CAA Workshop Problem
Proposer: Tom Hagstrom

Goal: Detailed study of the accuracy of boundary treatments with a range of incidence angles including shear and a sonic point.

Part 1: Uniform Flow We take the two-dimensional linearized Euler equations in dimensionless form, first assuming a uniform base flow:

$$\frac{\partial \rho}{\partial t} + (U \cdot \nabla)\rho + \nabla \cdot u = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + (U \cdot \nabla)u + \nabla p = 0, \tag{2}$$

$$\frac{\partial p}{\partial t} + (U \cdot \nabla)p + \nabla \cdot u = 0. \tag{3}$$

The base flow is given by:

$$U_1 = 0.3, \quad U_2 = 0.4, \tag{4}$$

and initial conditions, $\rho(x_1, x_2, 0)$, $u_j(x_1, x_2, 0)$ and $p(x_1, x_2, 0)$ are provided which lead to propagating sound, vorticity and entropy waves. The initial conditions are supported in $x_1 \in (-2, 2)$; that is

$$\rho(x_1, x_2, 0), u_j(x_1, x_2, 0), p(x_1, x_2, 0) = 0, \quad |x_1| \geq 2. \tag{5}$$

We take the computational domain to be:

$$(x_1, x_2) \in (-2, 2) \times (0, 1), \tag{6}$$

with periodic boundary conditions in x_2. That is we have the periodic boundary conditions:

$$\rho(x_1, 1, t) = \rho(x_1, 0, t), \quad u_j(x_1, 1, t) = u_j(x_1, 0, t), \quad p(x_1, 1, t) = p(x_1, 0, t). \tag{7}$$

The solvers can use their favorite radiation boundary conditions or absorbing layers at the inflow and outflow boundaries, $x_1 = \pm 2$.

The configuration of the problem is shown in Figure 1.

The problem is to solve up to $t = 64$, reporting relative L^2 errors measured on a uniform 129×33 mesh. The initial data and solution are 1-periodic in x_2 and will be provided on a uniform 513×129 mesh. (People who wish to use a finer mesh can interpolate. We have found that the data on a 257×65 mesh is accurate enough to generate numerical solutions with $7 - 8$ digits of accuracy.) Full details on obtaining the initial and solution data and reporting results is given below.

We note that the difficulty of the problem, from the point of view of boundary conditions, is that for late times sound waves incident on the boundary are produced by far away image sources, and thus the angles of incidence become more and more glancing.

Part 2: Subsonic Couette Flow

For the second part we make the base flow subsonic Couette flow:

$$U_1 = Mx_2, \quad M = 0.9, \quad U_2 = 0, \tag{8}$$

with the periodicity conditions replaced by the wall condition:

$$u_2 = 0, \quad x_2 = 0, 1. \tag{9}$$

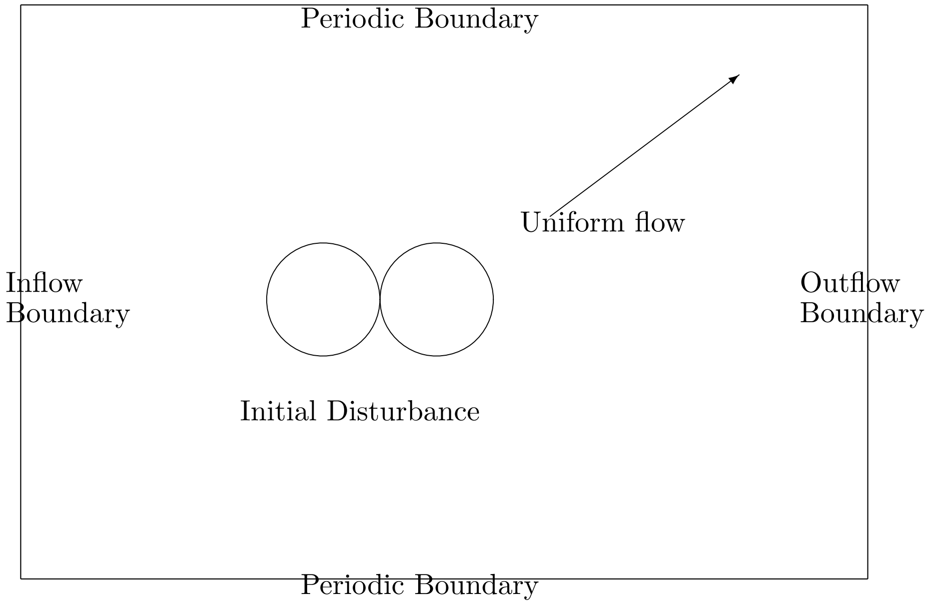

Figure 1: Computational Domain for Problem 1

Thus the governing equations are:

$$\frac{\partial \rho}{\partial t} + Mx_2 \frac{\partial \rho}{\partial x_1} + \nabla \cdot u = 0, \tag{10}$$

$$\frac{\partial u}{\partial t} + Mx_2 \frac{\partial u}{\partial x_1} + (Mu_2 \quad 0)^T + \nabla p = 0, \tag{11}$$

$$\frac{\partial p}{\partial t} + Mx_2 \frac{\partial p}{\partial x_1} + \nabla \cdot u = 0. \tag{12}$$

Again the solver may choose any method for specifying boundary conditions at inflow and outflow, $x_2 = \pm 2$. The domain configuration for Problems 2 and 3 is represented in Figure 2.

Part 3: Transonic Couette Flow

The third example is the same as the second except that we use (8) with:

$$M = 1.2. \tag{13}$$

We note that the new features introduced in Problems 2 and 3 are the presence of shear and a sonic point at the inflow and outflow boundaries.

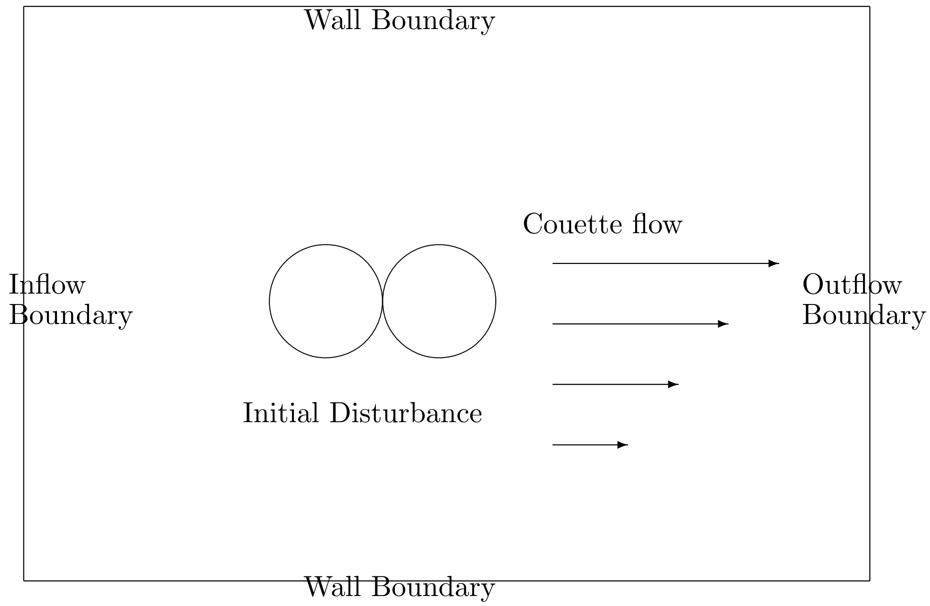

Figure 2: Computational Domain for Problems 2 and 3

DETAILED PROCEDURES

Initial and solution data can be obtained from the URL:

http://www.math.unm.edu/~ hagstrom/CAAWorkshop

Each line of each file contains 8 numbers:

$$i_1 \quad i_2 \quad x_1 \quad x_2 \quad u_1 \quad u_2 \quad p \quad \rho;$$

the first two are integer indices associated with a data point, the second two are the double precision coordinates of the data point, and the last four are the double precision values of the fields. Each file is identified by the subproblem: Pr1,Pr2,Pr3, and the time: t00,t01,t02,t04,...,t64. Thus the file Pr1.t00 contains the initial data for Problem 1 on the grid:

$$(x_{1,i_1}, x_{2,i_2}), \quad x_{1,i_1} = -2 + (i_1 - 1)/128, \quad x_{2,i_2} = (i_2 - 1)/128. \tag{14}$$

Here $1 \leq i_1 \leq 513$ and $1 \leq i_2 \leq 129$. Similarly, the file Pr3.t24 contains the solution of Problem 3 at $t = 24$ on the coarser grid:

$$(x_{1,i_1}, x_{2,i_2}), \quad x_{1,i_1} = -2 + (i_1 - 1)/32, \quad x_{2,i_2} = (i_2 - 1)/32, \tag{15}$$

$$1 \leq i_1 \leq 129, \quad 1 \leq i_2 \leq 33. \tag{16}$$

The times are:

$$t_1 = 1, \quad t_2 = 2, \quad t_k = 4(k-2), \quad k = 3, \ldots 18. \tag{17}$$

As the files are large and numerous, you may prefer to download the tarred versions, which are: Pr1.tgz, Pr2.tgz and Pr3.tgz.

You may use **any mesh you like** for solving the problem, interpolating between your mesh and the one on which I have given the data. Information which you should include in describing your solution is:

(i.) The discretization method used.

(ii.) The grid and time step.

(iii.) The detailed treatment of the artificial inflow and outflow boundaries, including the number of grid points used in any absorbing layers, absorption profiles, boundary condition orders, etc.

(iv.) Relative l_2 error data for all four fields at each of the eighteen time stations, t_k, computed on the 129×33 mesh. Precisely:

$$e_\rho(t_k) = \frac{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}\left(\rho_{\text{exact}}(x_{1,i_1},x_{2,i_2},t_k) - \rho_{\text{comp}}(x_{1,i_1},x_{2,i_2},t_k)\right)^2\right)^{1/2}}{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}\rho^2_{\text{exact}}(x_{1,i_1},x_{2,i_2},t_k)\right)^{1/2}}, \tag{18}$$

$$e_{u_1}(t_k) = \frac{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}\left(u_{1,\text{exact}}(x_{1,i_1},x_{2,i_2},t_k) - u_{1,\text{comp}}(x_{1,i_1},x_{2,i_2},t_k)\right)^2\right)^{1/2}}{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}u^2_{1,\text{exact}}(x_{1,i_1},x_{2,i_2},t_k)\right)^{1/2}}, \tag{19}$$

$$e_{u_2}(t_k) = \frac{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}\left(u_{2,\text{exact}}(x_{1,i_1},x_{2,i_2},t_k) - u_{2,\text{comp}}(x_{1,i_1},x_{2,i_2},t_k)\right)^2\right)^{1/2}}{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}u^2_{2,\text{exact}}(x_{1,i_1},x_{2,i_2},t_k)\right)^{1/2}}, \tag{20}$$

$$e_p(t_k) = \frac{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}\left(p_{\text{exact}}(x_{1,i_1},x_{2,i_2},t_k) - p_{\text{comp}}(x_{1,i_1},x_{2,i_2},t_k)\right)^2\right)^{1/2}}{\left(\sum_{i_1=1}^{129}\sum_{i_2=1}^{33}p^2_{\text{exact}}(x_{1,i_1},x_{2,i_2},t_k)\right)^{1/2}}. \tag{21}$$

Recall that the times are $t_1 = 1$, $t_2 = 2$, $t_k = 4(k-2)$, $k = 3, \ldots, 18$.

Solutions as well as any questions concerning the problem should be emailed to me at:

hagstrom@math.unm.edu.

At the URL mentioned above there is also a README file with additional information and contour plots of the solution fields at each time. PLEASE DO NOT HESITATE TO CONTACT ME IF YOU HAVE ANY QUESTIONS OR DIFFICULTIES.

Happy solving!