A NEW PROOF OF THE AGREEMENT BETWEEN THE CLASSES DEFINED BY M.-H. SCHWARTZ AND R. MACPHERSON

PAOLO ALUFFI AND JEAN-PAUL BRASSELET

Abstract. We give a short proof of the fact that the Chern classes for singular varieties defined by Marie-Hélène Schwartz by means of 'radial frames' agree with the functorial notion defined by Robert MacPherson.

Dedicated to Marie-Hélène Schwartz

1. Introduction

In the mid-60's, Marie-Hélène Schwartz ([Sch65a], [Sch65b]) defined a notion of *Chern classes for singular varieties* in relative cohomology, or, via Alexander duality, in homology with integer coefficients. For this purpose she used obstruction theory applied to frames of vector fields with controlled behavior along the singularities.

Soon thereafter, the work of Alexander Grothendieck aimed at proving a 'discrete Riemann-Roch theorem' was at the origin of a conjecture of existence (in characteristic zero) of a functorial theory of Chern classes, in terms of a natural transformation from the functor of constructible functions to a good homology theory. This conjecture is known under the name of 'Deligne-Grothendieck conjecture'. It was solved in the 70's by Robert MacPherson [Mac74], thereby yielding another notion of Chern classes for singular varieties.

MacPherson's work was independent from Schwartz's, yet the two notions had points in common: both classes specialize to the usual Chern class on non-singular varieties; and Schwartz's result, extending the Poincaré-Hopf theorem to singular varieties, may be viewed as a facet of the functoriality satisfied by MacPherson's classes.

It was then natural to conjecture that these notions should agree. In 1979 this was proved to indeed be the case, by Schwartz and the second-named author of the present article:

Theorem 1.1 ([BS81])**.** *MacPherson's and Schwartz's Chern classes are equal.*

The proof was obtained by relating indices of radial frames and Schwartz classes to key ingredients in MacPherson's construction of his classes, namely the *local Euler obstruction* and *Chern-Mather class.*

In this paper we recover the equality of Schwartz and MacPherson classes more directly, without the aid of these other invariants of singularities. The new point of view was inspired by a new expression for the functorial notion of Chern class in terms of classes defined for nonsingular (but possibly *noncomplete*) varieties, obtained by the first-named author [Alu06]. In fact, further improvements have made the argument independent of that reference, so the version presented here is self-contained.

In §2 we observe that a class defined for (possibly singular) varieties X necessarily agrees with the functorial notion if

– it may be written as a sum of contributions from pieces of a stratification of X;
– the contribution of a nonsingular stratum S is preserved through stratification-preserving morphisms; and
– the class agrees with the Chern class of the tangent bundle for nonsingular, complete varieties.

In §3 we observe that the classes defined by Marie-Hélène Schwartz satisfy these requirements, and the equality with MacPherson's classes follows then immediately.

2. A characterization of functorial Chern classes

In this section we work over an arbitrary algebraically closed field of characteristic zero, and in the Chow group A_*. The results hold *a fortiori* for complex varieties, in homology.

2.1. Let $\tilde{c}(X) \in A_*X$ be a class defined for all (possibly singular) varieties X, as a sum of contributions from a decomposition of X as a finite disjoint union of *nonsingular* (possibly noncomplete) varieties S_i:

$$X = \amalg_i S_i \rightsquigarrow \tilde{c}(X) = \sum_i \tilde{c}(S_i, X) \quad ;$$

we say that such decompositions are *admissible* (for $\tilde{c}$). As we will see, in some cases *every* decomposition of X as a finite disjoint union of nonsingular subvarieties may be admissible. In other situations, more restrictions may have to be placed on admissible decompositions: for example, the varieties S_i may be required to be elements of a Whitney stratification of X.

We assume that strata of a normal crossing divisor form an admissible decomposition. More precisely: if D is a divisor with simple normal crossings and nonsingular components D_j, $j \in J$ in a nonsingular variety Y, then we assume that

$$\amalg_{I \subset J} D_I^\circ$$

is an admissible decomposition of Y, where D_I° denotes

$$(\cap_{j \in I} D_j) \smallsetminus (\cup_{j \notin I} D_j) \quad .$$

(For example, $D_\emptyset^\circ$ is the complement of D in Y.)

This will be clearly satisfied for the decompositions we will consider.

Definition 2.1. We say that the datum of a class $\tilde{c}(X) \in A_*X$ as above for all algebraic varieties X is *locally determined* if the following condition holds:

– If $f : Y \to X$ is a proper morphism, S, resp. $T := f^{-1}(S)$ are members of admissible decompositions of X, resp. Y, and f restricts to an isomorphism $T \to S$, then

$$f_*\tilde{c}(T, Y) = \tilde{c}(S, X) \quad .$$

Example 2.2 (The functorial class). We denote by $c_*(X) \in A_*(X)$ the class defined by MacPherson in [Mac74] (see [Ful84], §19.1.7, for the adaptation of the definition to the Chow group $A_*(X)$, and [Ken90] for the extension to arbitrary algebraically closed fields of characteristic zero). Recall that this class is the value

$$c_*(X) := c_*(1\!\!1_X) \in A_*X$$

taken on the constant characteristic function $1\!\!1_X$ by a natural transformation

$$c_* : F \rightsquigarrow A_*$$

from the functor of constructible functions to the Chow group functor. Here, $F(X)$ is the group of constructible, integer valued functions on X; if $g : Y \to X$ is a proper map, the push-forward $g_*(\varphi)$ of a constructible function $\varphi = \sum_Z m_Z 1\!\!1_Z \in F(Y)$ is defined as the function on X whose value at $p \in X$ is

$$g_*(\varphi)(p) := \sum m_Z \chi(g^{-1}(p) \cap Z) \quad .$$

Here χ denotes topological Euler characteristic, over $\mathbb{C}$; for the extension to other fields of characteristic zero see [Ken90] or [Alu06].

For S any (in particular, any nonsingular) subvariety of a variety X, let

$$c_*(S, X) := c_*(1\!\!1_S) \in A_*X \quad .$$

Then *every* decomposition of X as a finite disjoint union of nonsingular subvarieties is admissible for c_*. Indeed, if $X = \amalg_i S_i$ then $1\!\!1_X = \sum_i 1\!\!1_{S_i}$; hence

$$c_*(X) = c_*(1\!\!1_X) = \sum_i c_*(1\!\!1_{S_i}) = \sum_i c_*(S_i, X) \quad .$$

Further, c_* is locally determined. Indeed, let $f : Y \to X$ be a proper map, restricting to an isomorphism $T \to S$. Then

$$f_* c_*(T, Y) = f_* c_*(1\!\!1_T) = c_* f_*(1\!\!1_T) = c_*(1\!\!1_S) = c_*(S, X)$$

since c_* is a natural transformation, and by definition of push-forward of constructible functions.

Theorem 2.3. *Suppose that $\tilde{c}$ is locally determined, and that*

$$\tilde{c}(V) = c(TV) \cap [V]$$

for every nonsingular complete *variety V. Then $\tilde{c}$ agrees with the functorial class c_*.*

Proof. Let $X = \amalg_i S_i$ be an admissible decomposition of X for $\tilde{c}$; it suffices to show that $\tilde{c}(S_i, X) = c_*(S_i, X)$.

If S is an element of the decomposition, let $\overline{S}$ be its closure in X, and let $f : Y \to \overline{S}$ be an embedded resolution of $\overline{S}$, such that the complement of $T := f^{-1}(S)$ in $Y = \overline{T}$ is a divisor D with normal crossings and nonsingular components D_j, $j \in J$. Since both $\tilde{c}$ and c_* are locally determined, it suffices to prove that

$$\tilde{c}(T, Y) = c_*(T, Y) \in A_*Y \quad .$$

Now

$$Y = \amalg_{I \subset J} D_I^\circ \quad ,$$

hence

$$\tilde{c}(Y) = \sum_{I \subset J} \tilde{c}(D_I^\circ, Y) \quad ,$$

from which the 'inclusion-exclusion' principle gives

$$\tilde{c}(T,Y) = \tilde{c}(D^\circ_\emptyset, Y) = \sum_{I\subset J} (-1)^{|I|}\tilde{c}(D_I, Y) \quad ,$$

where $D_I = \cap_{j\in I} D_j$, and $\tilde{c}(D_I,Y)$ denotes the sum $\sum_{K\supset I}\tilde{c}(D^\circ_K, Y)$. Each D_I is complete and nonsingular (since D is a divisor with simple normal crossings), and the inclusion $\iota : D_I \to Y$ is proper. For $K \supset I$,

$$\tilde{c}(D^\circ_K, Y) = \iota_*\tilde{c}(D^\circ_K, D_I)$$

since $\tilde{c}$ is locally determined, and hence

$$\tilde{c}(D_I, Y) = \sum_{K\supset I} \iota_*\tilde{c}(D^\circ_K, D_I) = \iota_*\tilde{c}(D_I) = \iota_*(c(TD_I)\cap [D_I]) \quad .$$

The same expression holds for $c_*(T,Y)$, by the same token, concluding the proof. □

3. Schwartz classes

We now assume that the ground field is $\mathbb{C}$, and work in homology with integer coefficients; the considerations in §2 apply to this context, via the canonical map from Chow group to homology.

3.1. Denote by $\tilde{c}$ the class defined by Marie-Hélène Schwartz in [Sch65a], [Sch65b]. This class agrees with the total (homology) Chern class of the tangent bundle for complete nonsingular varieties, and is computed in general by adding terms contributed by strata S_i of a Whitney stratification of X:

$$\tilde{c}(X) := \sum_i \tilde{c}(S_i, X) \quad .$$

That is, Whitney stratifications are admissible decompositions for $\tilde{c}$.

We briefly recall the definition of the Schwartz classes. The variety X is embedded in a nonsingular variety M, stratified by $M \smallsetminus X$ and by the strata S_i of X. For $x \in M$, let $E(x)$ be the subspace of the tangent space T_xM consisting of vectors which are tangent to the stratum of M containing x. The collection of the subspaces $E(x)$ determines a subspace E of the tangent bundle TM. It is no longer a bundle but the notion of section is well defined: a section of E is a section of TM with value in E. One considers then the space E_r of ordered r-frames of vectors tangent to the strata of M. It is a subspace of the bundle T_rM of ordered r-frames, associated to TM.

One also considers a triangulation K of M compatible with the stratification, and a cellular decomposition D of M, dual to K. The cells of D are transverse to the strata.

In dimension $r-1$, the Schwartz class is defined as the obstruction to the construction of a section of E_r on X. It is obtained by constructing a special section Z_r of E_r, called a *radial r-frame.* It is a section of T_rM without singularities on the skeleton of (real) dimension $(2m-2r+1)$ of D and with isolated singularities on the $2(m-r+1)$-skeleton, The construction is performed by induction on the dimension of the strata:

Strata of complex dimension (strictly) less than $(r-1)$ do not appear. Consider a stratum of (complex) dimension $(r-1)$. The cells of (real) dimension $2(m-r+1)$

intersect such a stratum in a point (or in the empty set). On such a cell, the r-frame is constructed as a radial field, and hence of index +1. Since $\dim_{\mathbb{C}} S = r-1$, one has then:

$$\tilde{c}(S,X) = \sum_{K_\alpha \subset S} K_\alpha^{r-1}$$

Suppose that the construction has already been performed on strata with dimension less than $\dim S$. The r-frame Z_r is defined on strata of the boundary ∂S with singularities a_j. Denote by U the "cellular" neighborhood of ∂S, consisting of cells of D which are dual to the simplices in ∂S. The construction is performed in such a way that in U the extension of Z_r, still denoted by Z_r, has no other singularities than the points a_j. By the construction of radial frames, as proved in Schwartz [Sch00], the r-frame Z_r satisfies the following properties:

1) The index $I(Z_r, a_j)$ is the same, computed as a section of the tangent bundle to the stratum of a_j or as a section of TM,

2) The frame Z_r is pointing outwards from the cellular neighborhood U. This means that on $\partial U \cap S$, the r-frame Z_r is tangent to S and pointing inwards towards $S \setminus U$,

3) Any two radial frames are homotopic on $\partial U \cap S$, as section of $T_r S$.

In particular, this last point is proved locally in [Sch00], §6.3 (Theorem 6.3.2, *Homotopie entre deux champs radiaux au voisinage d'une strate*) and globally in [Sch00], Chapter 9 (Theorem 9.1, *Théorème global d'homotopie*).

One then extends Z_r, which is defined in $S \cap U$, inside S with isolated singularities a lying in the intersection of S with the D-cells of dimension $2(m-r+1)$. In other words, the points a are in the intersection of S with the D-cells that are dual to the K-simplices of dimension $2(r-1)$, contained in $S \setminus U$.

The contribution $\tilde{c}(S,X)$ of a stratum S is computed in terms of indices $I(Z_r, a)$ at these singular points a of Z_r, lying in $S \setminus \partial S$:

$$\tilde{c}(S,X) := \sum_{K_\alpha \subset S \setminus \partial S} \mu_\alpha K_\alpha^{r-1}$$

where the simplices K_α have dimension $2(r-1)$ and

$$\mu_\alpha = \sum_{a \in D_\alpha \cap S} I(Z_r, a) \quad ,$$

where the cell D_α is dual to K_α.

According to Steenrod [Ste51], §34, two homotopic fields of frames produce homologous cycles. A consequence of properties 2) and 3) is therefore the following Lemma:

Lemma 3.1. *The contribution $\tilde{c}(S,X)$ does not depend on the construction of the radial frame on the strata of ∂S. In other words, any two radial frames Z_r and Z'_r defined on $S \cap \partial U$ lead equivalent cycles*

$$\sum_{K_\alpha \subset S} \mu_\alpha K_\alpha^{r-1} \sim \sum_{K_\alpha \subset S} \mu'_\alpha K_\alpha^{r-1}$$

Note that, via the Alexander isomorphism $H_{2(r-1)}(S) \cong H^{2p}(M, M \setminus S)$, with $p = m-r+1$, the class $\tilde{c}(S,X)$ corresponds to the class denoted $\hat{c}^{2p}(S)$ in Schwartz [Sch00].

By Theorem 2.3, in order to prove Theorem 1.1 it suffices to show that Schwartz's Chern classes are locally determined, which is the result of the following Lemma:

Lemma 3.2. *If $f : Y \to X$ is a proper morphism, S and $T := f^{-1}(S)$ resp. are members of admissible decompositions of X, Y resp., and f restricts to an isomorphism $T \to S$, then*

$$f_*\tilde{c}(T, Y) = \tilde{c}(S, X) \quad .$$

Proof. Denote by ∂S and ∂T the boundaries of S and T respectively, unions of strata of X and Y, and denote by k the common (complex) dimension of S and T. Assume that the construction of the Schwartz classes of X and Y has been performed by induction on the dimension of strata up to dimension $k-1$, included. This construction produces two r-frames: Z_r, defined on a cellular neighborhood U of ∂S pointing outwards from U; and W_r, defined on a cellular neighborhood V of ∂T pointing outwards from V. We therefore have the following situation:

On $S \cap \partial U$ the r-frame Z_r is tangent to S and pointing into $S \smallsetminus U$. On $T \cap \partial V$, the r-frame W_r is tangent to T and points into $T \smallsetminus V$. Under the hypothesis of the statement, one has an isomorphism of pairs $(S \smallsetminus U, S \cap \partial U) \cong (T \smallsetminus V, T \cap \partial V)$. The statement follows by applying Lemma 3.1. □

References

[Alu06] Paolo Aluffi. Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes. *Pure Appl. Math. Q.*, 2(4):915–941, 2006.

[BS81] Jean-Paul Brasselet and Marie-Hélène Schwartz. Sur les classes de Chern d'un ensemble analytique complexe. In *The Euler-Poincaré characteristic (French)*, volume 83 of *Astérisque*, pages 93–147. Soc. Math. France, Paris, 1981.

[Ful84] William Fulton. *Intersection theory*. Springer-Verlag, Berlin, 1984.

[Ken90] Gary Kennedy. MacPherson's Chern classes of singular algebraic varieties. *Comm. Algebra*, 18(9):2821–2839, 1990.

[Mac74] Robert D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.

[Sch65a] M.-H. Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. I. *C. R. Acad. Sci. Paris*, 260:3262–3264, 1965.

[Sch65b] M.-H. Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. II. *C. R. Acad. Sci. Paris*, 260:3535–3537, 1965.

[Sch00] M.-H. Schwartz. *Classes de Chern des ensembles analytiques*. Hermann, Paris, 2000.

[Ste51] Norman Steenrod. *The Topology of Fibre Bundles*. Princeton Univ. Press, 1951.

Mathematics Department, Florida State University, Tallahassee FL 32306, U.S.A.
E-mail address: aluffi@math.fsu.edu

IML – CNRS, Case 907, Luminy, 13288 Marseille Cedex 9, France
E-mail address: jpb@iml.univ-mrs.fr