{\; (\mathcal{W}(Y_i) \times \mathbf{R}) \cap F \;\;|\;\; F \in [F^*, g^{-1}(F^*)] \;\}.$$

The bottom of $\mathcal{P}_i$ is the quadrilateral $\mathcal{W}(Y_i)$ in F^*, the top is the quadrilateral $\mathcal{Z}_i$ in $g_i^{-1}(F^*)$ and there are 4 sides which are transverse to $\widetilde{\mathcal{F}}$, which are $\overline{b_1 b_2} \times \mathbf{R}$ and so on. The tops of the parallelepipeds will shrink to geodesics producing one lamination and likewise for the bottoms. We will change the parallelepipeds $\mathcal{P}_i$ in the next step.

Remark: Lemma 4.1 implies that for any geodesic α in a leaf of $\widetilde{\mathcal{F}}$, then the geodesics in $\alpha \times \mathbf{R}$ vary continuously in $\widetilde{M}$. Hence $\alpha \times \mathbf{R}$ is an embedded plane in $\widetilde{M}$. It follows that all objects constructed here ($\mathcal{D}_- \times \mathbf{R}$, $\mathcal{D}_+ \times \mathbf{R}$, $\mathcal{G}$, $\mathcal{G}_-$, $\mathcal{G}_+$, etc..) are continuous.

Step 3 - Convergence of the bottoms of the parallelepipeds.

We use the distortion parallelepipeds $\mathcal{P}_i$ from step 2. We take limits: First project to leaves of $\mathcal{F}$ in M. The quadrilaterals $\mathcal{W}(Y_i)$ have associated cross ratios $C(Y_i)$ converging to 0. Let x_i in the boundary of the waist of $\mathcal{W}(Y_i)$. Up to subsequence assume that $\pi(x_i)$ converges to x_0 in M and the directions of the geodesic sides of $\pi(\mathcal{W}_i)$ at $\pi(x_i)$ also converge. In M the quadrilaterals $\pi(\mathcal{W}(Y_i))$ shrink to geodesics in leaves. Lift x_0 to $\widetilde{x}_0$ in $\widetilde{M}$ with the limit geodesic in the leaf of $\mathcal{F}$ lifting to a geodesic l_0 in a leaf F' of $\widetilde{\mathcal{F}}$ through $\widetilde{x}_0$. This geodesic defines two points in $\mathcal{U}$ and hence a geodesic l_1 in F^* – the geodesic l_1 is exactly $l_0 \times \mathbf{R}$ intersected with F^*. We define two important sets:

$$\mathcal{D}_- \;=\; \bigcup_{h \in \pi_1(M)} \{\theta(h)(l_1)\} \;\subset\; F^* \quad \text{and} \quad \mathcal{D}_- \times \mathbf{R} \;=\; \bigcup_{l' \in \mathcal{D}_-} (l' \times \mathbf{R}) \;\subset\; \widetilde{M}.$$

We now change the $\mathcal{P}_i$ so that bottoms converge to l_1. Up to covering translations we may assume that the bottoms $\mathcal{B}_i$ of $\mathcal{P}_i$ converge to the geodesic l_0 in a leaf F' of $\widetilde{\mathcal{F}}$. This changes the $\mathcal{P}_i$ so the bottoms are not in F^* anymore - we adjust that as follows: For F in $\widetilde{\mathcal{F}}$ between F^* and F' let

$$A_i(F) \;=\; (\mathcal{B}_i \times \mathbf{R}) \cap F.$$

These are ideal quadrilaterals in F. The ideal endpoints of $\mathcal{B}_i$ converge to the 2 ideal points of l_0 as i grows, so the ideal points of $A_i(F^*)$ collapse to the 2 ideal points of l_1 - because the leaves of the vertical foliation in $\mathcal{A}$ vary continuously. This produces a thin wall from F^* to F'. Since the horizontal quadrilaterals in $\mathcal{P}_i$ eventually have cross ratio close to 1 (when going up), the $\mathcal{P}_i$ extends beyond F'. We can extend or contract the parallelepipeds $\mathcal{P}_i$ so that the bottoms are now always in F^* and they converge to l_1.

Conclusion 1 - Up to subsequences, covering translations and extension or contractions we may assume that the parallelepipeds $\mathcal{P}_i$ have bottoms $\mathcal{B}_i$ which are ideal quadrilaterals in F^* with cross ratio converging to 0 and $\mathcal{B}_i$ converging to the geodesic l_1 of $\mathcal{D}_-$, so that the waists of $\mathcal{B}_i$ converge to a fixed point of l_1. The tops of $\mathcal{P}_i$ have cross ratio converging to 1.

Step 4 - Convergence of the tops of the parallelepipeds.

We want to do the same approach for the tops of $\mathcal{P}_i$. Since the bottoms of the $\mathcal{P}_i$ will stay in F^* clearly the tops cannot do the same. As in step 3, up to another subsequence the waists of $\pi(\mathcal{T}_i)$ converge to a point in M and so do the directions of the sides of $\pi(\mathcal{T}_i)$. Lifting to $\widetilde{M}$ this defines a geodesic in a leaf of $\widetilde{\mathcal{F}}$ and using the vertical foliation this defines a geodesic l_2 in F^*. Define

$$\mathcal{D}_+ \;=\; \bigcup_{h \in \pi_1(M)} \{\theta(h)(l_2)\} \;\subset\; F^* \quad \text{and} \quad \mathcal{D}_+ \times \mathbf{R} \;=\; \bigcup_{l' \in \mathcal{D}_+} (l' \times \mathbf{R}) \;\subset\; \widetilde{M}.$$

In the same way as above we can extend or contract the tops $\mathcal{P}_i$ so that:

Conclusion 2 - In addition to conclusion 1: There are covering translations g_i in $\pi_1(M)$ so that the tops $g_i(\mathcal{T}_i)$ are contained in F^* and converge to a geodesic l_2 of F^*. The waists of $g_i(\mathcal{T}_i)$ converge to a single point of l_2. Finally in the case there is a leaf of $\mathcal{D}_-$ transversely intersecting a leaf of $\mathcal{D}_+$ then up to renaming l_2 by covering translations we can assume that these are l_1 and l_2. If this does not happen but $\mathcal{D}_-$ and $\mathcal{D}_+$ share a leaf assume that l_1 is equal to l_2. Fix the $\mathcal{P}_i, \mathcal{B}_i, \mathcal{T}_i, g_i$ from now on.

Using covering translations one gets the same conclusions for any leaves of $\mathcal{D}_-$ and $\mathcal{D}_+$. In fact using a diagonal process on sequences the same is true for any limit of leaves of $\mathcal{D}_-$ or $\mathcal{D}_+$. We stress this:

Lemma 5.5. *Using covering translations, extensions/contractions of parallelepipeds and limits; the following happens: suppose that l is either the intersection of a leaf of $\mathcal{D}_- \times \mathbf{R}$ with a leaf F of $\widetilde{\mathcal{F}}$ (a geodesic in a leaf of $\widetilde{\mathcal{F}}$) or a limit of such intersections. Then there is a sequence $\mathcal{Q}_i$ of parallelepipeds so that 2* *opposite sides of bottoms converge to l and so that cross ratio of bottoms (respectively tops) converges to 0 (respectively 1). The same holds if l comes from $\mathcal{D}_+ \times \mathbf{R}$ in which case the tops converge to l.*

Step 5 - Producing the (a priori only immersed) laminations.

We will eventually prove (in the next section) that each of $\mathcal{D}_- \times \mathbf{R}$, $\mathcal{D}_+ \times \mathbf{R}$ does not have transverse self intersections. But the first step is to obtain some embedded lamination which may not be one of these two a priori. There are 3 cases to consider (we will keep coming back to these options in the next section):

Option A - No leaf of $\mathcal{D}_-$ transversely intersects another leaf of $\mathcal{D}_-$ (similarly for $\mathcal{D}_+$).

Then $\mathcal{D}_- \times \mathbf{R} \subset \widetilde{M}$ is a collection of properly embedded planes without any transverse intersections which is invariant under covering translations. Its closure is a $\pi_1(M)$ invariant lamination in $\widetilde{M}$ which intersects leaves of $\widetilde{\mathcal{F}}$ in a union of geodesics. The image in M is a lamination transverse to $\mathcal{F}$.

Option B - No leaf of $\mathcal{D}_-$ transversely intersects a leaf of $\mathcal{D}_+$.

If no two leaves of $\mathcal{D}_-$ intersect transversely, then as in option A, we produce a lamination in M transverse to $\mathcal{F}$. Otherwise by option B, no leaf of $\mathcal{D}_-$ is also a leaf of $\mathcal{D}_+$. Consider a connected component in F^* of the union of leaves in $\mathcal{D}_-$. Then the convex hull C of this set (in the hyperbolic metric of F^*) is not all of $\mathbf{H}^2$. Let B be the boundary of C. The translates of B under $\theta(g)$ with g in $\pi_1(M)$ do not intersect B transversely. Therefore $\pi(B \times \mathbf{R})$ produces a lamination as in option A above. Notice that in this case maybe all leaves of the lamination are neither in $\mathcal{D}_-$ nor $\mathcal{D}_+$.

Option C - There is a leaf of $\mathcal{D}_-$ transversely intersecting a leaf of $\mathcal{D}_+$.

This is the most interesting case. The rest of the proof of theorem 5.1 is devoted to an analysis of this case. By conclusion 2, here we can choose the l_1 in $\mathcal{D}_-$ and l_2 in $\mathcal{D}_+$ with a transverse intersection. The goal is to show that leaves of $\mathcal{D}_-$ do not intersect transversely and likewise for $\mathcal{D}_+$, that is, option C implies option A. We stress that options A and B can happen concurrently, but B and C are contradictory.

An important remark here is that in all options A, B, C, these laminations are obtained as a union of $r \times \mathbf{R}$ for a collection of geodesics r in F^*. If $r_1 \times \mathbf{R}$ intersects $r_2 \times \mathbf{R}$ then there is F in $\widetilde{\mathcal{F}}$ so that

$$u_1 = (r_1 \times \mathbf{R}) \cap F \quad \text{intersects} \quad u_2 = (r_2 \times \mathbf{R}) \cap F,$$

both geodesics in F. If $u_1 = u_2$, then $r_1 \times \mathbf{R}$ is equal to $r_2 \times \mathbf{R}$. In particular $r_1 \times \mathbf{R}$ cannot be tangent to $r_2 \times \mathbf{R}$ at level F and then cross from one side of $r_2 \times \mathbf{R}$ to the other when passing through F. If on the other hand u_1 and u_2 intersect transversely in F then $r_1 \times \mathbf{R}$ and $r_2 \times \mathbf{R}$ will have transverse intersection for all G in $\widetilde{\mathcal{F}}$. This is one big advantage of producing these laminations using the universal circle.

To prove that option C implies option A, then by way of contradiction suppose there is l_3 in $\mathcal{D}_-$ which transversely intersects l_1. There is a covering translation h with $\theta(h)(l_1) = l_3$. Fix h for the rest of the analysis of option C. It follows that $\theta(h^{-1})(l_1)$ also intersects l_1 transversely. We use the setup in much more detail in this case. Notice that the bottoms $\mathcal{B}_i$ of the parallelepipeds $\mathcal{P}_i$ converge to l_1, in fact two opposite sides of $\mathcal{B}_i$ do and likewise two opposite sides of $\theta(g_i)(\mathcal{T}_i)$ converge to l_2 of $\mathcal{D}_+$.

For the rest of the proof fix i' big enough so that: If p_1, p_2, p_3, p_4 in $\mathcal{U}$ are the ideal points of $\mathcal{B}_{i'}$ then $\overline{p_1p_2}$ and $\overline{p_3p_4}$ are very close to l_1 and so

$$\overline{p_1p_2}, \ \overline{p_3p_4} \quad \text{intersect} \quad l_2, \ \ l_3 = \theta(h)(l_1) \quad \text{and} \quad \theta(h^{-1})(l_1) \quad \text{transversely.}$$

Also $\theta(g_{i'})(\mathcal{T}_{i'})$ has two opposite sides very close to l_2. Let

$$g = g_{i'}^2\ , \quad h_1 = \theta(h), \quad g_1 = \theta(g) \quad \text{and} \quad q_j = g_1(p_j) = \theta(g)(p_j) \ \in \ \mathcal{U}.$$

For simplicity we omit the notational dependence of p_j, q_j, h, h_1, g, g_1 on the index i'.

Step 6 - Analysing the dynamics of $h_1 = \theta(h)$ and $g_1 = \theta(g)$ in $\mathcal{U}$.

First we explain why we consider g the square of $g_{i'}$ instead of just $g_{i'}$. Let $\mathcal{I}$ be the open interval of $\mathcal{U}$ defined by p_1, p_2 and not containing p_3 and similarly $\mathcal{J}$ defined by p_3, p_4 and not containing p_1. One possibility is that $\theta(g_{i'})(p_1)$ and $\theta(g_{i'})(p_2)$ are in $\mathcal{I}$. Another possibility is that the quadrilateral $\theta(g_{i'})(\mathcal{T}_{i'})$ is rotated 180 degrees, that is, $\theta(g_{i'})(p_1)$ and $\theta(g_{i'})(p_2)$ are in $\mathcal{J}$. In any case $\theta(g^2_{i'})(p_1)$ (equal to q_1) and q_2 are in $\mathcal{I}$ and q_1 is the one closest to p_1, see fig. **??**, a. Also

$$g_1(\mathcal{T}_{i'}) \ = \ \theta(g^2_{i'})(\mathcal{B}_{i'})$$

is even thinner than $\theta(g_{i'})(\mathcal{B}_{i'})$. Similarly q_3, q_4 are in $\mathcal{J}$ and q_3 is closest to p_3. The dynamics of g_1 in $\mathcal{U}$ is as follows: $g_1(cl(\mathcal{I}))$ is contained in $\mathcal{I}$ producing at least one fixed point in $\mathcal{I}$ and similarly $g_1(cl(\mathcal{J}))$ a subset of $\mathcal{J}$ yields a fixed point in $\mathcal{J}$, where here cl denotes the closure in $\mathcal{U}$. Similarly there are at least two fixed points outside $\{\mathcal{I} \cup \mathcal{J}\}$: one near p_1, p_2 and another near p_3, p_4. In any case g_1 has at least 4 fixed points in $\mathcal{U}$. It follows that g acts freely in $\mathcal{H}$: if $g(F) = F$ for some leaf F of $\widetilde{\mathcal{F}}$, then g fixes only two points in $S^1_\infty(F)$ and similarly for g_1 acting in $\mathcal{U}$.

We define 8 points in $\mathcal{U}$ from the dynamics of $\theta(g)$: Let

$$a_j \ = \lim_{n \to -\infty} g_1^n(p_j), \quad b_j \ = \lim_{n \to +\infty} g_1^n(p_j).$$

Notice b_1, b_2 are in $\mathcal{I}$; b_3, b_4 are in $\mathcal{J}$ and none of the a_j are in $\mathcal{I}$ or $\mathcal{J}$. Let

- $\gamma = \overline{a_1 a_2}$ which is very close to $\overline{p_1 p_2}$ and to l_1,
- $\beta_1 = \overline{b_1 b_4}$ and $\beta_2 = \overline{b_2 b_3}$, both both very close to $\overline{q_1 q_4}$.

We define 8 closed intervals in $\mathcal{U}$: first I_1 is the interval of $\mathcal{U}$ from a_4 to a_1 not containing the other points a_j, b_j, then similarly

$$I_2 : \text{from } a_1 \to b_1, \quad I_3 : b_1 \to b_2; \quad I_4 : b_2 \to a_2; \quad I_5 : a_2 \to a_3; \quad I_6 : a_3 \to b_3; \quad I_7 : b_3 \to b_4 \text{ and } I_8 : b_4 \to a_4,$$

see fig. **??**, b, Notice that $g_1(I_k) = I_k$ for all k. In addition g_1 acts as a homeomorphism with only two fixed points in I_2, the repelling is a_1 and the attracting is b_1. Similarly for I_4, I_6, I_8. Any of the intervals I_1, I_3, I_5, I_7 may be a single point $-$ that is, if $a_1 = a_4$ then I_1 is a single point. On the other hand none of I_2, I_4, I_6, I_8 is a single point.

One key point here is that by choice of i', none of the endpoints of $h_1(l_1) = l_3$ are in I_1 or in I_5 and we may assume the same happens for the endpoints of $h_1(\gamma)$ and for $h_1(p_j)$ (make i' bigger if necessary). Let

$$A \ = \ \gamma \times \mathbf{R}, \qquad B = h(A) = l_3 \times \mathbf{R}.$$

Then $g(A) = A$ and A is a properly embedded plane in $\widetilde{M}$. Also B transversely intersects A, because l_3 transversely intersects l_1. Using the dynamics of g_1 we show that the intersections of the surfaces $\pi(A), \pi(B)$ stay in compact parts of both $\pi(A)$ and $\pi(B)$ and derive a contradiction.

Step 7 - Analysing intersection of walls $A = \gamma \times \mathbf{R}$ and $B = h(A)$.

Consider $\beta_1 \times \mathbf{R}$ a properly embedded plane which intersects A in an infinite curve ζ_1. Then

$$g(\beta_1 \times \mathbf{R}) = \beta_1 \times \mathbf{R}, \quad g(A) = A, \quad \text{so} \quad g(\zeta_1) = \zeta_1.$$

Hence ζ_1 projects to a closed curve α_1 in the annulus A/g. Similarly β_2 produces a closed curve α_2 in A/g. Notice α_1 is equal to α_2 if and only if $\beta_1 = \beta_2$. Let N be the annulus (possibly degenerate) in A/g bounded by α_1 and α_2 and let $\widetilde{N}$ be its lift to A. Recall the parametrization of $\widetilde{\mathcal{F}}$ as $\{F_t\}$ with t in $\mathbf{R}$ and F^* equal to F_0. The curve

$$\mu \quad = \quad (B \cap A) \ \cap \ (\bigcup_{t \geq 0} F_t)$$

is an infinite curve of transverse intersection of A and B.

The geodesic l_3 of $\mathbf{H}^2$ has one endpoint c_1 in I_2, I_3 or I_4 and the other c_2 in I_6, I_7 or I_8. Suppose first the endpoints are in I_2 and I_8 respectively. Notice that $g_{i'}(\mathcal{T}_{i'})$ is contained in F^*, so g sends a leaf above F^* to F^*, or g^{-1} acts as an increasing homeomorphism in $\mathcal{H}$. Let

$$s \quad = \quad h_1(\gamma) \ \cap \ \gamma \quad = \quad (B \ \cap \ A) \ \cap \ F_0 \quad \in \ \mu.$$

We consider how the points μ move in A as t increases. The action by $g_1^{-1} \ = \ (\theta(g))^{-1}$ brings the endpoints of l_3 closer to I_1 because the endpoints of l_3 are in I_2 and I_8, see fig. **??**, b. Looking at the action g^{-1} in A, it sends s to a point with same distance from $\widetilde{N}$, see fig. **??**, a ($\widetilde{N}$ is invariant under g^{-1}). Hence $s_1 = \mu \cap g^{-1}(F^*)$ is closer to $\widetilde{N}$ than $g^{-1}(s)$ is, see fig. **??**, a. This means that going up brings the intersection of B and A closer to $\widetilde{N}$ in A. In the same way let

$$s_n \ = \ \mu \cap g^{-n}(F^*)$$

Then $g^n(s_n)$ is in a geodesic in F^* with endpoints $\theta(g^n)(c_1)$ and $\theta(g^n)(c_2)$. But

$$\theta(g^n)(c_1) \ \rightarrow \ b_1, \quad \theta(g^n)(c_2) \ \rightarrow \ b_4 \quad \text{when} \ \ n \rightarrow +\infty.$$

As β_1 has ideal points b_1, b_4, the above convergence implies that μ is actually asymptotic to ζ_1 going up. Hence μ projects in A/g to a curve asymptotic to the closed curve ζ_1/g see fig. **??**, b.

Let α be the projection of ζ_1 to M, that is, $\alpha = \pi(\zeta_1)$. Since ζ_1 is invariant under g, it follows that α is a closed curve in M and since ζ_1/g is already closed it follows that

$$\alpha \ = \ g^n, \quad \text{for some} \ \ n \neq 0.$$

This means that the curve α represents the element g^n in the fundamental group.

Now reverse the roles of A and B. The points in $\mathcal{U} - \{I_1 \cup I_5\}$ get contracted towards I_3 and I_7 under the action of $g_1 = \theta(g)$. Let

$$\delta \ = \ hgh^{-1}.$$

Notice that the points a_j are not in the union of $h_1(I_1)$ and $h_1(I_5)$, by choice of i'. Hence the a_j are in the regions of $\mathcal{U}$ which get contracted by the action of $\theta(\delta)$ towards $h_1(I_3)$ and $h_1(I_7)$. From the point of view of B the same arguments as above show that the intersection of B with A going up (in the positive direction) is also trapped closer to a band of B unvariant under δ. Here we use the fact that $\mathcal{F}$ is transversely orientable - h preserves orientation in $\mathcal{H}$, so going up in A (action by g^{-1}) corresponds to going up in B (action by δ^{-1}) as well. An argument as the one done in for the curve μ as seen in A shows that there is a curve ζ_2 in this band invariant under δ and so that μ is asymptotic to ζ_2 in the positive direction. Let α_1 be the projection of ζ_2 in M. Similarly as above one shows that

$$\delta = hgh^{-1} = \alpha_1^m \quad \text{for some} \ \ m \neq 0.$$

Now μ is asymptotic to ζ_1 and to ζ_2 both of which project to closed curves in M. Therefore α is equal to α_1. Since the intersection of B and A is a single curve, then in fact ζ_1 is equal to ζ_2.

Step 8 - Incompatible actions in $\mathcal{U}$.

In the previous step we proved that

$$g = \alpha^n \quad \text{and also} \ \ hgh^{-1} = \alpha^m, \quad n, m \ \neq \ 0.$$

It follows that

$$g^m = \alpha^{nm} = (hgh^{-1})^n = hg^nh^{-1}.$$

This is obtained when l_3 has endpoints in I_2, I_8. The other cases are similar. An argument as above shows that the curve μ when viewed in A is always asymptotic to a curve ζ which is invariant under g. The curve ζ is obtained as the intersection

$$\zeta = (l \times \mathbf{R}) \cap A,$$

where l is a geodesic in F^* with endpoints

$$a = \lim_{i \to +\infty} \theta(g^i)(c_1), \qquad b = \lim_{i \to +\infty} \theta(g^i)(c_2).$$

Recall that c_1, c_2 are the endpoints of l_3. This occurs because c_1 is in the union of I_2, I_3 and I_4 and so is a. Similarly for b. Now the same arguments as above imply the same conclusion. The important fact is that the endpoints of l_3 are not in I_1 or I_5!

Conclusion: In all cases there are n, m non zero integers with $g^n = h^{-1}g^mh$.

We now prove this is impossible. Notice that g and hence $h^{-1}gh$ act freely in $\mathcal{H}$ and both act as decreasing homeomorphisms of $\mathcal{H}$. This again uses the fact that h preserves orientation of $\mathcal{H}$! Hence if n is positive then m is positive as well. Assume this is the case. Given u in $\mathcal{U}$, if

$$g_1(u) = u, \qquad \text{then} \qquad g_1^n(u) = u = h_1^{-1}g_1^mh_1(u) \qquad \text{or} \qquad g_1^mh_1(u) = h_1(u).$$

Since g_1 has fixed points in $\mathcal{U}$ this implies $g_1h_1(u) = h_1(u)$. The same applies to h_1^{-1} so h_1 leaves the set of fixed points of g_1 invariant. These fixed points are in I_1, I_3, I_5 and I_7. By construction $h_1(I_1)$ is disjoint from I_1 and I_5 therefore it is a subset of I_3 or I_7 and likewise for $h_1(I_5)$. Similarly $h_1^{-1}(I_1)$, $h_1^{-1}(I_5)$ are subsets of I_3 or I_7, so $h_1(I_3)$, $h_1(I_7)$ are contained in I_1 or I_5. Together these imply that

$$h_1(I_j) = I_{k(j)} \quad \text{for any} \quad j \in [1, 8].$$

There are 4 cases to consider all similar. Suppose first that $h_1(I_1) = I_3$ and h_1 preserves orientation of $\mathcal{U}$. Since $h_1(I_j) = I_{k(j)}$, this implies the very important consequence that $h_1(I_2) = I_4$.

Now consider the action of g_1^n and $h_1^{-1}g_1^mh_1$ in I_2. The key is that both n, m are positive!

- g_1^n only fixes ∂I_2 in I_2, with a_1 repelling fixed point for g_1^n and b_1 attracting,
- $h_1^{-1}g_1^mh_1$ conjugates the action of g_1^m in I_4 to act in I_2. As h_1 preserves orientation in $\mathcal{U}$ then $h_1(a_1) = b_2$ and $h_1(b_1) = a_2$. In I_4, g_1^m fixes only ∂I_4 and b_2 is attracting, a_2 is repelling. Hence the action of $h_1^{-1}g_1^mh_1$ on I_2 fixes only ∂I_2 and has a_1 attracting, b_1 repelling.

Hence the actions of g_1^n and $h_1^{-1}g_1^mh_1$ are incompatible in I_2 and therefore they cannot be equal. Consider the other 3 cases: When $h_1(I_1) = I_3$ but h_1 reverses orientation in $\mathcal{U}$, then $h_1(I_2) = I_2$ flipping the endpoints. The same argument produces a contradiction. When $h_1(I_1) = I_7$ and h_1 preserves the orientation then $h_1(I_2) = I_8$ but it sends the attracting fixed point in I_2 (of g_1) to the repelling one in I_8 (of g_1) again contradiction. Finally if $h_1(I_1) = I_7$ and h_1 reverses orientation in $\mathcal{U}$ then $h_1(I_8) = I_8$, flipping the endpoints, again a contradiction.

As all cases are impossible this finally shows that $h_1(l_1)$ and l_1 intersecting transversely is impossible. The same proof applied to $\mathcal{D}_+$ shows that $\theta(h')(l_2)$ transversely intersecting l_2 for some h' in $\pi_1(M)$ is impossible. Hence both $\mathcal{D}_-$ and $\mathcal{D}_+$ generate laminations in M:

Lemma 5.6. *If a leaf of $\mathcal{D}_-$ transversely intersects a leaf of $\mathcal{D}_+$ then both the sets $cl(\pi(\mathcal{D}_- \times \mathbf{R}))$ and $cl(\pi(\mathcal{D}_+ \times \mathbf{R}))$ are embedded laminations in M which are transverse to each other $-$ here cl denotes closure in M. In particular option C implies option A for both $\mathcal{D}_-$ and $\mathcal{D}_+$.*

This finishes the proof of theorem 5.1 □

6 The two transverse laminations

In the previous section we proved that if $\mathcal{F}$ is $\mathbf{R}$-covered with hyperbolic leaves, then either M is Seifert fibered, or a torus bundle over $\mathbf{S}^1$ or there is a lamination transverse to $\mathcal{F}$. We use the constructions and notations of the previous section. In this section we show that in the atoroidal case both $\mathcal{D}_-$ and $\mathcal{D}_+$ produce laminations transverse to $\mathcal{F}$ which are also transverse to each other. Unless otherwise stated, from non on assume that M is homotopically atoroidal. We first obtain some general results about laminations transverse to $\mathbf{R}$-covered foliations and then use these results to study the laminations constructed in the previous section.

We say that a lamination $\mathcal{G}$ transverse to a foliation $\mathcal{F}$ is a *lamination by geodesics* if leaves of $\mathcal{G}$ intersect leaves F of $\mathcal{F}$ in geodesics of F. Now restrict to $\mathbf{R}$-covered foliations with hyperbolic leaves. If in addition for each leaf G of $\widetilde{\mathcal{G}}$, the ideal points of $G \cap F$ as F varies in $\widetilde{\mathcal{F}}$ define two leaves of the vertical foliation of $\mathcal{A}$, then we say that $\mathcal{G}$ is a *universal lamination by geodesics* – like the laminations constructed in the last section. First we analyse complementary regions of general universal geodesic laminations in a series of results from proposition 6.1 till lemma 6.4.

If F is a leaf of $\widetilde{\mathcal{F}}$ (or of $\mathcal{F}$) let $\widetilde{\mathcal{G}}_F$ ($\mathcal{G}_F$ respectively) denote the lamination by geodesics induced by $\widetilde{\mathcal{G}}$ (or $\mathcal{G}$) in F.

Proposition 6.1. *Let $\mathcal{F}$ be $\mathbf{R}$-covered with hyperbolic leaves and M homotopically atoroidal. Suppose that $\mathcal{G}$ is an universal lamination by geodesics transverse to $\mathcal{F}$. Then for any leaf F of $\widetilde{\mathcal{F}}$, the complementary regions of $\widetilde{\mathcal{G}}_F$ (that is, $\widetilde{\mathcal{G}} \cap F$) are all finite sided ideal polygons in F with an upper bound on the number of boundary sides. Complementary regions of $\mathcal{G}$ in M are either solid tori or solid Klein bottles bounded by finitely leaves of $\mathcal{G}$.*

Proof. We first show that geodesics which are boundary leaves of $\widetilde{\mathcal{G}}_F$ and which get sufficiently close in F are asymptotic in F:

Lemma 6.2. *There is positive ϵ so that for any F in $\widetilde{\mathcal{F}}$ then any neck of size smaller than ϵ in a complementary region of $\widetilde{\mathcal{G}}_F$ will produce asymptotic leaves in F. Similarly for $\mathcal{G}$ and $\mathcal{F}$.*

Proof. Here a neck is a geodesic segment α in a leaf F of $\widetilde{\mathcal{F}}$ so that its boundary is in $\widetilde{\mathcal{G}}_F$ but the interior of α is disjoint from $\widetilde{\mathcal{G}}_F$. There are two leaves u_1, u_2 of $\widetilde{\mathcal{G}}_F$ through the endpoints of α and the goal is to show that if α has small length then u_1, u_2 are asymptotic. Suppose the lemma is not true. Find necks of size smaller than $1/i$ from points in leaves

$$s_i, r_i \ \text{ of } \ \widetilde{\mathcal{G}}_{L_i}, \ \text{ with } \ L_i \in \widetilde{\mathcal{F}}, \ \text{ but } \ s_i, r_i \ \text{ not asymptotic in } \ L_i.$$

Since s_i, r_i are not asymptotic we may assume these are the closest points in L_i from s_i to r_i. Since s_i, r_i are distinct they eventually diverge (in some direction), so find necks ξ_i of size 1 between s_i, r_i with angles between the necks and the geodesics s_i, r_i bounded away from 0 and π. Let p_i be the middle points of ξ_i. Let f_i in $\pi_1(M)$ with $f_i(p_i)$ converging to p with necks $f_i(\xi_i)$ also converging, and so that the geodesics $f_i(s_i), f_i(r_i)$ converge to leaves s, r in L_0 of $\widetilde{\mathcal{F}}$. Here L_0 is the limit of $f_i(L_i)$.

If s, r are not asymptotic L_0, there is a minimum positive distance b_0 between them and they diverge from each other in each direction. For nearby $f_i(p_i)$ the leaves $f_i(s_i), f_i(r_i)$ also get roughly b_0 away from each other and then start to diverge from each other - this is all happening in a compact set near the leaf L_0. But in hyperbolic geometry, once a pair of geodesics starts to diverge from each other they will never get close anymore. Hence for i big the minimum distance between $f_i(s_i)$ and $f_i(r_i)$ in $f_i(L_i)$ is close to b_0. This contradicts $d_{L_i}(s_i, r_i)$ converging to 0. We conclude that s, r are in fact asymptotic in L_0.

The leaves $f_i(s_i), f_i(r_i)$ are boundary leaves of $\widetilde{\mathcal{G}}_{f_i(L_i)}$. Using the universal circle identification,

$$(f_i(s_i) \times \mathbf{R}) \ \cap \ L_0, \qquad (f_i(r_i) \times \mathbf{R}) \ \cap \ L_0$$

are also boundary leaves of $\widetilde{\mathcal{G}}_{L_0}$ with necks of size very close to 1 near p_0. But the only boundary leaves of $\widetilde{\mathcal{G}}_{L_0}$ near p_0 and with neck size near 1 are s, r, hence $f_i(s_i)$ is contained in say $(s \times \mathbf{R})$ and $f_i(r_i)$ is contained in $(r \times \mathbf{R})$ for i big enough.

Equivalently this argument says that nearby gaps of $\widetilde{\mathcal{G}}$ in fact map to each other by moving transversaly to $\widetilde{\mathcal{F}}$. This uses the fact that the laminations are made of sets $l \times \mathbf{R}$ associated to the universal circle - a universal geodesic lamination. This argument that if necks are very near, then the geodesics through the endpoints are contained in the same leaves of $\widetilde{\mathcal{G}}$ will be often used here – we call it the *matching boundary effect.*

But now the leaves s, r define the same ideal point in $S^1_\infty(L_0)$ and again by hypothesis of the proposition this implies that leaves $f_i(s_i), f_i(r_i)$ also define the same ideal point in $S^1_\infty(f_i(L_i))$, that is, $f_i(s_i), f_i(r_i)$ are asymptotic for i big enough. This contradicts the fact that s_i, r_i are not asymptotic in L_i and finishes the proof of lemma 6.2. $\square$

We stress that this works for <u>boundary</u> leaves. In general there are infinitely many pairs of leaves s_i, r_i of $\widetilde{\mathcal{G}}_F$ which have necks of arbitrarily small size but are not asymptotic in F. But if they are boundary leaves of the <u>same</u> complementary region of $\widetilde{\mathcal{G}}_F$ then they have to be asymptotic.

We now return to the proof of proposition 6.1. If $\mathcal{G}$ is a foliation proposition 6.1 is trivial. Otherwise consider ϵ' much smaller than $\epsilon/2$ (ϵ as in lemma 6.2). We define an open set $B_{\epsilon'}$ in M: let u be a point in a leaf E of $\mathcal{F}$. Then

$$u \in B_{\epsilon'} \quad \text{if} \quad d_E(u, \mathcal{G}_E) < \epsilon',$$

where d_E is measured in E. Choose ϵ' small enough so that $B_{\epsilon'}$ is not M and let Z be a component of the boundary of $B_{\epsilon'}$. We consider how Z intersects the foliation $\mathcal{F}$. Let p be a point in Z which is in a leaf E of $\mathcal{F}$. If p is ϵ' away from two leaves l_1, l_2 of $\mathcal{G}_E$, then l_1, l_2 are $2\epsilon'$ away from each other, which is less than ϵ. By lemma 6.2, l_1 and l_2 are asymptotic in E and in that direction their distance decreases: any point between them is less than ϵ' from at least one of l_1 and l_2 and therefore Z does not intersect that direction anymore. Now consider the opposite direction: in that direction l_1, l_2 diverge from each other and become more than $2\epsilon'$ from each other, this means that the intersection of Z and E has a corner at p and in the diverging direction two arcs of

$$Z \cap E \quad \text{emanate from} \quad p.$$

On the other hand if p is not a corner then p is ϵ distant from a single boundary leaf l_1 of $\mathcal{G}_E$ and the intersection of Z with E tracks this leaf l_1 nearby. Conclusion: the intersections of Z with E track boundary leaves of $\mathcal{G}_E$ until they hit a corner and start to track another leaf of $\mathcal{G}_E$ (they can also never hit a corner).

What happens transversely to $\mathcal{F}$ in a nearby leaf L? If p is not a corner point, then for nearby L, the set Z intersects L in a curve ϵ' away from $\mathcal{G}_L$ and near the one in E. This means that Z is a two dimensional manifold near p. If on the other hand p is a corner point, then l_1, l_2 are asymptotic in E. For nearby L there are unique boundary leaves of $\mathcal{G}_L$ associated to l_1, l_2 – by the matching boundary effect, and these leaves in L are also asymptotic. The associated corner point in L is near the corner point in E, which shows that near p, the set Z is a two dimensional set. This also shows that Z is transverse to $\mathcal{F}$.

This is a crucial point: if boundary leaves of $\mathcal{G}_E$ could get close to each other without being asymptotic, there are <u>two</u> corners associated to these leaves. Moving transversely to $\mathcal{F}$ could push those boundary leaves apart from each other. In terms of Z this would mean two corners coming together and spliting to two curves without corners tracking the two boundary leaves, producing a saddle tangency of Z and $\mathcal{F}$.

In addition to being two dimensional it is easy to see by definition that Z cannot limit on itself transversely: one side would have to be closer to $\mathcal{G}$. Hence Z is compact surface transverse to $\mathcal{F}$, so Z is either the torus or the Klein bottle. Let $\mathcal{F}_Z$ be the induced foliation in Z by $\mathcal{F}$.

Lemma 6.3. *The leaves of $\mathcal{F}_Z$ are closed curves which are null homotopic in their respective leaves of $\mathcal{F}$. The set Z bounds a solid torus or solid Klein bottle in M.*

Proof. If Z is π_1-injective then there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup in $\pi_1(M)$, contrary to the atoroidal hypothesis. Hence there is a simple closed curve γ in Z which is null homotopic in M.

First we show there are no Reeb annuli in $\mathcal{F}_Z$. Suppose there is a Reeb annulus C bounded by leaves α, β, which are the limit of α_x with x converging to infinity. Consider lifts

$$\widetilde{C},\ \widetilde{\alpha},\ \widetilde{\beta},\ \widetilde{\alpha}_x, \quad \text{to} \quad \widetilde{M}.$$

Since C is a Reeb annulus, the curves $\widetilde{\alpha}, \widetilde{\beta}$ are in leaves of $\widetilde{\mathcal{F}}$ which are not separated from each other. But $\mathcal{F}$ is $\mathbf{R}$-covered, so they are in the same leaf of $\widetilde{\mathcal{F}}$, let it be F. Then $\widetilde{\beta}, \widetilde{\alpha}$ do not track the same geodesics of $\widetilde{\mathcal{G}}_F$, because they are distinct curves. Now look at nearby leaves $\widetilde{\alpha}_x$: by construction they track a chain of consecutively asymptotic boundary leaf geodesics of $\widetilde{\mathcal{G}}$ in the respective leaves of $\widetilde{\mathcal{F}}$. By the matching boundary effect this chain pulls to F to a chain of consecutively asymptotic geodesics in $\widetilde{\mathcal{G}}_F$. But some of then are tracked by $\widetilde{\beta}$ and some are tracked by $\widetilde{\alpha}$. This can only happen if $\widetilde{\beta}$ is equal to $\widetilde{\alpha}$ which is impossible. As in the previous lemma this is basically saying that the combinatorics of the intersections of Z with $\mathcal{F}$ do not change transversely to $\mathcal{F}$.

Given that there are no Reeb annuli in $\mathcal{F}_Z$ the curve γ is homotopic to one which is either a leaf of $\mathcal{F}_Z$ or transverse to it. A transversal to $\mathcal{F}_Z$ is transverse to $\mathcal{F}$ and as $\mathcal{F}$ is Reebless, the transversal is not null homotopic in M. Hence we can assume that γ is a leaf $\mathcal{F}_Z$ - by Reebless again, γ is null homotopic in its leaf. Nearby leaves of $\mathcal{F}_Z$ are also closed since γ has no holonomy. The limit of compact leaves is compact [Ha], so all leaves of $\mathcal{F}_Z$ are closed and bound disks in their respective leaves of $\mathcal{F}$. It now follows that Z bounds a solid torus or solid Klein bottle in M. This finishes the proof. □

We now finish the proof of proposition 6.1. If U is a complementary region of $\mathcal{G}$, let ϵ' be small enough so that $B_{\epsilon'}$ does not contain U and let Z be a component of $\partial B_{\epsilon'}$ contained in U. Let E be a leaf of $\mathcal{F}$ intersecting Z and β a component of the intersection of E and Z. By lemma 6.3, β is a closed curve in E which is null homotopic in E and tracks boundary leaves of $\mathcal{G}_E$. Hence the associated complementary region in E is a finite sided ideal polygon in E. Moving transversely does not change the combinatorics or the number of boundary sides in this polygon and since Z is closed, it follows that the complementary region U is a solid torus or solid Klein bottle. This also shows that for any F in $\widetilde{\mathcal{F}}$ any complementary region of $\widetilde{\mathcal{G}}_F$ is a finite sided ideal polygon.

We know show there are finitely many complementary regions of $\mathcal{G}$. Do the argument in $\widetilde{M}$. A complementary region of $\widetilde{\mathcal{G}}$ contains an ideal triangle in a leaf of $\widetilde{\mathcal{F}}$. Suppose for a moment there are infinitely many complementary regions $\mathcal{V}_i$ which are not equivalent under covering translations. Let v_i be the barycenter of an ideal triangle contained in a leaf L_i of $\widetilde{\mathcal{F}}$ and also in $\mathcal{V}_i$. Up to covering translations and a subsequence the v_i converge to a point v_0 in a leaf L_0. Also

$$d_{L_i}(v_i, \widetilde{\mathcal{G}}_{L_i}) \ > \ c_0,$$

for some positive constant c_0, because v_i is the barycenter of an ideal triangle in $L_i - \widetilde{\mathcal{G}}_{L_i}$. By continuity v_0 is not in $\widetilde{\mathcal{G}}$ so there is a complementary component $\mathcal{V}_0$ of $\widetilde{\mathcal{G}}$ with v_0 in $\mathcal{V}_0$. As $\mathcal{V}_0$ is open then $\mathcal{V}_i$ is equal to $\mathcal{V}_0$ for i sufficiently big, contradiction.

Hence there are only finitely many complementary regions of $\mathcal{G}$ in M and there is an upper bound to the number of sides in any complementary region of $\widetilde{\mathcal{G}}_F$ for any F in $\widetilde{\mathcal{F}}$. This finishes the proof of proposition 6.1. □

Lemma 6.4. *Let $\mathcal{F}$ be a transversely oriented foliation with hyperbolic leaves in M orientable and $\mathcal{G}$ a minimal, universal lamination by geodesics transverse to $\mathcal{F}$. If s_1, s_2 are asymptotic leaves of $\widetilde{\mathcal{G}}_F$ then s_1, s_2 are in the boundary of a complementary region of $\widetilde{\mathcal{G}}_F$. Similarly for asymptotic leaves of $\mathcal{G}$.*

Proof. Here we use the notation of the previous section where F^* is the distinguished leaf. In addition if g is in $\pi_1(M)$ then g acts in $\mathcal{U}$ and geodesics of F^* by $\theta(g)$. We also use the notation $l \times \mathbf{R}$ for any geodesic l in a leaf of $\widetilde{\mathcal{F}}$. Let V_0 be $\widetilde{\mathcal{G}}_{F^*}$. It suffices to prove the lemma for V_0.

If the lemma is false find s_1 boundary leaf of a complementary region Q of V_0 and s_2 not in ∂Q but asymptotic to s_1, see fig. **??**, a. Assume that s_1 separates s_2 from Q. Let u_1 be the common ideal point of s_1, s_2 and u_2 the other ideal point of s_1. Let s_3 be the other boundary leaf of Q with ideal point u_2.

Fix z in s_1. Let ϵ given by lemma 6.2. Here s_1 is not isolated in the s_2 side, since it is isolated in the s_3 side and $\mathcal{G}$ is minimal. Let w_i be a sequence in s_1 converging to u_2. As $\mathcal{G}$ is minimal, there are v_i in $(s_1 \times \mathbf{R})$ which are a bounded distance (depending on ϵ) from w_i in $(s_1 \times \mathbf{R})$ and

$$f_i \text{ in } \pi_1(M) \text{ with } d(f_i(v_i), z) < \epsilon/2, \quad \text{but } f_i(v_i) \text{ not in } (s_1 \times \mathbf{R}).$$

As s_1, s_3 are asymptotic and $d(v_i, w_i)$ is bounded, then there are y_i in $(s_3 \times \mathbf{R})$ with $d(y_i, v_i)$ converging to zero. So $d(f_i(y_i), z)$ is smaller than $\epsilon/2$ for i big enough. We may assume that $f_i(y_i)$ and $f_i(v_i)$ are in V_0. By the property of ϵ, the leaves of V_0 through $f_i(y_i)$ and $f_i(v_i)$ are asymptotic in the u_1 direction, see fig. **??**, a. Since s_1, s_3 are asymptotic in the u_2 direction this implies that

$$\theta(f_i)(u_2) = u_1 \quad \text{for } i \text{ big enough.}$$

To simplify notation fix i sufficiently big and let $h = \theta(f_i)$. By assumption h preserves the orientation of $\mathcal{U}$. It follows that $h(Q)$ is asymptotic to Q along $s_1, h(s_1)$ and in the same way $h^2(Q)$ is asymptotic to $h(Q)$ along $h(s_1), h^2(s_1)$, see fig. **??**, b. This shows that $h(s_1)$ is asymptotic to both s_1 and $h^2(s_1)$. Recall that $h(s_1)$ is not isolated in V_0. But it is isolated on the $h(Q)$ side. On the other side it is asymptotic to $s_1, h^2(s_1)$ both in V_0, so again it is isolated. This contradiction shows that s_1, s_2 cannot be asymptotic and finishes the proof of the lemma. □

Notice that M atoroidal is not needed for this lemma.

Definition 6.5. *Let $\mathcal{Q}_i$ be a sequence of distortion parallelepipeds in $\widetilde{M}$ so that bottoms are ideal quadrilaterals in leaves of $\widetilde{\mathcal{F}}$ with cross ratios converging to* 0, *while tops are ideal quadrilaterals in leaves of $\widetilde{\mathcal{F}}$ with cross ratios converging to* 1. *We call $\mathcal{Q}_i$ a shrinking sequence of distortion parallepipeds. For simplicity we sometimes omit the word shrinking and refer to $\mathcal{Q}_i$ as a sequence of distortion parallelepipeds.*

We now analyse the laminations constructed in the last section in detail in particularly in relation to the options in step 5 of the proof of theorem 5.1. The eventual goal of this section is

Goal: Show that option B does not occur.

Hence option C will occur and as shown in the previous section option A holds for both $\mathcal{D}_-$ and $\mathcal{D}_+$ and there will be two transverse laminations in M. We use the notations and constructions of section 5. In order to analyse this situation recall the options in step 5 to produce laminations by geodesics in M:

- If leaves of $\mathcal{D}_-$ do not self intersect transversely (option A which can also happen when option B occurs) then let $\mathcal{G}_-$ be the closure of $\pi(\mathcal{D}_- \times \mathbf{R})$ and this is a lamination.

- Suppose there are transverse self intersections of leaves of $\mathcal{D}_-$. Then since option C implies option A, we have that no leaf of $\mathcal{D}_-$ transversely intersects a leaf of $\mathcal{D}_+$. Let A be the connected component of the union of leaves of $\mathcal{D}_-$ containing l_1 and let C be its convex hull. Then C is not all of $\mathbf{H}^2$ by hypothesis here. Let B be ∂C and in this case let $\mathcal{G}_-$ be the closure of $\pi(C \times \mathbf{R})$, also a lamination. Here we are in option B.

  Similarly for $\mathcal{D}_+$ producing a lamination $\mathcal{G}_+$. Hence there are always two laminations, which a priori may be equal. In addition let

- $\mathcal{G}^m_-$ be a minimal sublamination of $\mathcal{G}_-$ and

- $\mathcal{G}^m_+$ a minimal sublamination of $\mathcal{G}_+$.

The 4 laminations $\mathcal{G}_-, \mathcal{G}_+, \mathcal{G}^m_-$ and $\mathcal{G}^m_+$ will be fixed from now on. As we will see later $\mathcal{G}^m_-$ and $\mathcal{G}^m_+$ are uniquely defined and in the end we will prove that $\mathcal{G}^m_-$ is equal to $\mathcal{G}_-$ and similarly for $\mathcal{G}^m_+$. Let

$$V_- \ = \ \widetilde{\mathcal{G}}_- \cap F^*, \ \ V_+ \ = \ \widetilde{\mathcal{G}}_+ \cap F^*, \ \ V^m_- \ = \ \widetilde{\mathcal{G}}^m_- \cap F^*, \ \ V^m_+ \ = \ \widetilde{\mathcal{G}}^m_+ \cap F^*,$$

all laminations in F^* (could be foliations too). Also V^m_- is contained in V_- and V^m_+ is contained in V_+.

We first derive general properties of leaves of V^m_-, V^m_+. Let l be a leaf of V^m_-. If $\mathcal{D}_-$ does not transversely self intersect, then $\mathcal{G}^m_-$ is contained in $cl(\pi(\mathcal{D}_- \times \mathbf{R}))$ so there are leaves u_i in $\mathcal{D}_-$ converging to l. On the other hand, if $\mathcal{D}_-$ has transverse self intersections, say l_1 with l_4; then l_1 does not intersect $\widetilde{\mathcal{G}}_-$, so l_1 has to be in a complementary region of V^m_- – a finite sided ideal polygon. Therefore a ray of l_1 is asymptotic to a leaf of V^m_-. But since $\mathcal{G}^m_-$ is minimal this implies that $\pi(l_1 \times \mathbf{R})$ limits on every leaf of $\mathcal{G}^m_-$, so again there are leaves u_i of $\mathcal{D}_-$ converging to l. The same happens for $\mathcal{D}_+$ so if l is a leaf of V^m_+ there are leaves v_i of $\mathcal{D}_+$ converging to l. Lemma 5.5 now implies:

Conclusion 3: - If l is a leaf of V^m_- there is a shrinking sequence of parallelepipeds $\mathcal{Q}_i$ with bottoms in F^* so that two opposite sides of the bottoms converge to l, the cross ratio of bottoms (respectively tops) converges to 0 (respectively 1). In the same way if l is a leaf of V^m_+ there are parallelepipeds $\mathcal{Q}_i$ with the same cross ratio characteristics with tops converging to l. The same holds for intersections of $\widetilde{\mathcal{G}}^m_+, \widetilde{\mathcal{G}}^m_-$ with any leaf F of $\widetilde{\mathcal{F}}$.

If Q is an ideal polygon in F^* (or $\mathbf{H}^2$) let $\partial_\infty Q$ be the ideal points of Q.

We will now prove a crucial technical lemma which will help in analysing option B later on and also help produce a transverse pseudo-Anosov flow to $\mathcal{F}$. Consider the case that $\mathcal{G}^m_-$ is not a foliation. Recall that if g is in $\pi_1(M)$, then $\theta(g)$ acts in $\mathcal{U}$ and in convex sets of F^*. Let C_1 be a complementary region of V^m_-. By proposition 6.1, $\pi(C_1 \times \mathbf{R})$ is a solid torus or solid Klein bottle, and the core is a curve transverse to $\mathcal{F}$. So there is a non trivial g in $\pi_1(M)$ with $\theta(g)(C_1)$ equal to C_1. Taking powers we may assume $\theta(g)$ fixes all points of $\partial_\infty C_1$, Here we are identifying $S^1_\infty(F^*)$ with $\mathcal{U}$ so $\theta(g)$ acts on $S^1_\infty(F^*)$. There are at least 3 points in $\partial_\infty C_1$, therefore g acts freely in $\mathcal{H}$. Up to taking inverse assume that g is monotone decreasing in $\mathcal{H}$, that is F^* is in the front of $g(F^*)$.

Lemma 6.6. *Suppose that $\mathcal{G}^m_-$ is not a foliation and C_1 is a complementary region of V^m_-. Let g in $\pi_1(M)$ non trivial with $\theta(g)(C_1)$ equal to C_1 and $\theta(g)$ fixing all ideal points of C_1. Suppose that g is monotone <u>decreasing</u> in $\mathcal{H}$. Let J_1 be a component of $\mathcal{U} - \partial C_1$. Then $\theta(g)$ acts as a contraction in J_1 with a single fixed point. Similarly if $\mathcal{G}^m_+$ is not a foliation and C_1 is a complementary region of V^m_+ with g, J_1 as above (g acts <u>decreasing</u> in $\mathcal{H}$) then $\theta(g^{-1})$ acts as a contraction in J_1 with a single fixed point.*

Proof. First we do the proof for $\mathcal{G}^m_-$. Let s_0 be a side of C_1, hence $\theta(g)(s_0)$ is equal to s_0. Let e_1, e_2 be the ideal points of s_0 and

$$J_1 \ = \ \text{closure of interval of } \ \mathcal{U} - \{e_1, e_2\} \ \text{ not containing other ideal points of } \ C_1.$$

We analyse the action of $\theta(g)$ in J_1. Notice $\theta(g)$ fixes C_1 hence fixes e_1, e_2. Notice s_0 is isolated on the C_1 side, so not isolated on the other side. Choose s_1 in V^m_- arbitrarily close to s_0 hence with ideal points in J_1. By conclusion 3 above there are parallepipeds $\mathcal{Q}_i$ with <u>bottoms</u> $\mathcal{R}_i$ having two opposite sides converging to s_1 and tops $\mathcal{S}_i$ with cross ratios of bottoms converging to 0 and cross ratios of tops converging to 1. Let F_t in $\widetilde{\mathcal{F}}$ with $\mathcal{S}_i$ contained in F_t, the t here depends on i. Since g is monotone decreasing in $\mathcal{H}$, there is a unique <u>positive</u> n so that either

$$F_t \ = \ g^{-n}(F^*) \quad \text{or} \quad F_t \ \text{ is between } \ g^{-n}(F^*) \ \text{ and } \ g^{-(n+1)}(F^*)$$

(notice that $g^{-n}(F^*)$ is above F^*). The quadrilateral $\mathcal{R}_i$ has ideal points e_3, e_4, e_5, e_6 in $\mathcal{U}$, so that $\overline{e_3e_4}$, $\overline{e_5e_6}$ are the sides close to each other and close to s_1, see fig. **??**, a. By lemma 6.4, if s_1 is asymptotic to s_0 then s_0 and s_1 are boundary sides of a complementary region of V_0. But then s_0 would be isolated on both sides contradiction to minimality of $\mathcal{G}$. Hence once s_1 is fixed, then for i big enough all the ideal points e_3, e_4, e_5, e_6 are in $J_1 - \partial J_1$. Choose the points so that $s_2 = \overline{e_5 e_6}$ is farther from s_0 than $\overline{e_3 e_4}$. Notice $t, n, e_3, ..., e_6$ all depend on i, which is omitted for simplicity.

Map $\mathcal{S}_i$ back by g^n, pushing it down in the leaf space. Then $g^n(\mathcal{S}_i)$ is either in F^* or is between F^* and $g^{-1}(F^*)$. Also $g^n(\mathcal{S}_i)$ is an ideal quadrilateral with cross ratio very close to 1. Suppose that both $\theta(g^n)(e_5)$ and $\theta(g^n)(e_6)$ are very close to e_5, e_6 respectively. Projecting $g^n(\mathcal{S}_i)$ to F^* using $\mathcal{U}$, that is,

$$(g^n(\mathcal{S}_i) \times \mathbf{R}) \ \cap \ F^*,$$

we get an ideal quadrilateral in F^* with cross ratio very close to 0. This is because $\theta(g^n)$ fixes e_1 and e_2 and $\theta(g^n)(e_3)$ is between e_1 and $\theta(g^n)(e_5)$ and $\theta(g^n)(e_4)$ between e_2 and $\theta(g^n)(e_6)$. The same will happen for the cross ratio of the intersection

$$(g^n(\mathcal{S}_i) \times \mathbf{R}) \ \cap \ F, \qquad \text{for any } \ F \ \text{ between } \ F^* \ \text{ and } \ g^{-1}(F^*).$$

All of them are very near the intersection of F and $s_0 \times \mathbf{R}$. This contradicts the fact that cross ratio of $g^n(\mathcal{S}_i)$ is very close to 1. Hence at least one of $\theta(g^n)(e_5)$ and $\theta(g^n)(e^6)$ is very far from e_5 or e_6 respectively.

One option is say that $\theta(g^n)(e_5)$ is very close to e_2. But if this keeps happening as s_1 gets closer to s_0 and i converges to infinity, then the only possibility for the dynamics of $\theta(g^n)$ in J_1 is that $\theta(g^n)$ has a repelling fixed point in e_1, an attracting fixed point in e_2 and no other fixed points. In fact since $\theta(g)(J_1)$ is equal to J_1, this implies that $\theta(g)$ acts in J_1 with the same dynamics. Take s leaf of V^m_- very close to s_0 with ideal points a, b, a close to e_1, b close to e_2. By lemma 6.4, a is not e_1 and b is not e_2. Hence for a, b sufficiently close to e_1, e_2 respectively, the points

$$e_1, \ a, \ \theta(g)(a), \ b, \ \theta(g)(b), \ e_2,$$

are circularly ordered in J_1. In other words the endpoints of s separate the endpoints of $\theta(g)(s)$ in $\mathcal{U}$, or s intersects $\theta(g)(s)$ transversely. This contradicts the fact that $\mathcal{G}^m_-$ is a lamination.

It follows that J_1 contains an interval J_2 bounded by e_5, e_6, with $\theta(g^n)(J_2)$ a subinterval of J_2. The cross ratio of $\theta(g^n)(\mathcal{S}_i)$ is very close to 1, so this quadrilateral is very thin in the other direction, see fig. **??**, b. Since $\theta(g^n)(e^5), \theta(g^n)(e^6)$ cannot be near e_5 or e_6, it follows that $\theta(g^n)(J_2)$ is a very small interval contained in J_2.

This is the fundamental property of a curve which is in the boundary of a complementary region of V^m_- so that it is the limit of <u>bottoms</u> of a shrinking sequence of distortion parallelepipeds. Choosing now s_1 closer and closer to s_0 and cross ratio of $\mathcal{R}_i$ converging to 0, cross ratio of $\mathcal{S}_i$ converging to 1, we can get e_5 arbitrarily close to e_1 and e_6 arbitrarily close to e_2. In addition the subinterval of J_1 from $\theta(g^n)(e_5)$ to $\theta(g^n)(e_6)$ is shrinking to a point (recall here that n varies with s_1 and i). This yields the conclusion:

Conclusion 4: – Using that s_0 is the limit of bottoms of distortion parallelepipeds, we obtain that $\theta(g)$ acts in $J_1 - \partial J_1$ as a contraction with a unique fixed point y not in ∂J_1.

This proves the first part of the lemma. If on the other hand C_1 is a complementary component of V^m_+ (assumed not to be F^*) and s_0, J_1 are defined as above, the same analysis applies. But now there is a sequence of distortion parallepipeds with the <u>tops</u> converging to s_0. The difference is that $\mathcal{R}_i$ is below $\mathcal{S}_i$, hence use a translate g^m to bring $\mathcal{R}_i$ closer to F^*, with m <u>negative</u>. Using the same arguments as above, we conclude that $\theta(g^{-1})$ acts as a contraction in $J_1 - \partial J_1$ or that $\theta(g)$ acts as an <u>expansion</u> in $J_1 - \partial J_1$. This finishes the proof of lemma 6.6. ◻

We will now show that option B does not occur.

Proposition 6.7. *Suppose that M is orientable and $\mathcal{F}$ is transversely orientable. There is a leaf of $\mathcal{D}_-$ intersecting a leaf of $\mathcal{D}_+$ transversely.*

Proof. Suppose that this is not true. Then option B holds and it implies that no leaf of $\mathcal{G}_-$ transversely intersects a leaf of $\mathcal{G}_+$. If $\mathcal{G}_-$ (or $\mathcal{G}_+$) is a foliation then $\mathcal{G}_-$ is minimal, for a non trivial sublamination would have complementary regions in leaves of $\widetilde{\mathcal{F}}$ which are finite sided ideal polygons and could not be filled with a foliation by geodesics. This also implies $\mathcal{G}_-, \mathcal{G}_+$ are equal. In this case all of the laminations

$$\mathcal{G}_-, \ \mathcal{G}_+, \ \mathcal{G}^m_- \ \text{ and } \ \mathcal{G}^m_+$$

are the same.

If none of $\mathcal{G}_-, \mathcal{G}_+$ are foliations then a complementary region of V^m_- is a finite sided ideal polygon Q in F^*. Since $\mathcal{G}^m_+$ does not intersect $\mathcal{G}^m_-$ transversely then V^m_+ cannot intersect ∂Q transversely. So any intersection of V^m_+ with Q is a geodesic in the interior of Q. There are only finitely many such geodesics and these would be isolated contradicting minimality of $\mathcal{G}^m_+$. Hence $\mathcal{G}^m_+$ does not intersect the interior of Q and so $\mathcal{G}^m_+$ is contained in $\mathcal{G}^m_-$. Similarly $\mathcal{G}^m_-$ is contained in $\mathcal{G}^m_+$ and so they are equal. These arguments also show that in general $\mathcal{G}^m_-$ and $\mathcal{G}^m_+$ are uniquely defined laminations. In any case we proved:

Fact: In option B, then $\mathcal{G}^m_-$ is equal to $\mathcal{G}^m_+$.

This is the fact that will lead to a contradiction.

Suppose first that $\mathcal{G}_-$ and $\mathcal{G}_+$ are not foliations. Then $\mathcal{G}^m_-, \mathcal{G}^m_+$ are also not foliations. Let C_1 be a complementary region of V^m_- and s_0 a boundary leaf of C_1. Let g be nontrivial in $\pi_1(M)$ with $\theta(g)$ fixing all points in $\partial_\infty C_1$ and let J_1 be a component of $\mathcal{U} - \partial_\infty C_1$ with endpoints the ideal points of s_0. Since s_0 is a leaf of V^m_-, the first part of lemma 6.6 shows that $\theta(g)$ acts as a contraction in J_1. As $\mathcal{G}^m_-$ is equal to $\mathcal{G}^m_+$, then s_0 is also a leaf of V^m_+. The second part of lemma 6.6 implies that $\theta(g^{-1})$ acts as a contraction in J_1.

These two conclusions are contradictory, so we obtain that $\mathcal{D}_-$ and $\mathcal{D}_+$ not intersecting transversely is impossible when at least one of $\mathcal{G}_-$ or $\mathcal{G}_+$ is not a foliation.

The next proposition shows that none of $\mathcal{G}_-, \mathcal{G}_+, \mathcal{G}$ can be foliations, so this finishes the proof of proposition 6.7. ☐

Proposition 6.8. *The lamination $\mathcal{G}_-$ (or $\mathcal{G}_+$) cannot be a foliation in M.*

Proof. This is presented separately because here we do not a priori assume that $\mathcal{D}_-$ and $\mathcal{D}_+$ have no transversal intersection. It will be used later as well, see remark 1. Also no orientability conditions here.

Suppose that say $\mathcal{G}_-$ is a foliation in M. Then as seen before $\mathcal{G}_-$ is minimal, hence $\mathcal{G}^m_- = \mathcal{G}_-$. Since

$$V_- \ = \ \widetilde{\mathcal{G}}_- \cap F$$

is a foliation by geodesics in F^* then its leaf space is Hausdorff and hence homeomorphic to the reals. We analyse this in detail. Fix q' in F^*, let s' be the leaf of V_- through q'. Let γ be a finite transversal to V_- in F^* starting at q' and look at all geodesics of V_- through γ. Suppose one is asymptotic to s' in the direction of the ray r' of s'. Then all leaves between these two are also asymptotic to s' and hence in any case there is a direction so no nearby leaf is asymptotic to s' (the opposite direction to r' if there is r'). As points in s' escape in that direction their distance to nearby leaves grows without bound. For any natural n there are points q_n in these nearby leaves of V_- which are centers of balls B_n of F^* of radius n so that all leaves in B_n will eventually intersect a subsegment of γ of length less than $1/n$. Up to subsequence choose f_n in $\pi_1(M)$ with $f_n(q_n)$ converging to q_0 and q_0 in a leaf L of $\widetilde{\mathcal{F}}$. Let s be the leaf through q_0 of the lamination $\widetilde{\mathcal{G}}_- \cap L$ of L. Let v another leaf of this lamination. If s and v are not asymptotic there is a minimum distance between them which is achieved in a fixed distance from starting points. Pairs of geodesics limiting on s and v will also have a minimum distance between them close to this distance, contradiction to the construction.

Conclusion $-$ All leaves of $\widetilde{\mathcal{G}}_- \cap L$ are asymptotic in L defining a unique ideal point in $S^1_\infty(L)$.

As $\mathcal{G}_-$ is a universal geodesic lamination the same holds for all leaves of $\widetilde{\mathcal{F}}$ and let u in $S^1_\infty(F^*)$ be the distinguished ideal point in F^*. Hence leaves of $\widetilde{\mathcal{G}}_-$ are described exactly as $s^* \times \mathbf{R}$, where s^* is an arbitrary geodesic in F^* with one ideal point u. Every point in $S^1_\infty(F^*)$ is an ideal point of some leaf in V_-.

We claim that this implies that $\mathcal{G}_-$ and $\mathcal{G}_+$ are equal. Let l^* be a leaf of V_+. Then l^* is asymptotic to a leaf l' of V_- so the angle between l^*, l' converges to 0. Take a limit in M and obtain F in $\widetilde{\mathcal{F}}$ and a common leaf of $\widetilde{\mathcal{G}}_- \cap F$ and $\widetilde{\mathcal{G}}_+ \cap F$, so a common leaf of $\widetilde{\mathcal{G}}_+$ and $\widetilde{\mathcal{G}}_-$. As $\mathcal{G}_-$ is minimal, then $\mathcal{G}_-$ is contained in $\mathcal{G}_+$, which implies that $\mathcal{G}_-$ is equal to $\mathcal{G}_+$ (because $\mathcal{G}_-$ is a foliation). Hence all laminations

$$\mathcal{G}_-, \ \mathcal{G}_+, \ \mathcal{G}^m_- \quad \text{and} \quad \mathcal{G}^m_+$$

are equal and they are foliations. In that case let V_0 be V_- (same as V_+ and so on).

The proof now is similar to arguments in lemma 6.6. However in that situation there were complementary regions of V_-, and these naturaly produced covering tranlations acting freely on $\mathcal{H}$ and leaving invariant the complementary region and one needs to first find appropriate g acting freely in $\mathcal{H}$. First lift to a finite regular cover so that the manifold is orientable and all foliations are transversely oriented and for simplicity in this proof we assume they are the original $M, \mathcal{G}_-, \mathcal{G}_+$. Notice $\mathcal{G}_-, \mathcal{G}_+$ are still minimal since they are foliations. Notice $\mathcal{U}$ is still the same and the action of $\pi_1(M)$ on $\mathcal{U}$ is still not uniformly quasisymmetric because of compactness of the finite cover.

Let ζ_1 be a leaf of V_-. Then ζ_1 is the limit of bottoms $\mathcal{R}_i$ of a shrinking sequence of distortion parallelepipeds with cross ratios of $\mathcal{R}_i$ converging to 0. There are 2 points in $\partial \mathcal{R}_i$ very close to the distinguished point u of $\mathcal{U}$.

Suppose first that u is in the interior of the small interval of $\mathcal{U}$ with these endpoints. We split the quadrilateral into thinner quadrilaterals: cover the quadrilateral $\mathcal{R}_i$ by two thin quadrilaterals $\mathcal{Q}_1, \mathcal{Q}_2$ both of which have ideal point in u and another in v, see fig. **??**, a. There are two corresponding parallelepipeds $\mathcal{V}_1, \mathcal{V}_2$ with tops $\mathcal{S}_1, \mathcal{S}_2$ in a leaf L of $\widetilde{\mathcal{F}}$. For simplicity we omit the dependence of L on i. We now choose the point v carefully so that at least one of $\mathcal{S}_1, \mathcal{S}_2$ has cross ratio very close to 1. Let u'_1, u'_2, u', v' be the points in $S^1_\infty(L)$ corresponding to u_1, v_1, u, v respectively under the universal circle $\mathcal{U}$, see fig. **??**, b. The top of the original parallelepiped is a quadriletaral $\mathcal{S}$ very thin in the other direction, see fig. **??**, b. If for example u' is not very near either of u'_1, u'_2, then choose v' very close to one of the other ideal points of $\mathcal{T}$, see fig. **??**, b. Then one of $\mathcal{S}_1$ or $\mathcal{S}_2$ is thin in the other direction, in the picture $\mathcal{S}_2$ is thin. Conversely if u' is very near u'_1 or u'_2 choose v' in the middle between the other ideal points of $\mathcal{T}$. Regardless one obtains a thin quadrilateral in the other direction.

If on the other hand the small interval from u_1 to u_2 does not contain u, then enlarge $\mathcal{R}_i$ to include u. For high enough i, the new $\mathcal{R}_i$ will still have cross ratio very close to 0. But clearly the top of the associated parallelepiped is an ideal quadrilateral which is even thinner, so its cross ratio is even closer to 1.

A priori this process has changed $\mathcal{G}_-, \mathcal{G}_+$, so let us consider this carefully. The splitting or enlarging does not distort the bottoms $\mathcal{R}_i$ substantially, so $\mathcal{G}_-$ is not changed. Therefore $\mathcal{G}_-$ is still a foliation. Since $\mathcal{G}_-$ is a foliation, then regardless of what the new $\mathcal{D}_+$ is, the first part of the proof implies that $\mathcal{G}_+$ is equal to $\mathcal{G}_-$. Hence $\mathcal{G}_+$ is not changed either. This fact will be used in the arguments below.

Conclusion $-$ Any leaf of V_- is a limit of bottoms $\mathcal{R}_i$ of a shrinking sequence of distortion parallepipeds with all ideal quadrilaterals $\mathcal{R}_i$ having a vertex in u.

In order to mimic the proof of lemma 6.6 we first construct a suitable g in $\pi_1(M)$ with at least 3 fixed points in $\mathcal{U}$. Put any orientation in $\mathcal{U}$. Let ideal points of $\mathcal{R}_i$ as positively oriented in $\mathcal{U}$ be denoted by

$$u_1, \ u_2, \ u_3, \ u_4,$$

where u_1 is equal to u. Up to subsequence and choosing orientation to $\mathcal{U}$ assume u_2 is very close to u_1, see fig. **??**, a. We are stil using the fixed leaf ζ_1 introduced before. The points u_j are fixed in this proof.

Now choose a leaf ζ_2 of V_+ having an ideal point in the interval defined by u_2 and u_3, not containing u and not near u_2 or u_3, see fig. **??**, a. This is possible since $\mathcal{G}_+$ is equal to $\mathcal{G}_-$. The tops $\mathcal{S}_i$ of the sequence of distortion parallelepipeds have cross ratio very close to 1. For i big enough there is a covering translation g so that $\theta(g)(\mathcal{R}_i)$ is very close to ζ_2 and with cross ratio very close to 1. This is because ζ_2 is a leaf of $\mathcal{G}_+$. The element g depends on ζ_1, ζ_2 and i. Notice that $\theta(g)$ fixes u_1 and preserves the orientation in $\mathcal{U}$. The fact that $\theta(g)(\mathcal{R}_i)$ is very close to ζ_2 and has cross ratio very close to 1 together with $\theta(g)(u_1)$ being equal to u_1 implies the following:

- $\theta(g)(u_4)$ is very close to u_1,
- $\theta(g)$ moves u_4 counterclockwise (as seen in fig. **??**, a);
- $\theta(g)(u_2)$, $\theta(g)(u_3)$ are very close to the ideal point of ζ_2 (the other one besides u_1) and
- $\theta(g)$ moves u_2 counterclockwise, moves u_3 clockwise, see fig. **??**, a.

This implies there are at least 3 fixed points of $\theta(g)$ in $\mathcal{U}$: u_1, plus one fixed point near u_3 and one near $\theta(g)(u_3)$. Hence g acts freely in $\mathcal{H}$. Assume it is decreasing in $\mathcal{H}$.

Once the suitable g is found the argument follows the analysis in the proof of lemma 6.6. We only sketch the main ideas. The element g is fixed here. Let ζ_3 be another a leaf in V_- invariant under $\theta(g)$ with an ideal point u_5 not equal to u and u_5 near u_3 (u_3 defined above). Let J be the interval of $\mathcal{U} - \{u_1, u_5\}$ containing $\theta(g)(u_2)$. By the above there is a fixed point of $\theta(g)$ in J. There is a sequence of ideal quadrilaterals which are bottoms of distortion parallelepipeds, for simplicity still denoted by $\mathcal{R}_i$, so that $\mathcal{R}_i$ converges to ζ_3 and $\mathcal{R}_i$ has ideal points u, v_2, v_3, v_4 positively circularly oriented in $\mathcal{U}$ and v_2, v_3, v_4 all in J (v_2, v_3, v_4 depend on i). This is possible because u_2 (in the prior construction) was chosen in J and using covering translates we can map ζ_3 to arbitrarily near ζ_4 with ideal point in J. Going up transversely to $\widetilde{\mathcal{F}}$ using the universal circle we obtain quadrilaterals $\mathcal{S}_i$ with cross ratios very close to 1 and map back by g^n, n positive.

We check the action of $\theta(g^n)$ in J: first of all it fixes the boundary of J. Notice $\theta(g^n)$ has a fixed point in J, hence $\theta(g^n)$ cannot map v_2 very near u_5 and cannot map v_3, v_4 very near u. Hence the arguments in lemma 6.6 show that $\theta(g^n)$ moves v_2, v_3, v_4 close together. In the limit one obtains that the action of $\theta(g)$ in J is a contraction with a single fixed point.

Using the same arguments with ζ_3 a leaf of $\mathcal{G}_+$, that is ζ_3 being the limit of tops of a sequence of distortion parallelepipeds we obtain $\theta(g)$ acts as an expansion in J.

This is contradiction and shows that $\mathcal{G}_-$ (or $\mathcal{G}_+$) cannot be a foliation. This finishes the proof of proposition 6.8. $\square$

These results imply the following:

Corollary 6.9. *Suppose that M is orientable and $\mathcal{F}$ transversely orientable. Neither $\mathcal{G}_-$ nor $\mathcal{G}_+$ can be a foliation. Also by proposition 6.7, there is a leaf of $\mathcal{D}_-$ intersecting a leaf of $\mathcal{D}_+$ transversely. This shows that option C in step 5 of theorem 5.1 occurs. As seen in the proof of theorem 5.1 this implies that both $\mathcal{D}_-$ and $\mathcal{D}_+$ have no transverse self intersections. This means that option A occurs for both of them. Therefore $\mathcal{G}_-$ is $cl(\pi(\mathcal{D}_- \times \mathbf{R}))$. Similarly $\mathcal{G}_+$ is $cl(\pi(\mathcal{D}_+ \times \mathbf{R}))$. In addition there is positive r_1 so that for every G in $\widetilde{\mathcal{G}}_-$, F in $\widetilde{\mathcal{F}}$ and p in the intersection l of G and F, there is a point in the intersection of $\widetilde{\mathcal{G}}_+$ and l at most r_1 distant from p in l. Otherwise taking limits we find one such l not intersecting $\widetilde{\mathcal{G}}_+$ and a leaf of $\widetilde{\mathcal{G}}_-$ not intersecting $\widetilde{\mathcal{G}}_+$ which was disallowed. Finally if l' is a leaf of the intersection of $\widetilde{\mathcal{G}}_+$ and F and l' intersects l in q then the angle between l, l' in F^* at q is bounded away from 0 and π. Similarly for $\widetilde{\mathcal{G}}^m_-$, $\widetilde{\mathcal{G}}^m_+$.*

To sum up what we have obtained so far:

Corollary 6.10. *Suppose that $\mathcal{F}$ is an $\mathbf{R}$-covered foliation with hyperbolic leaves, M is atoroidal and not a Seifert fibered space. Suppose that $\mathcal{F}$ is transversely oriented and M orientable. Let $\mathcal{G}_-$ and $\mathcal{G}_+$ be the universal geodesic laminations constructed in the previous section. Then neither $\mathcal{G}_-, \mathcal{G}_+$ is a foliation. They are transverse to each other and with solid torus complementary components.*

Remark: At this point it is useful to make the following remark: In some situations it may seem at first that the main theorem is trivial: Consider φ an Anosov flow so that its stable foliation $\mathcal{F}^s$ is $\mathbf{R}$-covered and transversely orientable (there are many examples [Fe2]). The flow φ is tangent to $\mathcal{F}^s$ and since $\mathcal{F}^s$ is transversely orientable one can perturb φ slightly to a new flow φ' transverse to $\mathcal{F}^s$. By Anosov's fundamental results φ' is also an Anosov flow [An]. However if in addition M is atoroidal then the flow Φ transverse to $\mathcal{F}^s$ which will be constructed here is not a perturbation of φ as above. It has substantially different properties. In particular Φ is not an Anosov flow $-$ it has singularities. This is because Φ is obtained by blowing down complementary regions of $\mathcal{G}_-, \mathcal{G}_+$ and the solid torus complementary components will produce singularities. In fact this is the key property that implies the weak hyperbolization conjecture for M. Also the flow Φ will be regulating for $\mathcal{G}$: every orbit of the flow $\widetilde{\Phi}$ in $\widetilde{M}$ intersects every leaf of $\widetilde{\mathcal{F}}$ and vice versa, as opposed to what happens for small perturbations of the Anosov flows above [Fe2, Fe5].

We now relate $\widetilde{\mathcal{G}}_-$ and $\widetilde{\mathcal{G}}_+$.

Proposition 6.11. *For every complementary region Q of V_- there is an unique associated complementary region Q' of V_+ having the same number of sides as Q. Let g in $\pi_1(M)$, non trivial with $\theta(g)$ fixing all points in $\partial_\infty Q$. Suppose that g acts as a decreasing homeomorphism in $\mathcal{H}$. Then the fixed point set of $\theta(g)$ in $\mathcal{U}$ is exactly the union of $\partial_\infty Q$ and $\partial_\infty Q'$; the points in $\partial_\infty Q$ are repelling fixed points, those in $\partial_\infty Q'$ are attracting and they alternate in $\mathcal{U}$. There is a unique compact complementary region of the union of Q and Q' in F^* which is a compact finite sided polygon.*

Proof. First we will prove this result for the minimal sublaminations $\mathcal{G}^m_-$, $\mathcal{G}^m_+$ of $\mathcal{G}_-, \mathcal{G}_+$. Then we show that $\mathcal{G}_-, \mathcal{G}_+$ are minimal, that is $\mathcal{G}^m_-$ is equal to $\mathcal{G}_-$ proving the result for $\mathcal{G}_-$ and $\mathcal{G}_+$ as well.

Let Q be a complementary region of V^m_- (which is the intersection of $\widetilde{\mathcal{G}}^m_-$ with F^*) and j the number of ideal points of Q. The proof of lemma 6.6 shows that the action of $\theta(g)$ on each component of

$$\mathcal{U} - \partial_\infty Q$$

is a contraction with a single fixed point. This is because each component of the boundary of Q (a geodesic in F^*) is a leaf of V^m_-. This shows that there are exactly $2j$ fixed points of $\theta(g)$ in $\mathcal{U}$. Let l_1 be a boundary leaf of Q and l_2 another leaf asymptotic to it, defining the common ideal point w_1 $-$ a repelling fixed point of $\theta(g)$, see fig. **??**. Let J_1 be the complementary interval of $\partial_\infty Q$ in $\mathcal{U}$ defined by l_1 and similarly define J_2. Let

$$p_j \ \text{ in } \ l_1 \cap V^m_+ \quad \text{with} \quad p_j \ \rightarrow \ w_1.$$

Let γ_j be the leaves of V^m_+ through p_j. Since the angle between γ_j and l_1 is bounded away from 0 and π the endpoints of γ_j are eventually in the union of J_1 and J_2 and close to w_1. Let γ be one such leaf. Then $\theta(g^n)(\gamma)$ is a leaf of V^m_+ and as n converges to $+\infty$, the endpoints of $\theta(g^n)(\gamma)$ converge to the fixed points y_1, y_2 of $\theta(g)$ in J_1, J_2 respectively, see fig. **??**, a. As V^m_+ is a lamination in F^* this produces s_1 a leaf of V^m_+ with ideal points y_1, y_2. Also $\theta(g)(s_1)$ is equal to s_1.

In the same way associated to any ideal point w_j of Q we find s_j leaf of V^m_+. Consecutive s_j's are asymptotic, creating a finite sided ideal polygon Q' with the same number of sides as Q and Q' invariant under $\theta(g)$. If there is any leaf of V^m_+ in the interior of Q' it would be isolated in V^m_+ contradiction. Hence Q' is a complementary region of V^m_+. This proves the conclusion of of lemma 6.11 for V^m_- and V^m_+.

This analysis implies that $\mathcal{G}_-$ is minimal and so equal to $\mathcal{G}^m_-$. Otherwise there is Q complementary region of V^m_- left invariant under $\theta(g)$ with g non trivial in $\pi_1(M)$ and l a leaf of V_- in the interior of Q. Hence Q has at least 4 ideal points. Let Q' be the associated complementary region of V^m_+ $-$ by construction it has at least two ideal points on each component of $\mathcal{U} - l$, let p_1, p_2 in one component and p_3, p_4 in the other component. The p_j are all fixed by $\theta(g)$. Consider a sequence of ideal quadrilaterals $\mathcal{R}_i$ converging to l which are the bottoms of a sequence of distortion parallepipeds. Then one can show that the associated tops $\mathcal{S}_i$ have cross ratio bounded away from 1 $-$ the fixed points p_1, p_2 and p_3, p_4 of

$\theta(g)$ in $\partial_\infty Q'$ keep the quadrilateral $\mathcal{S}_i$ from being too thin in the other direction. The $\mathcal{S}_i$ are trapped between two walls

$$(\overline{p_1p_2} \times \mathbf{R}) \quad \text{and} \quad (\overline{p_3p_4} \times \mathbf{R}).$$

These walls are invariant under g, so the cross ratios (with the correct order) cannot get too close to 1. This contradiction shows that $\mathcal{G}^m_-$ is equal to $\mathcal{G}_-$, that is $\mathcal{G}_-$ is a minimal lamination.

Similarly $\mathcal{G}_+$ is minimal. As we already proved the results for $\mathcal{G}^m_-$ and $\mathcal{G}^m_+$ this finishes the proof of proposition 6.11. □

Before producing the transversal flow we check the non orientable situations:

Proposition 6.12. *Suppose that $\mathcal{F}$ is transversely oriented and $\mathbf{R}$-covered with hyperbolic leaves and that M is homotopically atoroidal. Then there are laminations by geodesics $\mathcal{G}_+, \mathcal{G}_-$ transverse to $\mathcal{F}$ and transverse to each other and which satisfy the conclusions of lemma 6.11.*

Proof. The difference here is that M may be non orientable. The covering translations reversing orientation in M are exactly those which reverse orientation in $\mathcal{U}$, because $\mathcal{F}$ is transversely orientable. If M is orientable previous results apply. Otherwise let M_2 be the orientable double cover so $\pi_1(M_2)$ is a normal subgroup of index 2 in $\pi_1(M)$. Using that M_2 is homotopically atoroidal construct the laminations $\mathcal{G}^*_-, \mathcal{G}^*_+$ in M_2 as before. Let f in $\pi_1(M)$ which is orientation reversing. It induces an involution f_2 of M_2 so that $M = M_2/f$.

We claim that f leaves invariant the laminations $\widetilde{\mathcal{G}}^*_-, \widetilde{\mathcal{G}}^*_+$. Lift $\mathcal{F}$ to $\mathcal{F}_2$ in M_2. Notice that the universal circle is the same for $\mathcal{F}$ and $\mathcal{F}_2$. Let

$$V_- \;=\; \widetilde{\mathcal{G}}^*_- \cap F^*, \qquad V_+ \;=\; \widetilde{\mathcal{G}}^*_+ \cap F^*$$

as before. Suppose that $\theta(f)(V_-)$ is not equal to V_-. Since V_- is a lamination in F^* with no isolated leaves and finite sided complementary regions, then $\theta(f)(V_-)$ has some transverse intersection with V_-. Let Q be a complementary region of V_- with a boundary leaf l and a leaf l' of $\theta(f)(V_-)$ intersecting l transversely. Let g be a non trivial covering translation in $\pi_1(M_2)$ with $\theta(g)(Q)$ equal to Q and fixing all ideal points of Q. Assume that g acts as a decreasing homeomorphism of $\mathcal{H}$. The arguments in the proof of proposition 6.11, show that the sequence of geodesics $\theta(g^n)(l')$ converges to a geodesic l" which is asymptotic to a leaf r of V_+. This is because the ideal points of l" are attracting fixed points of $\theta(g)$, so the rays in l" are asymptotic to leaves in V_+. We do not know a priori that l'' is a leaf of V_+, it could be a diagonal in a complementary component of V_+.

But since $\pi_1(M_2)$ is normal in $\pi_1(M)$, then for any h in $\pi_1(M_2)$ it follows that $hf = fh'$ for some h' in $\pi_1(M_2)$. Hence

$$hf(\widetilde{\mathcal{G}}^*_-) \;=\; fh'(\widetilde{\mathcal{G}}^*_-) \;=\; f(\widetilde{\mathcal{G}}^*_-).$$

So $\pi_1(M_2)$ preserves $f(\widetilde{\mathcal{G}}^*_-)$ and therefore $\theta(\pi_1(M_2))$ preserves $f(V_-)$. Hence

$$\theta(g^n)(l') \;\in\; f(V_-) \quad \text{and} \quad l\text{"} \;\in\; f(V_-)$$

also. Since r in V_+ and l" in $\theta(f)(V_-)$ are asymptotic, then taking limits this implies that $\theta(f)(V_-)$ and V_+ share a leaf. But $f(\mathcal{G}^*_-)$ and $\mathcal{G}^*_+$ are minimal so $\theta(f)(V_-)$ is equal to V_+. However leaves in $\theta(f)(V_-)$ are still limits of bottoms of sequences of distortion parallelepipeds. As in the proof of proposition 6.7 this contradicts the properties of leaves of V_+ (that is they are limits of tops of sequences of distortion parallelepipeds), because $\theta(f)(V_-)$ is equal to V_+. We conclude that

$$\theta(f)(V_-) \;=\; V_-, \quad \theta(f)(V_+) \;=\; V_+, \quad \text{so} \quad f(\widetilde{\mathcal{G}}^*_-) \;=\; \widetilde{\mathcal{G}}^*_-, \quad f(\widetilde{\mathcal{G}}^*_+) \;=\; \widetilde{\mathcal{G}}^*_+.$$

Therefore $\mathcal{G}^*_-, \mathcal{G}^*_-$ are invariant under f_2 and induce laminations $\mathcal{G}_-, \mathcal{G}_+$ in M which are transverse to $\mathcal{F}$ and to each other and satisfy the properties of lemma 6.11. This finishes the proof of lemma 6.12. □

Remark 1 – By proposition 6.8 the lamination $\mathcal{G}_-$ is never a foliation. Also in the atoroidal case the complementary regions of $\mathcal{G}_-$ are solid tori (or solid Klein bottles). Therefore $\mathcal{G}_-$ is a an essential lamination [Ga-Oe] and it is a genuine lamination [Ga-Ka], that is, complementary regions are not I-bundles. A theorem of Gabai and Kazez [Ga-Ka] then implies that:

Corollary 6.13. *If M is aspherical and has an $\mathbf{R}$-covered foliation, then M satisfies the weak hyperbolization conjecture: either there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$ or $\pi_1(M)$ is Gromov negatively curved.*

Since M is irreducible [Ro] there are many important consequences for the geometry of M. In particular M is conjecturally hyperbolic [Th4].

Remark 2 – The final case to be considered is $\mathcal{F}$ not transversely orientable. Lift to double cover M' so that the lifted $\mathcal{F}'$ is transversely orientable. Using the proposition 6.12 produce laminations $\mathcal{G}_-, \mathcal{G}_+$ in M'. Let $\mathbf{B}$ be $\pi_1(M')$ and f in $(\pi_1(M) - \mathbf{B})$. Then f reverses orientation in $\mathcal{H}$. Hence leaves of $f(V_-)$ are now the limit of thin quadrilaterals which get distorted when moving <u>down</u> transverse to $\widetilde{\mathcal{F}}$. The same arguments as in the previous proposition show that

$$\theta(f)(V_-) \;=\; V_+ \quad \text{and} \quad \theta(f)(V_+) \;=\; V_-,$$

that is, f switches the invariant laminations. This is because f reverses the orientation to $\mathcal{H}$ so something which is limit of bottoms of sequences of distortions parallelepipeds, has image under $\theta(f)$ which is the limit of tops of sequences of distortion parallelepipeds. This would produce a nonorientable line field in the intersection of the two laminations in M, which is tranverse to $\mathcal{F}$. We think this situation in fact cannot occur, but at this point we cannot rule it out. In any case M has a finite regular cover M' with an essential lamination $\mathcal{G}_-$. Remark 1 shows that $\pi_1(M')$ is negatively curved in the large and so is $\pi_1(M)$, because $\pi_1(M')$ has finite index in $\pi_1(M)$ [Gr].

Remark 3 – If there is a $\mathbf{Z} \oplus \mathbf{Z}$ subgroup of $\pi_1(M)$, it can be represented by an <u>immersed</u> incompressible torus T which is in general position with respect to $\mathcal{F}$. Following classical ideas of Thurston [Th1], Roussarie [Rou] and more recently Gabai [Ga5], it follows that T can be put in tight position with respect to $\mathcal{F}$. As $\mathcal{F}$ is $\mathbf{R}$-covered and Reebless it follows that $\mathcal{F}$ is taut [Fe5, Go]. Given that $\mathcal{F}$ is taut Gabai [Ga5] showed that T can be homotoped to be either contained in a leaf of $\mathcal{F}$ or transverse to $\mathcal{F}$ (here T may fail to be embedded!). Taut is used to avoid circles of tangency. In the first case not all leaves of $\mathcal{F}$ are hyperbolic. In the second case T represents a region in leaves whose geometry is only boundedly distorted moving tranversely to the foliation – this explains the dichotomy mentioned in the introduction.

7 The transverse pseudo-Anosov flow

Here $\mathcal{F}$ is a transversely oriented, $\mathbf{R}$-covered foliation with hyperbolic leaves; M homotopically atoroidal. By section 6 there are universal laminations by geodesics $\mathcal{G}_+, \mathcal{G}_-$, transverse to $\mathcal{F}$ and to each other. We use the notations and constructions from the previous sections. A complementary region Q of V_- is an ideal polygon and has associated complementary region Q' of V_+, producing a complementary region

$$P \;=\; Q \cap Q' \quad \text{of} \quad F^* - (V_+ \cup V_-)$$

with compact closure. This region has at least 6 boundary sides, see fig. **??**, a; and there is g in $\pi_1(M)$ with $\theta(g)$ leaving both Q and Q' invariant, hence also leaving their intersection invariant. In M these produce complementary regions of $\mathcal{G}_+ \cup \mathcal{G}_-$ which are solid tori or solid Klein bottles. They are homeomorphic to

$$P \times I/\eta,$$

where η is a homeomorphism of P. All other complementary regions P' of $V_- \cup V_+$ are in "cusps" of V_- and V_+, hence are relatively compact quadrilaterals in F^*. Up to the action of $\pi_1(M)$ there are finitely many complementary components of $(V_- \cup V_+)$ in F^* with 6 sides or more,

There is a flow transverse to $\mathcal{F}$ defined in the intersection of $\mathcal{G}_+$ and $\mathcal{G}_-$: just consider the orientable line field which is the intersection of leaves of $\mathcal{G}_+, \mathcal{G}_-$. Now collapse the complementary regions of $\mathcal{G}_+ \cup \mathcal{G}_-$ along leaves of $\mathcal{F}$ to produce 2 invariant singular foliations in M. In F^* each closure of complementary region of $V_- \cup V_+$ collapses to a point. This produces a flow Φ in M which is transverse to $\mathcal{F}$. The collapsing of $\mathcal{G}_+$ produces the singular foliation $\mathcal{F}^u$ (unstable) and $\mathcal{G}_-$ produces $\mathcal{F}^s$ (stable). This operation of collapsing along leaves was described in great detail in Mosher's articles [Mo1, Mo2]. A complementary region of V_- with p sides (see fig. **??**, a) blows down to a p-prong singular leaf of $\widetilde{\mathcal{F}}^s$ in F^*, see fig. **??**, b. These complementary regions are periodic under some $\theta(g)$ with g in $\pi_1(M)$ and therefore produce closed orbits of Φ in M. The local cross section is given in figure **??**, b. All other points in M are topologically non singular for the flow Φ and foliations $\mathcal{F}^s, \mathcal{F}^u$. The flow lines of Φ are "tangent" to $\mathcal{F}^u, \mathcal{F}^s$. Since the laminations $\mathcal{G}_-, \mathcal{G}_+$ are minimal it follows that all leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ are dense in M. There is a homotopy equivalence $\xi : M \to M$ preserving leaves of $\mathcal{F}$ and sending $\mathcal{G}_+$ to $\mathcal{F}^u$, $\mathcal{G}_-$ to $\mathcal{F}^s$. There is a lift $\widetilde{\xi} : \widetilde{M} \to \widetilde{M}$ preserving leaves of $\widetilde{\mathcal{F}}$ and moving points a bounded distance.

We now show that Φ is a pseudo-Anosov flow.

Definition 7.1. *(topological pseudo-Anosov) A flow φ in a manifold N^3 is a topological pseudo-Anosov flow if there are no point orbits of φ and orbits of φ are contained in two (possibly singular) foliations $\mathcal{E}^s, \mathcal{E}^u$ stable and unstable satisfying:*

1 – All flowlines in a leaf of $\mathcal{E}^s$ are forward asymptotic, all flow lines in a leaf of $\mathcal{E}^u$ are backward asymptotic.

2 – The (topological) singularities of $\mathcal{E}^s, \mathcal{E}^u$ are all of p-prong type. The singular locus is a finite union of closed orbits of φ and p-local leaves of $\mathcal{E}^s$ abut this singular orbit and similarly for $\mathcal{E}^u$.

3 – The foliations $\mathcal{E}^s, \mathcal{E}^u$ are transverse to each other and intersect exactly along the flow lines of φ.

The flow Φ constructed above is transverse to $\mathcal{F}$ and its flow lines are contained in leaves of $\mathcal{F}^s, \mathcal{F}^u$. Under a small perturbation so that Φ is still transverse to $\mathcal{F}$ we can assume that: the orbits of Φ are C^1 and leaves of $\mathcal{F}^s, \mathcal{F}^u$ are C^1 submanifolds in the complement of the singularities and in the singularities we have a standard topological p-prong picture. We stress that is not clear whether these flows can be made "smooth" pseudo-Anosov as defined by Mosher in [Mo2]. In particular it is not clear whether one can define the strong stable/unstable foliations associated to the flow.

Notice that for any g in $\pi_1(M)$, $\theta(g)$ acts on V_- and V_+ hence acts on the points of the intersection. This action is still denoted by $\theta(g)$. A leaf of $\mathcal{F}^s$ or $\widetilde{\mathcal{F}}^s$ is *periodic* if it contains a periodic orbit of Φ or the lift of a periodic orbit. Given $x \in M$ let $W^s(x)$ be the leaf of $\mathcal{F}^s$ containing x and likewise define $W^u(x)$. Let $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u, \widetilde{\Phi}$ be the lifts to $\widetilde{M}$. If $y \in \widetilde{M}$ define $\widetilde{W}^u(y), \widetilde{W}^s(y)$ similarly to the above.

Proposition 7.2. *(topological hyperbolicity) For any two points x, y in a leaf E of $\widetilde{\mathcal{F}}^s$ their orbits are asymptotic in future time. In negative time the distance between orbits converges to infinity, in the intrinsic metric of E. The opposite behavior occurs in leaves of $\mathcal{F}^u$.*

Proof. The dynamics of Φ is entirely encoded by the dynamics of the orientable line field $\mathcal{G}_+ \cap \mathcal{G}_-$. This is what is going to be used here. We first analyse the case that $\pi(E)$ contains a singular orbit of Φ. Then E is a blow down of p leaves of $\widetilde{\mathcal{G}}_-$ in $\widetilde{M}$. Let L in $\widetilde{\mathcal{G}}_-$ be one of them. The intersection l of L and F^* is in the boundary a complementary region Q of V_- and there is an associated complementary region Q' of V_+. There is g in $\pi_1(M)$ non trivial, with g acting as a decreasing homeomorphism of $\mathcal{H}$ and $\theta(g)$ fixing only $\partial_\infty Q$, $\partial_\infty Q'$ in $S^1_\infty(F^*)$, so that points in $\partial_\infty Q'$ are attracting and points in $\partial_\infty Q$ are repelling. This dynamics of $\theta(g)$ is the fundamental point here. There are two boundary leaves of Q' intersecting L, let s be one of them. Let

$$S \;=\; (s \times \mathbf{R}), \;\; \widetilde{\alpha} \;=\; S \cap L \quad \text{and} \quad z \;=\; (S \cap L) \cap F^* \;=\; s \cap l.$$

Also

$$l \subset \partial Q \quad \text{is a leaf of} \quad V_-, \quad s \subset \partial Q' \quad \text{is a leaf of} \quad V_+, \quad s \cap l \neq \emptyset.$$

Then $\theta(g)(s) = s$ and $g(S) = S$. The map g is associated to the closed orbit $\alpha = \pi(\widetilde{\alpha})$ in $\pi(L)$. Orbits of $\widetilde{\Phi}$ in L correspond to leaves of $\widetilde{\mathcal{G}}_+$ intersecting L, so let

$$H \in \widetilde{\mathcal{G}}_+ \quad \text{with} \quad H \cap L = \widetilde{\gamma},$$

orbit of the flow. Start with the intersection a_0 of H and L at level F^* - which is a point in l. Go up to $g^{-1}(F^*)$ along the flow line $\widetilde{\gamma}$. Mapping the intersection of $\widetilde{\gamma}$ and $g^{-1}(F^*)$ down by g produces a point a_1 in l - this is like the first return map associated to the closed orbit α. The action of $\theta(g)$ in $S^1_\infty(F^*)$ moves the ideal points of of $H \cap F^*$ closer to the ideal points of s, because the ideal points of s are in $\partial_\infty Q'$ and are attracting for $\theta(g)$. This implies that a_1 is closer to z than a_0 is. This is exactly the same argument as in step 7 of the proof of theorem 5.1. Iterating this procedure the images in l converge to z, that is,

$$a_n = g^n(\widetilde{\gamma} \cap g^{-n}(F^*))$$

converges to z. Hence in $\widetilde{M}$ the orbit $\widetilde{\gamma}$ is asymptotic to the orbit $\widetilde{\alpha}$ in the forward direction as one moves up. All orbits in L on that side of S are asymptotic to the orbit $\widetilde{\alpha}$. Orbits through the corners of $Q \cap Q'$ collapse to a single orbit in the blow down. Hence all orbits are asymptotic to the closed orbit in $\pi(E)$ after the collapsing. This proves the result for singular leaves. The key is the action of $\theta(g)$ in $\mathcal{U}$.

If now ζ is any periodic orbit of Φ, which is non singular then $\widetilde{\zeta}$ is the intersection of two leaves of $\widetilde{\mathcal{F}}^s$ and of $\widetilde{\mathcal{F}}^u$ which come from unique leaves

$$L \text{ of } \widetilde{\mathcal{G}}_- \quad \text{and} \quad S \text{ of } \widetilde{\mathcal{G}}_+.$$

Let

$$l = L \cap F^*, \quad s = S \cap F^*, \quad \widetilde{\alpha} = S \cap L$$

and let b_1, b_2 the ideal points of l. Let g in $\pi_1(M)$ non trivial with $g(\widetilde{\alpha})$ equal to $\widetilde{\alpha}$ so that g leaves both components of $L - S$ invariant. Then $\theta(g)$ has at least 4 fixed points and therefore acts freely in $\mathcal{H}$, assume as a decreasing homeomorphism. Given that, use an analysis similar to the proof of lemma 6.6: let $\mathcal{R}_i$ be quadrilaterals converging to a leaf l_0 of V_- near but not equal to l. The analysis of lemma 6.6 shows that

$$\theta(g) \text{ is a contraction in that interval of } \mathcal{U} - \{b_1, b_2\}.$$

Therefore $\theta(g)$ has exactly 4 fixed points in $\mathcal{U}$ which are the ideal points of l and s. Then the same analysis as in the singular case yields that all orbits in L are forward asymptotic to $\widetilde{\alpha}$ so after collapse all orbits are forward asymptotic to $\widetilde{\zeta}$.

This takes care of periodic leaves of $\widetilde{\mathcal{F}}^s, \widetilde{\mathcal{F}}^u$. We now deal with general leaves. Notation: if x, y are in the same leaf of the intersection of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}^s$, let $d'(x, y)$ be their distance along that leaf. Consider x, y in the same leaf $F \cap E$ where F is a leaf of $\widetilde{\mathcal{F}}$ and E a leaf of $\widetilde{\mathcal{F}}^s$. If the orbits of $\widetilde{\Phi}$ through x, y are not asymptotic in future time we can find positive a_0 so that

$$\widetilde{\Phi}_{t_i}(x), \ \widetilde{\Phi}_{s_i}(y) \in F_i \in \widetilde{\mathcal{F}}, \quad d'(\widetilde{\Phi}_{t_i}(x_i), \widetilde{\Phi}_{s_i}(y)) > a_0 \quad \text{and} \quad t_i, s_i \to +\infty.$$

Hence we can find segments β_i in the intersection of leaves of $\widetilde{\mathcal{F}}^s$ with leaves of $\widetilde{\mathcal{F}}$ with endpoints x_i, y_i in leaves of $\widetilde{\mathcal{F}}$ which have length of β_i converging to a_0 and so that

$$\widetilde{\Phi}_{-t_i}(x_i), \widetilde{\Phi}_{-s_i}(y_i) \in F, \quad \text{and} \quad d'(\widetilde{\Phi}_{-t_i}(x_i), \widetilde{\Phi}_{-s_i}(y_i)) < a_1, \quad \text{for some fixed} \quad a_1 > 0.$$

Up to subsequence there are covering translations h_i with $h_i(\beta_i)$ converging to β_0.

Since a periodic leaf of $\mathcal{F}^u$ is dense in M let γ orbit of $\widetilde{\Phi}$ with $\pi(\gamma)$ periodic and non singular so that $\widetilde{W}^u(\gamma)$ intersects β_0, see fig. ??. Hence for i big $\widetilde{W}^u(\gamma)$ and $h_i(\beta_i)$ intersect. Making β_i, β_0 smaller if necessary we may assume that: for any p in β_i, the leaf $\widetilde{W}^u(p)$ does not intersect a singular orbit between $\widetilde{W}^s(\beta_i)$ and $\widetilde{W}^s(\gamma)$. Equivalently $\widetilde{W}^u(p)$ intersects $\widetilde{W}^s(\gamma)$. Now flow back. The points in β_0 flow back very near $\widetilde{W}^s(\gamma)$ and from then on always near $\widetilde{W}^s(\gamma)$, see fig. ??. So for i big $h_i(\beta_i)$ also does. Flowing backwards in $\widetilde{W}^s(\gamma)$, the segments which intersect γ in the interior blow up, because all orbits in $\widetilde{W}^s(\gamma)$ are forward asymptotic to γ and $\widetilde{W}^s(\gamma)$ is periodic $-$ hence the lengths will blow up past $2a_1$ and will never again be smaller than $2a_1$. Therefore nearby segments obtained flowing back pieces of $h_i(\beta_i)$ will also have big length. But the segments $h_i(\beta_i)$ flow back to segments of length smaller than a_1 for arbitrarily long time when i is big $-$ hence this is a contradiction. Hence orbits in E are forward asymptotic.

The same argument shows that flowing in the negative direction blows up distance along stable leaves without bound $-$ because this happens in periodic leaves and then use the argument above of the intersection with $\widetilde{W}^u(\gamma)$. This finishes the proof of the proposition. $\square$

Finally we get the metric pseudo-Anosov property for the flow Φ.

Proposition 7.3. *(metric hyperbolic) For every positive a_2, there is positive a_3 so that: let β be a segment in the intersection of a leaf of $\widetilde{\mathcal{F}}$ with a leaf of $\widetilde{\mathcal{F}}^u$ of length at least a_2. Flow forward every point of β to obtain another segment β' in another leaf of $\widetilde{\mathcal{F}}$. If every point of β moves at least a_3 flow length, then the length of the final segment is at least double the length of β.*

Proof. Otherwise get segments β_i so that there are longer and longer times so that length of flow of β_i is smaller than $2a_2$. Call these segments in leaves F_i of $\widetilde{\mathcal{F}}$ to be ζ_i. Then the ζ_i have length less than $2a_2$ and it takes longer and longer for them in the negative direction to decrease to length a_2. The arguments in the previous proposition disallow this. This finishes the proof. $\square$

Remark: A very important question is to analyse geometric properties of the transverse flow Φ. For instance is the flow quasigeodesic? That means flow lines of $\widetilde{\Phi}$ are uniformly efficient in measuring distance [Fe2]. This has several important consequences, for intance the continuous extension property for leaves of $\widetilde{\mathcal{F}}$ [Ca-Th]. When $\mathcal{G}$ is an uniform foliation, it is very easy to see that Φ is quasigeodesic, because Φ is regulating [Th7]. In general this is an open question.

References

[An] D. V. Anosov, *Geodesic flows on closed Riemannian manifolds with negative curvature*, Proc. Steklov Inst. Math. **90** (1969).

[An-Si] D. V. Anosov and Y. Sinai, *Some smoothly ergodic systems*, Russian Math. Surveys **22** (1967) 5 103-167.

[Ba1] T. Barbot, *Caractérization des flots d'Anosov en dimension 3 par leurs feuilletages faibles*, Ergod. Th. Dynam. Sys. **70** (1995) 247-270.

[Ba2] T. Barbot, *Flots d'Anosov sur les variétés graphées au sens de Waldhausen"*, Ann. Inst. Fourier Grenoble **46** (1996) 1451-1517.

[Ba3] T. Barbot, *Generalizations of the Bonatti-Langevin example and their classification up to topological conjugacy*, Comm. Anal. Geom. **6** (1998) 749-798.

[Bl-Ca] S. Bleiler and A. Casson, *Automorphisms of surfaces after Nielsen and Thurston*, Cambridge Univ. Press, 1988.

[Br] M. Brittenham, *Essential laminations in Seifert fibered spaces*, Topology **32** (1993) 61-85.

[Cal1] D. Calegari, *The geometry of* **R***-covered foliations*, Geom. Topol. **4** (2000) 457-515 (eletronic).

[Cal2] D. Calegari, **R***-covered foliations of hyperbolic* 3*-manifolds*, Geom. Topol. **3** (1999) 137-153 (eletronic).

[Can] A. Candel, *Uniformization of surface laminations*, Ann. Sci. Ecole Norm. Sup. (4) **26** (1993) 489-516.

[Ca-Th] J. Cannon and W. Thurston, *Group invariant Peano curves*, preprint, 1987.

[Ca-Co] J. Cantwell and L. Conlon, *Smoothability of proper foliations*, Ann. Inst. Fourier (Grenoble) **38** (1988) 219-244.

[Ca-Ju] A. Casson and D. Jungreis, *Convergence groups and Seifert fibered manifolds*, Inv. Math. **118** (1994) 441-456.

[De-Ro] C. Delman and R. Roberts, *Alternating knots satisfy strong property P*, Comm. Math. Helv. **74** (1999) 376-397.

[Di] P. Dippolito, *Codimension one foliations of closed manifolds*, Ann. of Math. **107** (1978) 403-453.

[Ep] D. Epstein, *Transversely hyperbolic* 1*-dimensional foliations. Transversal structure of foliations*, Astérisque **116** (1984) 53-69.

[FLP] A. Fathi, Laudenbach, Poenaru, *Travaux de Thurston sur les surfaces*, Astérisque, Societé Mathématique de France, **66-67** (1979).

[Fe1] S. Fenley, *Quasi-isometric foliations*, Topology **31** (1992) 667-676.

[Fe2] S. Fenley, *Anosov flows in* 3*-manifolds*, Ann. of Math. **139** (1994) 79-115.

[Fe3] S. Fenley, *Quasigeodesic Anosov flows and homotopic properties of closed orbits*, Jour. Diff. Geo. **41** (1995) 479-514.

[Fe4] S. Fenley, *The structure of branching in Anosov flows of* 3*-manifolds*, Comm. Math. Helv. **73** (1998) 259-297.

[Fe5] S. Fenley, *Regulating pseudo-Anosov flows and topology of foliations*, submitted.

[Fe6] S. Fenley, *Pseudo-Anosov flows and incompressible tori*, submitted.

[Fe7] S. Fenley, *Topological rigidity of pseudo-Anosov flows transverse to foliations*, in preparation.

[Ga1] D. Gabai, *Foliations and the topology of 3-manifolds*, J. Diff. Geo. **18** (1983) 445–503.

[Ga2] D. Gabai, *Foliations and the topology of 3-manifolds II*, J. Diff. Geo. **26** (1987) 461–478.

[Ga3] D. Gabai, *Foliations and the topology of 3-manifolds III*, J. Diff. Geo. **26** (1987) 479–536.

[Ga4] D. Gabai, *Convergence groups are Fuchsian groups*, Ann. of Math. **136** (1992) 447-510.

[Ga5] D. Gabai, *Combinatorial volume preserving flows and taut foliations*, Comm. Math. Helv. **75** (2000) 109-124.

[Ga-Ka] D. Gabai and W. Kazez, *Group negative curvature for* 3*-manifolds with genuine laminations*, Geom. Topol. **2** (1998) 65-77.

[Ga-Oe] D. Gabai and U. Oertel, *Essential laminations and* 3*-manifolds*, Ann. of Math. **130** (1989) 41-73.

[Ge-Ma] F. Gehring and G. Martin, *Discrete quasiconformal groups*, Proc. London Math. Soc. **55** (1987) 331-358.

[Gh] E. Ghys, *Flots d'Anosov sur les* 3*-variétés fibrés en cercles*, Erg. Th. Dyn. Sys. **4** (1984) 67-80.

[Go] S. Goodman, *Closed leaves in foliations of codimension 1*, Comm. Math. Helv. **50** (1975) 383-388.

[Go-Sh] S. Goodman and S. Shields, *A condition for the stability of* **R***-covered foliations of* 3*-manifolds*, Trans. A.M.S., to appear.

[Gr] M. Gromov, *Hyperbolic groups*, in Essays on Group theory, Springer-Verlag, 1987, pp. 75-263.

[Ha] A. Haefliger, *Variétës feuilleteés*, Ann. Scuola Norm. Sup. Pisa (3) **16** (1962) 367-397.

[He] J. Hempel, *3-manifolds*, Ann. of Math. Studies 86, Princeton University Press, 1976.

[Hi1] Hinkkanen, *The structure of certain quasisymmetric groups*, Mem. A.M.S. **83** (1990) number 422.

[Hi2] Hinkkanen, *Uniformly quasisymmetric groups*, Proc. London Math. Soc. **51** (1985) 318-338.

[Le] O. Lehto, *Univalent functions and Teichmuller spaces*, Graduate Texts in Mathematics **109** Springer, 1987.

[Li] R. Lickorish, *A foliation for* 3*-manifolds* Ann. of Math. **82** (1965) 414-420.

[Mo1] L. Mosher, *Dynamical systems and the homology norm of a* 3*-manifold I. Efficient interesection of surfaces and flows*, Duke Math. Jour. **65** (1992) 449–500.

[Mo2] L. Mosher, *Laminations and flows transverse to finite depth foliations*, Part I: Branched surfaces and dynamics – available from http://newark.rutgers.edu:80/ mosher/, Part II in preparation.

[No] S. P. Novikov, *Topology of foliations*, Trans. Moscow Math. Soc. **14** (1963) 268-305.

[Pa] F. Palmeira, *Open manifolds foliated by planes*, Ann. of Math., **107** (1978) 109-131.

[Pl1] J. Plante, *Foliations with measure preserving holonomy*, Ann. of Math. **107** (1975) 327-361.

[Pl2] J. Plante, *Anosov flows, transversely affine foliations and a conjecture of Verjovsky*, Jour. London Math. Soc. **83** (1981) 359-362.

[Re] G. Reeb, *Sur certaines propriétés topologiques de variétés feuilleteées*, Actualités Sci. Indust. **1183** Herman Paris (1952) 91-158.

[Ro] H. Rosenberg, *Foliations by planes*, Topology **7** (1968) 131-138.

[Rob] R. Roberts, *Constructing taut foliations*, Comm. Math. Helv. **70** (1995) 516-545.

[Rou] Roussarie, *Plongements dans les variétées feuilletées et classification de feuilletages sans holonomie*, Publ. Math. I.H.E.S. **43** (1973) 101-142.

[Sc] P. Scott, *There are no fake Seifert fibered spaces*, Ann. of Math. **117** (1983) 35-70.

[So] V. Solodov, *Components of topological foliations*, Mat. Sb. **119** (161) (1982) 340-354. English translation in Math. U.S.S.R. Sb **47** (1984) 329-343.

[Sul] D. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976) 225-255.

[Th1] W. Thurston, *Foliations of* 3*-manifolds which are circle bundles*, Thesis, University of California, Berkeley, 1972.

[Th2] W. Thurston, *On the geometry and dynamics of diffeomormphisms of surfaces*, Bull. A.M.S. **19** (1998) 417-431.

[Th3] W. Thurston, *The geometry and topology of 3-manifolds*, Princeton University Lecture Notes, 1982.

[Th4] W. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. A.M.S. **6** (1982) 357-381.

[Th5] W. Thurston, *Hyperbolic structures on* 3*-manifolds II: Surface groups and* 3*-manifolds that fiber over the circle*, preprint.

[Th6] W. Thurston, *Earthaquakes in two dimensional hyperbolic geometry*, in London Math. Soc. Lecture Notes series, **113** (1986) 91-112.

[Th7] W. Thurston, *Three manifolds, foliations and circles I*, preprint 1997.

[Th8] W. Thurston, *Three manifolds, foliations and circles, II. Transverse asymptotic geometry of foliations*, in preparation.

[Th9] W. Thurston, *Private communication, 1997.*

[Th10] W. Thurston, *The universal circle for foliations and transverse laminations*, lectures in M.S.R.I., 1997.

[Tu] P. Tukia, *Homeomorphic conjugates of Fuchsian groups*, J. Reigne Angew. Math. **391** (1988) 1-54.

[Ve] A. Verjovsky, *Codimension one Anosov flows*, Bol. Soc. Mat. Mex. **19** (1977) 49-77.

[Wa] K. Waldhausen, *On irreducible* 3*-Manifolds which are sufficiently large*, Ann. of Math. **87** (1968) 56-88.

[Wo] J. Wood, *Foliations on* 3*-manifolds*, Ann. of Math. **89** (1969) 336-358.

Florida State University
Tallahassee, FL 32306-4510