ON A BURGERS' TYPE EQUATION

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ABSTRACT. In this paper we study the dynamics of a Burgers' type equation (1.1). First, we use a new method called attractor bifurcation introduced by Ma and Wang in [4, 6] to study the bifurcation of Burgers' type equation out of the trivial solution. For Dirichlet boundary condition, we get pitchfork attractor bifurcation as the parameter λ crosses the first eigenvalue. For periodic boundary condition, we get bifurcated S^1 attractor consisting of steady states. Second, we study the long time behavior of the equation. We show that there exists a global attractor whose dimension is at least of the order of $\sqrt{\lambda}$. Thus it provides another example of extended system (see (1.2)) whose global attractor has a Hausdorff/fractal dimension that scales at least linearly in the system size while the long time dynamics is non-chaotic.

1. INTRODUCTION

In this paper, we study the dynamics of the following Burgers' type equation on $[0, 1]$

$$\begin{cases} u_t = u_{xx} + \lambda u - \lambda u u_x, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where λ is a positive real parameter. This Burgers' type equation can be casted into an extended system on the interval $[0, \sqrt{\lambda}]$ via a change of variable $X = \sqrt{\lambda}x, \quad \tau = \lambda t, \quad U = \sqrt{\lambda}u,$

$$U_\tau = U_{XX} + U - UU_X. \tag{1.2}$$

Equation (1.1) can be derived by differentiating the following Burgers' type equation with respect to spatial variable x and changing a variable $t = \tau/\lambda$

$$\frac{\partial v}{\partial \tau} - \frac{1}{2}v_x^2 = \frac{1}{\lambda}v_{xx} + v - < v >, \tag{1.3}$$

where $< v >= \int_0^1 v(x, \tau)dx$. The interested reader is referred to ([1]) for the physical relevance of this equation in flame front propagation and results on steady states and their stabilities.

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There are two goals in this manuscript. First, we use a new method called attractor bifurcation developed in [4, 6] to study the bifurcation of this Burgers' type equation (1.1) with either the Dirichlet boundary condition

$$u(0,t) = u(1,t) = 0, \tag{1.4}$$

or the periodic boundary condition

$$u(x+1,t) = u(x,t). \tag{1.5}$$

Second, we study the long time dynamics of the Burgers' type equation. We show that there exists a global attractor whose dimension grows at least linearly to the square root of the parameter λ. This tells us that the dimension of the global attractor scales at least linearly with the length (volume) of the system in the alternative form (1.2). On the other hand, it is known that the Burgers' type equation cannot possess chaos ([8]). Thus, the Burgers' type equation provides an example of extended system whose dimension of attractor scales at least linearly with the volume (length) of the system while the long time dynamics is non-chaotic. This tells us that the criterion of dimension of attractor proportional to the system size itself is not sufficient to guarantee chaos, let alone extensive chaos ([2], [10]).

We recall the general setting and definition of attractor bifurcation in Section 2.1. In Section 2.2, we recall a criterion to determine the asymptotic stability which is useful when attempting to apply the attractor bifurcation theorem, Theorem 2.3. In Section 2.3, we recall generalized center manifold theory as an important tool to reduce the dynamic bifurcation equations and we demonstrate the skill to compute the approximations up to third order terms which makes things clear when we want to specify the bifurcation type and determine the structure of the bifurcation.

The main results of the first part are put in Section 3. Theorem 3.1 states that pitch-fork attractor bifurcated from trivial solution occurs when the parameter λ crosses π^2 in Dirichlet boundary condition case. For periodic boundary condition, we get S^1 attractor consisting of steady states bifurcated from trivial solution when λ crosses first eigenvalue $4\pi^2$. We achieve this result in Theorem 3.2 , 3.3. Namely, in Theorem 3.2, we look for solutions in the odd function space to get the approximation of bifurcated solutions. In Theorem 3.3, we prove the existence of the attractor bifurcation in general case, and get the bifurcation structure based on the result of Theorem 3.2 and the translation invariance of Burgers' type equation. Finally, we extend this result to Theorem 3.4 which asserts S^1 invariance set bifurcation occurs when λ crosses each eigenvalue as a completion of the story.

The main results of the second part are contained in Section 4 where we demonstrate that the system is dissipative and possess a global attractor utilizing a technique similar to those used by Nicolaenko, Scheurer and Temam ([9]). We also show that the Hausdorff and fractal dimension of the attractor grows at least linearly in $\sqrt{\lambda}$. In another word, in the form of extended system of (1.2) on the interval $[0, \sqrt{\lambda}]$, the dimension of the global attractor scales at least linearly in the volume of the system. This is achieved via an upper bound utilizing the Constantin-Foias form of the Kaplan-Yorke formula in terms of global Lyapunov exponents ([9]), and a lower bound in terms of the dimension of the unstable manifold associated with the trivial solution. Nevertheless, the dynamics of the extended system is non-chaotic due to a result of Matano ([8])

2. Dynamic Bifurcation Theory

2.1. **Attractor Bifurcation.** We recall in this section a general theory on attractor bifurcation developed in [4, 6].

Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the following nonlinear evolution equations.

$$\begin{cases} \dfrac{du}{dt} = L_\lambda u + G(u,\lambda), \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $u : [0,\infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$, is the system parameter, and $L_\lambda : H_1 \to H$ are parametrized linear completely continuous fields continuously depending on $\lambda \in \mathbb{R}^1$, which satisfy

$$\begin{cases} L_\lambda = -A + B_\lambda & \text{a sectorial operator,} \\ A : H_1 \to H & \text{a linear homeomorphism,} \\ B_\lambda : H_1 \to H & \text{the parametrized linear compact operators.} \end{cases} \tag{2.2}$$

It is easy to see that L_λ generates an analytic semi-group $\{e^{-tL_\lambda}\}_{t\geq 0}$. Then we can define fractional power operators L_λ^α for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_1 > \alpha_2$, and $H_0 = H$.

Furthermore, we assume that the nonlinear terms $G(\cdot,\lambda) : H_\alpha \to H$ for some $1 > \alpha \geq 0$ are a family of parametrized C^r bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}^1$ such that

$$G(u,\lambda) = o(\|u\|_{H_\alpha}), \quad \forall\ \lambda \in \mathbb{R}^1. \tag{2.3}$$

In general, we are interested in the sectorial operator $L_\lambda = -A + B_\lambda$ such that there exist an eigenvalue sequence $\{\rho_k\} \subset \mathbb{C}$ and an eigenvector sequence $\{e_k, h_k\} \subset H_1$ of A:

$$\begin{cases} Az_k = \rho_k z_k,\ z_k = e_k + ih_k, \\ Re\rho_k \to \infty,\ \text{as } k \to \infty, \\ |Im\rho_k/(Re\rho_k + a)| \leq C,\ \text{for some constants } a, C > 0, \end{cases} \tag{2.4}$$

such that $\{e_k, h_k\}$ is a basis of H.

Condition (2.4) implies that A is a sectorial operator, hence we can define fractional power operator A^α with domain $H_\alpha = D(A^\alpha)$. For the operator

$B_\lambda : H_1 \longrightarrow H$, we assume that there is a constant $0 < \theta < 1$ such that

$$B_\lambda : H_\theta \longrightarrow H \text{ bounded, } \forall\, \lambda \in \mathbb{R}. \tag{2.5}$$

Let $\{S_\lambda(t)\}_{t\geq 0}$ be an operator semi-group generated by the equation (2.1), then the solution of (2.1) can be expressed as

$$u(t) = S_\lambda(t)u_0, \qquad t \geq 0.$$

Definition 2.1. *A set $\Sigma \subset H$ is called an invariant set of (2.1) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (2.1) is said to be an attractor if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\varphi \in U$ we have*

$$\lim_{t\to\infty} \operatorname{dist}_H(u(t,\varphi), \Sigma) = 0. \tag{2.6}$$

The largest open set U satisfying (2.6) is called the basin of attraction of Σ.

Definition 2.2. (1) *We say that the equation (2.1) bifurcates from $(u,\lambda) = (0,\lambda_0)$ to an invariant set Ω_λ, if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (2.1) such that $0 \notin \Omega_{\lambda_n}$, and*

$$\lim_{n\to\infty} \lambda_n = \lambda_0,$$
$$\lim_{n\to\infty} \max_{x\in\Omega_{\lambda_n}} |x| = 0.$$

(2) *If the invariant sets Ω_λ are attractors of (2.1), then the bifurcation is called attractor bifurcation.*
(3) *If Ω_λ are attractors and are homotopically equivalent to an m–dimensional sphere S^m, then the bifurcation is called S^m–attractor bifurcation.*

A complex number $\beta = \alpha_1 + i\alpha_2 \in \mathbb{C}$ is called an eigenvalue of $L_\lambda : H_1 \to H$ if there are $x, y \in H_1$ such that

$$L_\lambda z = \beta z,\ z = x + iy.$$

Now let the eigenvalues (counting the multiplicity) of L_λ be given by

$$\beta_1(\lambda), \beta_2(\lambda), \cdots, \beta_k(\lambda), \cdots \in \mathbb{C}.$$

Suppose that

$$Re\beta_i(\lambda) = \begin{cases} < 0 & \text{if } \lambda < \lambda_0 \\ = 0 & \text{if } \lambda = \lambda_0 \\ > 0 & \text{if } \lambda > \lambda_0 \end{cases} \qquad (1 \leq i \leq m) \tag{2.7}$$

$$Re\beta_j(\lambda_0) < 0. \qquad \forall\, m+1 \leq j. \tag{2.8}$$

Let the eigenspace of L_λ at λ_0 be

$$E_0 = \bigcup_{1\leq j\leq m} \bigcup_{k=1}^{\infty} \{u, v \in H_1 \mid (L_{\lambda_0} - \beta_j(\lambda_0))^k w = 0, w = u + iv\}.$$

It is known that $\dim E_0 = m$.

Theorem 2.3 (T. Ma and S. Wang [4]). *Assume that the conditions (2.2)-(2.8) hold true, and $u = 0$ is a locally asymptotically stable equilibrium point of (2.1) at $\lambda = \lambda_0$. Then the following assertions hold true.*

(1) *(2.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to attractors Σ_λ for $\lambda > \lambda_0$, with $m - 1 \leq dim\Sigma_\lambda \leq m$, which is connected as $m > 1$;*
(2) *The attractor Σ_λ is a limit of a sequence of m-dimensional annulus M_k with $M_{k+1} \subset M_k$; especially if Σ_λ is a finite simplicial complex, then Σ_λ has the homotopy type of S^{m-1};*
(3) *For any $u_\lambda \in \Sigma_\lambda$, u_λ can be expressed as*

$$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}),\ v_\lambda \in E_0;$$

(4) *There is an open set $U \subset H$ with $0 \in U$ such that the attractor Σ_λ bifurcated from $(0, \lambda_0)$ attracts $U \backslash \Gamma$ in H, where Γ is the stable manifold of $u = 0$ with co-dimension m.*

Next, we mention a sufficient condition which implies that the bifurcated attractor Σ_λ of (2.1) from an eigenvalue with multiplicity two is homeomorphic to a circle S^1. Let v be a two-dimensional $C^r(r \geq 1)$ vector field given by

$$v_\lambda(x) = \lambda x - G(x, \lambda), \tag{2.9}$$

for $x \in \mathbb{R}^2$. Here

$$G(x, \lambda) = G_k(x, \lambda) + o(|x|^k),$$

where G_k is a $k-$multilinear field, which satisfies

$$C_1|x|^{k+1} \leq < G_k(x, \lambda), x >_H \leq C_2|x|^{k+1}, \tag{2.10}$$

for some constants $C_2 > C_1 > 0$, and $k = 2m + 1, m \geq 1$.

Theorem 2.4 (T. Ma and S. Wang [6]). *Under the condition (2.10), the vector field (2.9) bifurcates from $(x, \lambda) = (0, 0)$ on $\lambda > 0$ to an attractor Σ_λ, which is homeomorphic to S^1. Moreover, one and only one of the following is true.*

(1) *Σ_λ is a periodic orbit,*
(2) *Σ_λ consists of only singular points, or*
(3) *Σ_λ consists at most 2(k+1)=4(m+1) singular points, and has $4N+n$ $(N + n \geq 1)$ singular points, $2N$ of which are saddle points, $2N$ of which are stable node points (possibly degenerate), and n of which have index zero.*

2.2. **Global Stability.** In this section, we recall a useful theorem proved in [5] to check the asymptotic stability of the equation

$$\frac{du}{dt} = Lu + G(u) \tag{2.11}$$

where $L : H_1 \to H$ is symmetric, therefore all eigenvalues of L are real.

Let the eigenvalues β_k of L satisfy

$$\begin{cases} \beta_i = 0, \ 1 \leq i \leq m(m \geq 1), \\ \beta_j < 0, \ m+1 \leq j < \infty. \end{cases} \tag{2.12}$$

Set

$$E_0 = \{u \in H_1 \mid Lu = 0\},$$
$$(E_0)^{\perp} = \{u \in H_1 \mid < u, v >_H = 0, \forall v \in E_0\}.$$

Theorem 2.5 (T. Ma and S. Wang [5]). *Let $L : H_1 \to H$ be symmetric with spectrum given by (2.12), and $G : H_1 \to H$ satisfy the following orthogonal condition*

$$< G(u), u >_H = 0, \ \forall u \in H_1. \tag{2.13}$$

Then one and only one of the following two assertions holds true:

(1) *There exists a sequence of invariant sets $\{\Gamma_n\} \subset E_0$ of (2.11) such that*
$$0 \notin \Gamma_n, \ \lim_{n\to\infty} \sup_{x\in\Gamma_n} |x|_H = 0.$$
(2) *The trivial equilibrium point $u = 0$ of (2.11) is locally asymptotically stable under the H-norm.*

Furthermore, if (2.11) has no invariant sets in E_0 except the trivial one $\{0\}$, then $u = 0$ is globally asymptotically stable.

2.3. **Center Manifold Approximation.** In this section, we derive an approximation formula of central manifold reduction, which was used in [6]. For convenience, we introduce the center manifold theorem in infinite dimensional spaces. Let H_1 and H be decomposed into

$$\begin{cases} H_1 = E_1^{\lambda} \oplus E_2^{\lambda}, \\ H = \widetilde{E}_1^{\lambda} \oplus \widetilde{E}_2^{\lambda}, \end{cases} \tag{2.14}$$

for λ near $\lambda_0 \in \mathbb{R}^1$, where E_1^{λ}, E_2^{λ} are invariant subspaces of L_λ, such that

$$\dim E_1^{\lambda} < \infty,$$
$$\widetilde{E}_1^{\lambda} = E_1^{\lambda},$$
$$\widetilde{E}_2^{\lambda} = \text{ the closure of } E_2^{\lambda} \text{ in } H.$$

In addition, L_λ can be decomposed into $L_\lambda = \mathcal{L}_1^{\lambda} \oplus \mathcal{L}_2^{\lambda}$ such that for any λ near λ_0,

$$\begin{cases} \mathcal{L}_1^{\lambda} = L_\lambda \mid_{E_1^{\lambda}} : E_1^{\lambda} \to \widetilde{E}_1^{\lambda}, \\ \mathcal{L}_2^{\lambda} = L_\lambda \mid_{E_2^{\lambda}} : E_2^{\lambda} \to \widetilde{E}_2^{\lambda}, \end{cases} \tag{2.15}$$

where the eigenvalues of $\mathcal{L}_2^\lambda$ possess negative real parts, and the eigenvalues of $\mathcal{L}_1^\lambda$ possess nonnegative real parts at $\lambda = \lambda_0$. Thus, for λ near λ_0, equation (2.1) can be written as

$$\begin{cases} \dfrac{dx}{dt} = \mathcal{L}_1^\lambda x + G_1(x, y, \lambda), \\ \dfrac{dy}{dt} = \mathcal{L}_2^\lambda x + G_2(x, y, \lambda), \end{cases} \tag{2.16}$$

where $u = x + y \in H_1$, $x \in E_1^\lambda$, $y \in E_2^\lambda$, $G_i(x, y, \lambda) = P_i G(u, \lambda)$, and $P_i : H \to \widetilde{E}_i^\lambda$ are canonical projections. Furthermore, let

$$E_2^\lambda(\alpha) = \text{the closure of } E_2^\lambda \text{ in } H_\alpha$$

where $\alpha < 1$ given by (2.3). The following center manifold theorem is classical.

Theorem 2.6 (See D. Henry [3])**.** *Assume (2.2), (2.3), (2.14) and (2.15). Then there exists a neighborhood of λ_0 given by $|\lambda - \lambda_0| < \delta$ for some $\delta > 0$, a neighborhood $U_\lambda \subset E_1^\lambda$ of $x = 0$, and a C^1 function $\Phi(\cdot, \lambda) : U_\lambda \to E_2^\lambda(\alpha)$ depending continuously on λ, such that*

(1) $\Phi(0, \lambda) = 0$, $D_x\Phi(0, \lambda) = 0$*;*

(2) *the set*

$$M_\lambda = \{(x, y) \in H_1 \mid x \in U_\lambda, y = \Phi(x, \lambda) \in E_2^\lambda(\alpha)\},$$

called the center manifolds, are locally invariant for (2.1), i.e. for each $u_0 \in M_\lambda$

$$u_\lambda(t, u_0) \in M_\lambda, \quad \forall\, 0 \le t < t(u_0)$$

for some $t(u_0) > 0$, where $u_\lambda(t, u_0)$ is the solution of (2.1);

(3) *if $(x_\lambda(t), y_\lambda(t))$ is a solution of (2.16), then there is a $\beta_\lambda > 0$ and k_λ depending on $(x_\lambda(0), y_\lambda(0))$ such that*

$$\|y_\lambda(t) - \Phi(x_\lambda(t), \lambda)\|_H \le k_\lambda e^{-\beta_\lambda t}.$$

We now recall an approximation of the equations restricted on the center manifold which will be used later; see [6]. In our application, L_λ is symmetric and $G(\cdot, \lambda)$ is bilinear. Hence the eigenvalues of L_λ are real and the eigenvectors of L_λ form an orthogonal basis of H. Suppose that the eigenvalues (counting the multiplicity) and the eigenvectors are

$$\beta_1(\lambda) = \beta_2(\lambda) = \cdots = \beta_m(\lambda) > \beta_{m+1}(\lambda) \ge, \cdots,$$

$$e_1(x, \lambda), e_2(x, \lambda), \cdots, e_m(x, \lambda), e_{m+1}(x, \lambda), \cdots,$$

and $E_1^\lambda = \text{span}\{e_1, e_2, \cdots, e_m\}$, $E_2^\lambda = (E_1^\lambda)^\perp$, $x = \displaystyle\sum_{j=1}^m x_j e_j$.

With help of Theorem 2.6, the bifurcation equation of (2.1) can be reduced as

$$\frac{dx_k}{dt} = \beta_1 x_k + \frac{1}{< e_k, e_k >_H} < G(x + \Phi(x), \lambda), e_k >_H \tag{2.17}$$

for $k = 1, 2, \cdots, m$. It is known that the center manifold function

$$\Phi(x, \lambda) = \sum_{n=m+1}^{\infty} \Phi_n(x, \lambda) e_n$$

satisfies the following defining equation

$$\Phi_n(x, \lambda) = \int_{-\infty}^{0} e^{-\beta_n \tau} \rho_\epsilon < G(z(\tau, x) + \Phi, \lambda), e_n >_H d\tau, \tag{2.18}$$

where ρ_ϵ is a C^∞ cut-off function and $z(t, x) = \sum_{i=1}^{m} z_i(t, x) e_i$ satisfies

$$\begin{cases} \dfrac{dz_i}{dt} = \beta_1 z_i + \dfrac{\rho_\epsilon(z)}{< e_i, e_i >_H} < G(z + \Phi, \lambda), e_i >_H, \\ z_i(0) = x_i. \end{cases} \tag{2.19}$$

Hence, we have

$$z_i(t, x) = x_i e^{\beta_1 t} + o(|x|). \tag{2.20}$$

Inserting (2.20) in (2.18), by tangency of Φ we get

$$\begin{aligned} \Phi_n(x, \lambda) = \sum_{i,l=1}^{m} x_i x_l \Big(\int_{-\infty}^{0} e^{(2\beta_1 - \beta_n)\tau} d\tau \Big) \times \\ < G(e_i, e_l, \lambda), e_n >_H + o(|x|^2). \end{aligned} \tag{2.21}$$

By (2.17) and (2.21), we conclude that

$$\frac{dx_k}{dt} = \beta_1 x_k + \sum_{i,j=1}^{m} b_{i,j}^k x_i x_j + \sum_{i,j,l=1}^{m} a_{i,j,l}^k x_i x_j x_l + o(|x|^3), \tag{2.22}$$

where

$$b_{i,j}^k = \frac{1}{< e_k, e_k >_H} < G(e_i, e_j, \lambda), e_k >_H,$$

$$a_{i,j,l}^k = \sum_{n=m+1}^{\infty} \frac{1}{(2\beta_1 - \beta_n) < e_n, e_n >_H < e_k, e_k >_H} < G(e_i, e_l, \lambda), e_n >_H$$

$$\times < G(e_j, e_n, \lambda) + G(e_n, e_j, \lambda), e_k >_H,$$

for $k = 1, 2, \cdots, m$.

3. Attractor Bifurcations of the Burgers' Type Equations

3.1. The Case with Dirichlet Boundary Condition. Consider the Burgers' type equation with the Dirichlet boundary condition. We define

$$
\begin{aligned}
&H = L^2(0,1),\\
&H_{1/2} = H_0^1(0,1),\\
&H_1 = H^2(0,1) \cap H_{1/2}.
\end{aligned}
$$

Then the original Burgers' type equation (1.1) with Dirichlet boundary condition (1.4) can be written into the following abstract form in H :

$$
\begin{cases}
\dfrac{du}{dt} = L_\lambda u + G(u,\lambda),\\
u(x,0) = u_0(x),
\end{cases}
\tag{3.1}
$$

where $L_\lambda : H_1 \to H$ is the operator defined by

$$L_\lambda u = \frac{\partial^2 u}{\partial x^2} + \lambda u,$$

and $G(\cdot,\lambda) : H_{1/2} \to H$ is a bilinear operator defined by

$$G(u,\lambda) = -\lambda u \frac{\partial u}{\partial x}.$$

The main result in this case is

Theorem 3.1. *Burgers' type equation (3.1) defined in H possesses pitchfork attractor bifurcations from trivial solutions to attractors Σ_λ when the parameter λ crosses the critical value $\lambda_0 = \pi^2$. Namely,*

(1) *$u = 0$ is a globally asymptotically stable solution of (3.1), for $\lambda \le \pi^2$.*
(2) *When $\lambda > \pi^2$, $u = 0$ is not asymptotically stable and the system bifurcates to two steady state solutions u_1^λ, u_2^λ which are local attractors. Both bifurcation solutions can be written as*

$$
\begin{aligned}
&u_1^\lambda = \alpha(\lambda)\sin \pi x + o(|\alpha(\lambda)|),\\
&u_2^\lambda = -\alpha(\lambda)\sin \pi x + o(|\alpha(\lambda)|),\\
&\alpha(\lambda) = \sqrt{2(4\pi^2 - \lambda)(\lambda - \pi^2)}/\lambda\pi.
\end{aligned}
$$

(3) *There exist an open set $U \subset H$ with $0 \in U$ and a number $\pi^2 < \eta \le 4\pi^2$ such that if $\pi^2 < \lambda < \eta$, the stable manifold $\Gamma \subset H$ of $u = 0$ with codimension one separates U into two open sets U_λ^1 and U_λ^2, which are basins of attraction of u_1 and u_2 respectively, i.e. $\overline{U} = \overline{U}_\lambda^1 \cup \overline{U}_\lambda^2$.*

Proof. We shall prove this theorem by using Theorem 2.3, 2.5 together with the Lyapunov-Schmidt reduction procedure. We proceed in several steps as follows.

STEP 1. It's easy to see that the symmetric operator $L_\lambda : H_1 \to H$ is a sectorial operator. It is known that the eigenvalues and eigenfunctions of L_λ are given by

$$\beta_k(\lambda) = \lambda - (k\pi)^2,$$
$$e_k(x) = \sqrt{2}\sin(k\pi x),$$

for $k = 1, 2, \cdots$, and satisfy the conditions (2.7) and (2.8) with $m = 1$ at $\lambda = \lambda_0 = \pi^2$. Next, if u is a solution of (3.1), then

$$(G(u,\lambda), u)_{L^2} = -\lambda \int_0^1 u^2 u_x dx = -\frac{1}{3}\lambda u^3(t,x) \mid_{x=0}^{x=1} = 0.$$

Hence $G(u,\lambda)$ meets condition (2.13). Moreover, when $\lambda_0 = \pi^2$ the equation (3.1) has no nontrivial invariant set in $E_0 = \mathrm{span}\{\sin \pi x\}$. Hence by Theorem 2.5, u=0 is a globally asymptotically equilibrium point of (3.1) at $\lambda = \lambda_0 = \pi^2$. This proves Assertion (1).

STEP 2. For $u \in H_{1/2}$, by Kondrachov compactness theorem, we have

$$\|G(u,\lambda)\|_{L^2}^2 = \lambda^2 \int_0^1 u^2 u_x^2 dx \le C_1 \lambda^2 \|u\|_{C_0}^2 \|u\|_{H_{1/2}}^2 \le C_2 \lambda^2 \|u\|_{H_{1/2}}^4,$$

for some constants C_1 and C_2. Hence, $G(u,\lambda) = o(\|u\|_{H_{1/2}})$ is a C^∞ compact operator. Thus, by Theorem 2.3, (3.1) bifurcates from $(u,\lambda) = (0,\pi^2)$ to attractors Σ_λ for $\lambda > \pi^2$. By the third assertion of Theorem 2.3, for any $u_\lambda \in \Sigma_\lambda$, u_λ can be expressed as

$$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}),\ v_\lambda \in E_0.$$

Assertion (3) follows from Assertion (4) of Theorem 2.3.

STEP 3. In this step, we use the Lyapunov-Schmidt procedure to calculate the approximation of the attractor Σ_λ. The steady state equation of (3.1) is given by

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \lambda u - \lambda u \dfrac{\partial u}{\partial x} = 0, \\ u(0) = u(1) = 0. \end{cases} \tag{3.2}$$

Let $u \in H$ be expressed as $u = \sum_{n=1}^\infty x_n e_n(x)$. By the Lyapunov-Schmidt reduction procedure, the bifurcation equation of (3.2) can be expressed as

$$\beta_n(\lambda)x_n - \lambda \int_0^1 u\frac{du}{dx}(\sqrt{2}\sin n\pi x)dx = 0,\ n = 1, 2, 3, \cdots. \tag{3.3}$$

We know that

$$u\frac{du}{dx} = \pi \sum_{k,j=1}^\infty j x_k x_j (\sin(k+j)\pi x + \sin(k-j)\pi x). \tag{3.4}$$

It follows from (3.3)-(3.4) that

$$\gamma_n x_n - \pi \sum_{k=1}^{n-1} k x_k x_{n-k} + \pi n \sum_{k=1}^{\infty} x_k x_{n+k} = 0, \tag{3.5}$$

where $\gamma_n = \frac{\sqrt{2}\beta_n(\lambda)}{\lambda}$, for $n = 1, 2, 3, \cdots$. Inductively, we infer from (3.5) that

$$\begin{cases} x_2 = \dfrac{\pi}{\gamma_2} {x_1}^2 + o(|x_1|^2), \\ x_m = O(|x_1|^m), \qquad \text{for } m \geq 2. \end{cases} \tag{3.6}$$

Plugging (3.6) into (3.5) for n=1, we get the bifurcation equation

$$\gamma_1 x_1 + \frac{\pi^2}{\gamma_2} {x_1}^3 + o(|x_1|^3) = 0. \tag{3.7}$$

Since $\gamma_2 < 0$ for λ near π^2, and γ_1 changes signs when λ crosses π^2, by Krasnoselski's bifurcation theorem, the steady state equation of (3.1) bifurcates from the trivial solution to steady state solutions determined by (3.7). Comparing this with the third assertion of Theorem 2.3, we obtain that the bifurcated attractor Σ_λ of equation (3.1) coincides with the bifurcated steady state solutions of (3.7). Thus, the bifurcation solution of (3.1) is given by

$$\begin{aligned} u_1^\lambda &= \alpha(\lambda) \sin \pi x + o(|\alpha(\lambda)|), \\ u_2^\lambda &= -\alpha(\lambda) \sin \pi x + o(|\alpha(\lambda)|), \\ \alpha(\lambda) &= \sqrt{2(4\pi^2 - \lambda)(\lambda - \pi^2)}/\lambda\pi. \end{aligned}$$

This completes the proof. □

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3.2. **The Case With Periodic Boundary Condition.** The Burgers' type equation in one dimensional space with the periodic boundary condition is given by

$$\begin{cases} \dfrac{du}{dt} = \dfrac{\partial^2 u}{\partial x^2} + \lambda u - \lambda u \dfrac{\partial u}{\partial x}, \\ u(x, 0) = u_0(x), \\ u(x+1, t) = u(x, t), \end{cases} \tag{3.8}$$

with constraint

$$\int_0^1 u(x, t) dx = 0. \tag{3.9}$$

We shall study the attractor bifurcation of (3.8) and (3.9) in the following two cases:

(1) CASE WITH ODD SOLUTIONS. In this case, we look for solutions of (3.8) and (3.9), which are odd functions with respect to x. Hence we set

$$\widetilde{H}(odd) = \{v \in L^2(0,1) \mid \int_0^1 v(x)dx = 0\},$$
$$\widetilde{H}_{1/2}(odd) = \{v \in \widetilde{H}(odd) \cap H^1_{per}(0,1) \mid v(-x) = -v(x)\},$$
$$\widetilde{H}_1(odd) = \widetilde{H}_{1/2}(odd) \cap H^2_{per}(0,1).$$

(2) GENERAL CASE. In this case, we look for solutions without oddness assumption. Let

$$\widetilde{H} = \{v \in L^2(0,1) \mid \int_0^1 v(x)dx = 0\},$$
$$\widetilde{H}_{1/2} = \widetilde{H} \cap H^1_{per}(0,1),$$
$$\widetilde{H}_1 = \widetilde{H} \cap H^2_{per}(0,1).$$

In both cases, we define the operators L_λ and $G(\cdot,\lambda)$ by

$$L_\lambda u = \frac{\partial^2 u}{\partial x^2} + \lambda u, \quad G(u,\lambda) = -\lambda u u_x.$$

Thus, the problem (3.8) and (3.9) can be written as

$$\begin{cases} \dfrac{du}{dt} = L_\lambda u + G(u,\lambda), \\ u(0) = u_0(x), \end{cases} \tag{3.10}$$

in space $\widetilde{H}(odd)$ or $\widetilde{H}$. The results for odd solutions case is:

Theorem 3.2. *For Burgers' type equation (3.8) and (3.9) defined in $\widetilde{H}(odd)$, the following assertions hold true.*

(1) *$u = 0$ is a globally asymptotically stable solution of (3.8) and (3.9), for $\lambda \le 4\pi^2$.*
(2) *When $\lambda > 4\pi^2$, $u = 0$ is not asymptotically stable and the system bifurcates to two steady state solutions u_1^λ, u_2^λ which are local attractors. Both bifurcation solutions can be written as*

$$u_1^\lambda = \alpha(\lambda)\sin 2\pi x + o(|\alpha(\lambda)|),$$
$$u_2^\lambda = -\alpha(\lambda)\sin 2\pi x + o(|\alpha(\lambda)|),$$
$$\alpha(\lambda) = \sqrt{(16\pi^2 - \lambda)(\lambda - 4\pi^2)}/\sqrt{2}\lambda\pi.$$

(3) *There exist an open set $U \subset H$ with $0 \in U$ and a number $4\pi^2 < \eta \le 16\pi^2$ such that if $4\pi^2 < \lambda < \eta$, the stable manifold $\Gamma \subset H$ of $u = 0$ with codimension one separates U into two open sets U_λ^1 and U_λ^2, which are basins of attraction of u_1 and u_2 respectively, i.e. $\overline{U} = \overline{U}_\lambda^1 \cup \overline{U}_\lambda^2$.*

Proof. The eigenvalues and eigenfunctions of the sectorial operator L_λ : $\widetilde{H}_1(odd) \to \widetilde{H}(odd)$ are given by

$$\beta_k(\lambda) = \lambda_k = \lambda - (2k\pi)^2,$$
$$e_k(x) = \sqrt{2}\sin(2k\pi x),$$

for $k = 1, 2, \cdots$. The rest part of the proof is similar with the proof of Theorem 3.1. □

For general periodic case, we have the following result.

Theorem 3.3. *In space $\widetilde{H}$, the Burgers' type equation (3.8) and (3.9) bifurcates from $(u, \lambda) = (0, 4\pi^2)$ on $\lambda > 4\pi^2$ to an attractor Σ_λ. Σ_λ is homeomorphic to S^1, which attracts any bounded set in $U\backslash\Gamma$, Γ the stable manifold of $u = 0$ with codimension two in H and U some open neighborhood o f $u = 0$ in $\widetilde{H}$. Moreover, Σ_λ consists of steady states of (3.8) and (3.9).*

Proof. Step 1. The eigenvalues and eigenfunctions of sectorial operator L_λ : $\widetilde{H}_1 \to \widetilde{H}$ are given by

$$\beta_{2n-1}(\lambda) = \beta_{2n}(\lambda) = \lambda - (2n\pi)^2, \tag{3.11}$$

$$e_{2n-1}(x) = \sqrt{2}\sin 2n\pi x,\ \ e_{2n}(x) = \sqrt{2}\cos 2n\pi x,\ \ n \geq 1. \tag{3.12}$$

Condition (3.11) satisfies the conditions (2.7) and (2.8) with $m = 2$ at $\lambda = \lambda_0 = 4\pi^2$. It's easy to see that the conditions of Theorem 2.5 are satisfied by L_{λ_0} and $G(u, \lambda_0)$. Since the equation (3.8) and (3.9) has no nontrivial invariant set in $E_0 = \text{span}\,\{e_1(x), e_2(x)\}$, u=0 is globally asymptotically stable at $\lambda = 4\pi^2$. Thus Theorem 2.3 asserts the existence of the attractor bifurcation.

Step 2. In this step, we apply the center manifold reduction to calculate the structure of the attractor. Let $E_1^\lambda = E_0 = \text{span}\{e_1, e_2\}, E_2^\lambda = (E_0)^\perp$ and $\Phi_\lambda(x_1, x_2)$ be the center manifold as mentioned in Theorem 2.6. Let $u = \sum_{n=1}^{\infty} x_n(t)e_n(x)$, the reduction equations can be expressed as

$$\begin{cases} \dfrac{dx_1}{dt} = \beta_1(\lambda)x_1 + < G(x + \Phi(x)), e_1 >_{\widetilde{H}}, \\ \dfrac{dx_2}{dt} = \beta_1(\lambda)x_2 + < G(x + \Phi(x)), e_2 >_{\widetilde{H}}, \end{cases} \tag{3.13}$$

where $x = x_1e_1 + x_2e_2$. By formula (2.22), equation (3.13) can be expressed as

$$\begin{cases} \dfrac{dx_1}{dt} = \beta_1(\lambda)x_1 + \displaystyle\sum_{i,j,l=1}^{2} a^1_{i,j,l}x_ix_jx_l + o(|x|^3), \\ \dfrac{dx_2}{dt} = \beta_1(\lambda)x_2 + \displaystyle\sum_{i,j,l=1}^{2} a^2_{i,j,l}x_ix_jx_l + o(|x|^3), \end{cases} \tag{3.14}$$

where

$$a^k_{i,j,l} = \sum_{n=3}^{\infty} \frac{1}{(2\beta_1 - \beta_n) < e_n, e_n >_{\widetilde{H}} < e_k, e_k >_{\widetilde{H}}} < G(e_j, e_l, \lambda), e_n >_{\widetilde{H}} \times$$
$$< G(e_i, e_n, \lambda) + G_2(e_n, e_i, \lambda), e_k >_{\widetilde{H}}, \text{ for } i, j, l, k = 1, 2.$$

Note that

$$< G(e_j, e_l, \lambda), e_n >_{\widetilde{H}} = -\lambda \int_0^1 e_j \frac{de_l}{dx} e_n dx = 0 \text{ for } n \geq 5.$$

By direct calculation, we obtain that

$$\begin{aligned}
& a^1_{111} = \frac{-2\pi^2\lambda^2}{\lambda + 8\pi^2}, \\
& a^1_{222} = 0, \\
& a^1_{211} + a^1_{121} + a^1_{112} = 0, \\
& a^1_{122} + a^1_{212} + a^1_{221} = \frac{-2\pi^2\lambda^2}{\lambda + 8\pi^2},
\end{aligned}$$

and

$$\begin{aligned}
& a^2_{111} = 0, \\
& a^2_{222} = \frac{-2\pi^2\lambda^2}{\lambda + 8\pi^2}, \\
& a^2_{211} + a^2_{121} + a^2_{112} = \frac{-2\pi^2\lambda^2}{\lambda + 8\pi^2}, \\
& a^2_{122} + a^2_{212} + a^2_{221} = 0.
\end{aligned}$$

Hence equation (3.14) translates into

$$\begin{cases} \dfrac{dx_1}{dt} = (\lambda - 4\pi^2)x_1 - \dfrac{2\pi^2\lambda^2}{\lambda + 8\pi^2}(x_1^3 + x_1 x_2^2) + o(|x|^3), \\ \dfrac{dx_2}{dt} = (\lambda - 4\pi^2)x_2 - \dfrac{2\pi^2\lambda^2}{\lambda + 8\pi^2}(x_1^2 x_2 + x_2^3) + o(|x|^3). \end{cases} \tag{3.15}$$

It's obvious that equation (3.15) fits the conditions of Theorem 2.4, it follows that the attractor of (3.15) bifurcated from $(x, \lambda) = (0, 4\pi^2)$ on $\lambda > 4\pi^2$ is a circle S^1.

STEP 3. Finally we verify that Σ_λ consists of singular points of (3.8) and (3.9). By Theorem 3.2, the problem (3.8) and (3.9) has a steady state solution $u_\lambda = \alpha(\lambda) \sin 2\pi x + h(x)$, where $h(x) = o(|\alpha(\lambda)|)$. Because of the invariance of (3.8) and (3.9) for the translation

$$u(x, t) \to u(x + \theta, t),$$

the functions

$$u_\lambda(x+\theta) = \alpha(\lambda)\sin(x+\theta) + h(x+\theta),\ \theta \in \mathbb{R}^1$$

are steady state solutions of (3.8) and (3.9), and the set

$$\Gamma = \{\alpha(\lambda)\sin(x+\theta) + h(x+\theta) \mid -\infty < \theta < +\infty\}$$

is a circle S^1 in H. Therefore $\Sigma_\lambda = \Gamma$. This completes the proof. □

3.3. Bifurcation From General Eigenvalues.

Theorem 3.4. *In the space $\widetilde{H}$, the problem (3.8) with (3.9) bifurcates from $(u,\lambda) = (0, 4n^2\pi^2)$ on $\lambda > 4n^2\pi^2$ to an invariant set $\Sigma_\lambda = S^1$, which consists of steady state solutions of (3.8) and (3.9).*

Proof. For $u \in \widetilde{H}$, it can be expressed as

$$u = \sum_{n=1}^{\infty} x_n(t) e_n(x), \tag{3.16}$$

e_n as given in (3.12). Then the reduction equation of (3.10) to the center-manifold near $\lambda = 4n^2\pi^2$ is in the following form

$$\begin{cases} \dfrac{dx_{2n-1}}{dt} = \beta_{2n} x_{2n-1} + < G(u,\lambda), e_{2n-1} >_{\widetilde{H}}, \\ \dfrac{dx_{2n}}{dt} = \beta_{2n} x_{2n} + < G(u,\lambda), e_{2n} >_{\widetilde{H}}\ . \end{cases} \tag{3.17}$$

The same as the proof of Theorem 3.3, (3.17) can be expressed as

$$\begin{cases} \dfrac{dx_{2n-1}}{dt} = \beta_{2n} x_{2n-1} + \displaystyle\sum_{i,j,l=2n-1}^{2n} c_{i,j,l}^{2n-1} x_i x_j x_l + o(|x|^3), \\ \dfrac{dx_{2n}}{dt} = \beta_{2n} x_{2n} + \displaystyle\sum_{i,j,l=2n-1}^{2n} c_{i,j,l}^{2n} x_i x_j x_l + o(|x|^3), \end{cases} \tag{3.18}$$

where $x = (x_{2n-1}, x_{2n})$, and

$$c_{i,j,l}^k = \sum_{m=1, m\neq 2n-1, 2n}^{\infty} \frac{1}{(2\beta_{2n} - \beta_m) < e_m, e_m >_{\widetilde{H}} < e_k, e_k >_{\widetilde{H}}} \times$$
$$< G(e_i, e_l, \lambda), e_m >_{\widetilde{H}} < G(e_j, e_m, \lambda) + G_2(e_m, e_j, \lambda), e_k >_{\widetilde{H}},$$

for $i, j, k, l = 2n-1, 2n$.

By direct computation, we get

$$\begin{cases} \dfrac{dx_{2n-1}}{dt} = (\lambda - 4n^2\pi^2) x_{2n-1} - \dfrac{2n^2\pi^2\lambda^2}{\lambda + 8n^2\pi^2}(x_{2n-1}^3 + x_{2n-1}x_{2n}^2) + o(|x|^3), \\ \dfrac{dx_{2n}}{dt} = (\lambda - 4n^2\pi^2) x_{2n} - \dfrac{2n^2\pi^2\lambda^2}{\lambda + 8n^2\pi^2}(x_{2n-1}^2 x_{2n} + x_{2n}^3) + o(|x|^3). \end{cases}$$

The rest part of the proof is the same as the proof of Theorem 3.3, and the theorem is complete.

□

4. Dimension of Attractor

In this section, we investigate the long time behavior of the Burgers' type equation (1.1). In particular, we are interested in the existence of global attractor and the estimate of its dimension. In the first subsection, we show that the system is dissipative utilizing a technique developed by Nicolaenko, Scheurer and Temam for the Kuramoto-Sivashinsky equation ([9]). This technique utilizes the nonlinear advection term to stabilize the linearly unstable low modes (with the help of the 2nd order dissipative operator of course). The first subsection also contains (optimal) estimates on the size of absorbing ball in the L^2 space and estimates on the long time average of the leading order dissipation. These estimates imply the existence of a global attractor (see for instance [9]) and will then be used in the second subsection to derive an upper bound on the Hausdorff and fractal dimension. The derivation of the upper bound relies on the Constantin-Foias form of the Kaplan-Yorke formula with global Lyapunov exponents (see for instance ([9])). In the third subsection, we derive a lower bound on the dimension of the global attractor via estimating the dimension of the unstable manifold associated with the trivial solution ($u \equiv 0$). The number of unstable modes associated with the trivial solution is at least $\sqrt{\lambda}$ for large λ. Thus the dimension of the global attractor scales at least linearly in $\sqrt{\lambda}$. In terms of the alternative formulation as extended system on $[0, \sqrt{\lambda}]$ (1.2), we see that the dimension of the global attractor scales at least linearly in the system volume (in this case, length). Nevertheless, the long time dynamics is non-chaotic (let alone extensive chaotic) since all trajectories converge either to a time periodic orbit or steady states ([8]). Therefore, the Burgers' type equation studied here suggests that the mere criterion of dimension of global attractor scale linearly in the system volume is not sufficient to guarantee extensive chaos as suggested early (see for instance [2], [10]).

4.1. **Dissipativity and Basic Estimates.**

In this section we show that the Burgers' type equation (1.1) is dissipative in the sense that it possesses bounded absorbing balls in L^2 and H^1 which implies the existence of a global attractor (see for instance ([9])). The Burgers' type equation (1.1) resembles the Kuramoto-Sivashinsky equation ([9]) in the sense that the linearized (around the trivial solution) dynamics is not stable global in time (see section 4.3 below), and as we shall demonstrate below, the nonlinear advection term stablizes (together with higher order linear dissipative term) the system.

Just as in the case of Kuramoto-Sivashinsky equation studied by Nicolaenko, Scheurer and Temam ([9]), we consider a translation of the unknown by a translational function ψ,

$$u = v + \psi \tag{4.1}$$

where ψ is stationary and will be determined later.

Here we have focused on the (simpler) Dirichlet boundary condition only. The Dirichlet boundary condition is equivalent to periodic boundary condition with odd symmetry. The general periodic boundary condition can be handled similarly allowing time dependence of ψ (see for instance ([9]) and the references therein for the Kuramoto-Sivasinsky equation case).

It is easy to see that the translated unknown v satisfies the following equation

$$v_t = v_{xx} + \lambda v - \lambda v v_x - \lambda \psi v_x - \lambda v \psi_x + \psi_{xx} + \lambda \psi - \lambda \psi \psi_x. \tag{4.2}$$

Multiplying the above equation by v and integrating over the domain $\Omega = [0, 1]$, we have

$$\frac{1}{2}\frac{d}{dt}|v|_{L^2}^2 = -|v_x|_{L^2}^2 + \lambda |v|_{L^2}^2 - \frac{\lambda}{2}\int_\Omega \psi_x v^2\,dx - \int_\Omega (\psi_x v_x + \lambda \psi v - \lambda \psi \psi_x v)\,dx \tag{4.3}$$

Therefore, we naturally hope that $\psi_x \approx 4$ (in fact any number greater than 2 will work) in order to ensure dissipativity (uniform in time boundedness of $|v|_{L^2}$). This is equivalent to the desire of $\psi \approx 4x$. Unfortunately $4x$ violates the given boundary condition. We then propose the following modification of $4x$ as our translational function

$$\psi(x) = \begin{cases} 4x, & 0 \le x \le 1-\delta, \\ \dfrac{4(1-\delta)(1-x)}{\delta}, & 1-\delta \le x \le 1 \end{cases} \tag{4.4}$$

where $\delta > 0$ is a positive constant and is specified as

$$\delta = \frac{1}{\lambda}. \tag{4.5}$$

With this choice of translational function ψ, we can easily derive, with the help of Poincaré inequality etc, the following estimates

$$\frac{d}{dt}|v|_{L^2}^2 \le -\frac{1}{4}|v_x|_{L^2}^2 + \frac{\lambda}{4}|v|_{L^2}^2 + \kappa\lambda \tag{4.6}$$

where here and below κ denotes a generic constant independent of λ.

This leads to, with the help of Gronwall type inequality, the existence of an absorbing ball in L^2 for v of the size independent of λ and an upper bound on the time averaged leading order dissipation, i.e.

$$\begin{aligned} |v(t)|_{L^2} &\le \kappa, \quad \text{for } t \ge T, \\ \limsup_{T\to\infty} \frac{1}{T}\int_0^T |v_x|_{L^2}^2 &\le \kappa\lambda. \end{aligned}$$

This further implies, thanks to the translational relation (4.1) and the explicit form of translational function ψ (4.4), the following absorbing ball (in L^2) and bound on the time averaged leading order dissipation

Proposition 4.1. *There exists a generic constant κ independent of λ such that for any bounded ball B_R in the phase space $H = L^2(\Omega)$, there exists a T_R so that*

$$|u(t)|_{L^2} \leq \kappa, \quad \text{for } t \geq T_R, \tag{4.7}$$

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T |u_x|_{L^2}^2 \leq \kappa\lambda. \tag{4.8}$$

Remark 4.2. *It is worthwhile to point out that the estimates above in proposition 4.1 are optimal in terms of dependence on λ for large λ. The optimality can be inferred from earlier results on the asymptotic behavior of steady states for large λ (see Theorem 2 in* [1]*). Roughly speaking, there exists steady state solution very much similar to our translational function ψ which saturates the estimates above.*

4.2. **Upper Bound on the Global Attractor.** Once we have established bounded absorbing ball in L^2 for the Burgers type equation (1.1), it is easy to show the existence of bounded absorbing ball in H^1 via multiplying (1.1) by u, integrating over $\Omega = [0, 1]$ and applying integration by parts as well as classical Sobolev and interpolation inequalities. This then leads to the existence of a global attractor which is a compact invariant set in L^2 that attracts all bounded sets in L^2 (see for instance ([9])). In fact, it is possible to show that the attractor is analytic in space and in time by the Foias-Temam technique of Gevrey regularity and complexification of the time.

In order to estimate the Hausdorff and fractal dimension of the global attractor, we use a Kaplan-Yorke type formula derived by Constantin and Foias (see for instance ([9])). Roughly speaking, if an arbitrary m-volume under the linearized (along any trajectory on the global attractor) dynamics is contracting on average in the long time, then the dimension of the attractor must be bounded by m. Whether a particular m-volume is contracting is related to global Lyapunov exponents and can be estimated via a formula.

For an abstract dynamical system

$$\frac{dv}{dt} = F(v), \quad v|_{t=0} = v_0.$$

The first variation (linearization) along a trajectory $v(t)$ is given by

$$\frac{dV}{dt} = F'(v)V, \quad V|_{t=0} = V_0. \tag{4.9}$$

where

$$F'(v)V = V_{xx} + \lambda V - \lambda v V_x - \lambda V v_x \tag{4.10}$$

in our case.

Whether a given m-volume is contracting under the linearized dynamics is then related to the trace of the linear operator $F'(v)$. The key result that we rely on is the following one due to Constantin, Foias and Temam

Lemma 4.3 ([9], Chapter V, section 3.4). *Let X be an invariant set of the dynamical system. Suppose*

$$\limsup_{T\to\infty} \sup_{v_0\in X} \sup_{\phi_j} \frac{1}{T}\int_0^T tr\ (F'(v)\circ Q_m(t))\,dt < 0 \tag{4.11}$$

for any $v_0 \in X \subset H$ and any orthonormal basis $\{\phi_j(t), j \geq 1\}$ of $H = L^2$ with $\phi_j \in H_0^1$ for all t where $Q_m(t)$ is the orthogonal projection in H onto the linear span of $\{\phi_j(t), m \geq j \geq 1\}$. Then there exists an absolute constant κ such that

$$\dim_H(X) \leq m, \quad \dim_f(X) \leq \kappa m \tag{4.12}$$

where $\dim_H, \dim_f$ denotes the Hausdorff and fractal dimension respectively.

Next, we proceed to estimate the left hand side of (4.11).
By definition, we have

$$\begin{aligned} \text{tr }(F'(v)\circ Q_m(t)) &= \sum_{j=1}^{\infty}(F'(v(t))\circ Q_m(t)\phi_j(t), \phi_j(t)) \\ &= \sum_{j=1}^{m}(F'(v(t))\phi_j(t), \phi_j(t)) \end{aligned} \tag{4.13}$$

Noticing

$$(F'(v(t))\phi_j(t), \phi_j(t)) = -|\phi_{jx}|_{L^2}^2 + \lambda|\phi_j|_{L^2}^2 - \frac{\lambda}{2}\int_\Omega v_x\phi_j^2, \tag{4.14}$$

denoting

$$\rho_m = \sum_{j=1}^{m} |\phi_j(x,t)|^2 \tag{4.15}$$

we have

$$
\begin{aligned}
\sum_{j=1}^{m}(F'(v(t))\phi_j(t),\phi_j(t)) &= -\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2+\lambda m-\frac{\lambda}{2}\int_\Omega v_x\rho_m \\
&\le -\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2+\lambda m+\frac{\lambda}{2}|\rho_m|_{L^3}|v_x|_{L^{3/2}} \\
&\le -\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2+\lambda m+\frac{\lambda}{2}|\rho_m|_{L^3}|v_x|_{L^2} \\
&\le -\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2+\lambda m+\kappa\lambda(\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2+m)^{1/3}|v_x|_{L^2} \\
&\le -\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2+\lambda m+\kappa\lambda(\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2)^{1/3}|v_x|_{L^2} \\
&\le -\frac{1}{2}\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2+\lambda m+\kappa\lambda^{3/2}|v_x|_{L^2}^{3/2} \\
&\le -\kappa m^3+\lambda m+\kappa\lambda^{3/2}|v_x|_{L^2}^{3/2} \qquad (4.16)
\end{aligned}
$$

where we have used Holder's inequality, the Ghidaglia-Marion-Temam version of the Sobolev-Lieb-Thierring inequality ([9], Appendix, Theorem 3.1), and the fact that, since $\{\phi_j, j\ge 1\}$ is an o.n.b. for H and each belongs to H_0^1,

$$
\begin{aligned}
\sum_{j=1}^{m}|\phi_{jx}|_{L^2}^2 &= -\sum_{j=1}^{m}(\phi_{jxx},\phi_j) \\
&\ge \sum_{j=1}^{m} j^2\pi^2 \\
&= \kappa m^3 \qquad (4.17)
\end{aligned}
$$

where κ is an absolute constant independent of m.

Combining (4.8), (4.11), (4.12) and (4.16) we have

$$
\limsup_{T\to\infty}\sup_{v_0\in X}\sup_{\phi_j}\frac{1}{T}\int_0^T \mathrm{tr}\,(F'(v)\circ Q_m(t))\,dt \le -\kappa m^3+\lambda m+\kappa\lambda^{9/4}. \qquad (4.18)
$$

This then leads to the following upper bound on the Hausdorff and fractal dimension of the global attractor for the Burgers' type equation (1.1).

Proposition 4.4. *The Burgers type equation (1.1) possess a global attractor $\mathcal{A}_\lambda$ for all positive parameter λ. Moreover, there exists a constant κ_u independent of λ such that*

$$
\dim_H(\mathcal{A}_\lambda)+\dim_f(\mathcal{A}_\lambda)\le\kappa_u\lambda^{3/4}. \qquad (4.19)
$$

4.3. **Lower Bound on the Global Attractor.** We now derive a lower bound which grows linearly in $\sqrt{\lambda}$. The desired lower bound is achieved via estimating the dimension of the unstable manifold associated with the trivial solution $u \equiv 0$ since the unstable manifold must be a part of the global attractor (at least in the generic case of $\lambda \neq j^2\pi^2$).

The linearized equation takes the form

$$v_t = v_{xx} + \lambda v \tag{4.20}$$

equipped with the homogeneous Dirichlet or periodic boundary condition on the interval $[0, 1]$.

It is easy to see that for a fixed (large) λ, there are roughly $\sqrt{\lambda}$ number of unstable modes to the linearized equation. Therefore the unstable manifold is at least (roughly) $\sqrt{\lambda}$. This further implies

$$\dim \mathcal{A}_\lambda \geq \kappa_l \sqrt{\lambda} \tag{4.21}$$

Combining the upper bound (4.19) and the lower bound (4.21) we have the following result

Theorem 4.5. *For fixed (large) λ, the Burgers' type equation (1.1) possesses a global attractor whose dimension can be estimated as*

$$\kappa_l \sqrt{\lambda} \leq \dim \mathcal{A}_\lambda \leq \kappa_u \lambda^{3/4} \tag{4.22}$$

where κ_l and κ_u are generic constants independent of λ.

An immediate corollary of this result is

Corollary 4.6. *The extended system (1.2) on the extended domain $\widetilde{\Omega} = [0, \sqrt{\lambda}]$, possesses a global attractor $\widetilde{\mathcal{A}}_\lambda$ whose dimension can be estimated as*

$$\kappa_l |\widetilde{\Omega}| \leq \dim \widetilde{\mathcal{A}}_\lambda \leq \kappa_u |\widetilde{\Omega}|^{3/2} \tag{4.23}$$

i.e., the dimension of the attractor scales at linearly in the volume (in our one dimensional case, the length) of the extended system with other parameters of the system held fixed.

Again, we reiterate that this extended system has no chaotic dynamics since each trajectory converges to either a periodic orbit or steady states [8]. Of course, there are dissipative systems whose attractor dimensions scale linearly in the system volume while the dynamics in non-chaotic. For instance, the 1D Chaffee-Infante equation $u_t - u_{xx} + u^3 - u = 0$ on $[0, L]$ is such an example. See Theorem 2.1, Chapter VI in [9] for an upper bound. The lower bound can be derived in the same way as above.

Finally, we remark that there is still a discrepancy between the upper and lower bound. It is still an open question to derive an optimal estimate on the dimension of the attractor (although estimates in proposition 4.1 are optimal). We tend to believe that the lower bound is optimal. The lower bound also agrees with the heuristic Landau-Lifschitz argument. However, it seems that the approach here will not lead to the speculated optimal bound, since the last term on the first of in equation (4.16) involves v_x which is not bounded in $L^p, p > 1$ (see remark 4.2).

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