Option Pricing with Selfsimilar Additive Processes

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Abstract

The use of time-inhomogeneous additive processes in option pricing has become increasingly popular in recent years due to the ability of these models to adequately price across both strike and maturity with relatively few parameters. In this paper we use the property of selfsimilarity to construct two classes of models whose time one distributions agree with those of prespecified Lévy processes. The pricing errors of these models are assessed for the case of Standard and Poor's 500 index options from the year 2005. We find that both classes of models show dramatic improvement in pricing error over their associated Lévy processes. Furthermore, with regard to the average of the pricing errors over the quote dates studied, one such model yields a mean pricing error significantly less than that implied by the bid-ask spreads of the options, and also significantly less than those given by the less parsimonious Lévy stochastic volatility models.

1 Introduction

The purpose of this paper is to study the ability of certain non-Lévy, additive processes to price options on the Standard and Poor's 500 index (SPX) across both strike and maturity. As noted by Carr and Wu [9] and Eberlein and Kluge [14], Lévy models are incapable of adequately fitting implied volatility surfaces of equity options across both strike and maturity. Although stochastic volatility models with jumps have success in pricing both across strike and maturity [11], the use of such models likely incurs the cost of computing with a significantly larger number of parameters than in the Lévy case. By using additive models which allow for nonstationarity of increments, one obtains greater flexibility in pricing across maturity than with Lévy processes. Furthermore, the additive processes developed in this paper require only one more parameter than a Lévy model of like time one distribution. This additional parameter is known as the Hurst exponent.

One class of models studied consists of price processes whose logarithm is given by a H-selfsimilar additive process of Sato. These models were previously studied by Nolder in 2003 [22]. Recently Carr, Geman, Madan, and Yor [6] demonstrated that exponential H-selfsimilar additive models could be calibrated with great success using options on the Standard and Poor's 500 index, as well as for individual equity options. One reason for their success is due to the fact that these processes may be constructed from any of a multitude of selfdecomposable distributions already used in finance. In contrast, selfsimilar Lévy models consist solely of Brownian motion and Paretian stable processes [20]. The selfsimilar additive models studied in this paper are those whose time one distributions coincide with those of the Variance Gamma and Normal Inverse Gaussian processes, denoted by "HssVG" and "HssNIG," respectively.

The other class of models studied consists of those processes whose logarithm is given by a Lévy process time-changed by an independent, increasing H-selfsimilar additive process. Examples from this class of models are constructed by time-changing a Brownian motion by an independent, increasing, selfsimilar additive process. When the directing process used for the time-change is a selfsimilar Gamma process (resp. selfsimilar Inverse Gaussian process) the resulting process is denoted by "VHG" (resp. "NHIG"). In order that this research may be viewed in historical context, a brief history of pricing with additive processes now follows.

The use of additive processes to price options dates prior to 1900 when Louis Bachelier, a student of Henri Poincaré, began the development of the theory of Brownian motion [13]. In his dissertation, "Theorie de la Speculation," changes in the price of a stock were modeled with what is now known as a zero mean, arithmetic Brownian motion, a $\frac{1}{2}$-selfsimilar additive process [20]. Later, Paul Lévy studied and further developed the theory associated with a class of stochastic processes which included Brownian motion as a special case, the stable processes [17]. As in the case of Brownian motion, the increments of these stable processes were both independent and stationary. Unlike the associated one-dimensional distributions of Brownian motion, the non-Gaussian stable distributions permitted existence of, at most, the first moment [23].

The forthcoming financial application came in 1963 when Mandelbrot coined the phrase "stable Paretian" to denote the set of non-Gaussian stable distributions. By modeling the change in the logarithm of the closing spot price of cotton [21] as a Paretian stable distributed random variable, he was able to demonstrate that the tail behavior of his model was consistent with that corresponding to the observed change in the logarithm of the spot price [20]. A special case of this stable Paretian distribution would later be revived by Carr and Wu for the purpose of option pricing [9].

In January of 1973 Peter Clark introduced a new pricing model, the lognormal-normal process. Given $\sigma_1, \sigma_2, \mu > 0$, the time one distribution of the process being time–changed was Normal$(0, \sigma_2^2)$ distributed while the directing process (subordinator) had a time one distribution which was Lognormal(μ, σ_1^2) distributed. Clark attributed the nonuniformity of the rate of change in cotton futures prices to the nonuniformity of the rate at which information became available to traders. In his argument, fast trading days were caused by the influx of new information inconsistent with previously held expectations, while slow trading days were caused from a lack of new information [10]. This idea would prove useful in the subsequent development of the Lévy stochastic volatility models [5]. During that same year Fisher Black and Myron Scholes produced their closed-form expression for the price of a European call option, where the underlying price process was modeled by a geometric Brownian motion [2].

Another closed-form option pricing formula appeared in 1998 in which the logarithm of the underlier was distributed according to a non-stable Lévy process, known as the Variance Gamma process. In this model, a Brownian motion was time-changed by an independent Gamma process, yielding a model which could better accommodate the leptokurtic and negatively skewed risk-neutral distributions of the logarithm of the return data [18]. Although this process lacked selfsimilarity, its increments were independent and stationary, with notably, finite moments. Other well-known Lévy processes include: Normal Inverse Gaussian, Meixner, Generalized Hyperbolic [24], the five parameter CGMY process with drift term included [4], and the more general 6 parameter KoBoL process with drift term included [3]. As noted by Carr and Wu, Lévy models have associated model implied volatility surfaces as a function of maturity, T, and moneyness, $\frac{\ln(K/S)}{\sqrt{T}}$, which tend to a constant value as maturity gets large. This was contrary to the observed behavior associated with SPX options where, as maturity increased, the market implied volatility tended to a function which appeared to be independent of maturity alone and decreasing with moneyness [9].

One answer to the problem of fitting option prices across both strike and maturity came in 2003 when Carr, Madan, Geman, and Yor showed that one may subordinate a Lévy process by the time integral of a mean-reverting, positive process in order to obtain a process which was quite flexible across both strike and maturity [5]. Since closed-form characteristic functions exist where the rate of time change is given by the solution to the Cox-Ingersoll Ross or Ornstein-Uhlenbeck stochastic differential equation with one-dimensional Gamma or Inverse Gausssian distribution, one could use Fourier transform methods to quickly price options under such models.

Another answer to this problem came with the simpler Finite Moment Log Stable process, a stable Paretian process in which the skewness parameter, β, was set to an extreme value of -1. With β fixed and the drift term set so that the martingale condition was satisfied, the FMLS model was able to fit SPX option data relatively well with only two free parameters [9]. In an effort to find models which had more flexibility than the Finite Moment Log Stable process and were more parsimonious than the stochastic volatility models, researchers pursued the development of time-inhomogeneous additive processes. In 2003 Nolder proposed the modeling of prices of certain financial instruments with the H-selfsimilar additive processes of Sato [22]. Eberlein and Kluge in 2004 used piecewise-stationary additive processes to price swaptions [14]. More recently in 2007 Carr, Madan, Geman, and Yor used exponential H-selfsimilar additive models of Sato to price options written on the Standard and Poor's 500 index, along with various equity options. [6].

2 Construction of Additive Processes

Definition 1 *A stochastic process* $\{X_t : t \geq 0\}$ *is selfsimilar if, for any* $a > 0$*, there exists* $b > 0$*, such that* $\{X_{at}\}$ *and* $\{bX_t\}$ *are identical in law.*

Furthermore, if $\{X_t\}$ is selfsimilar, then there exists $H > 0$ such that $b = a^H$ [23, p. 73].

Definition 2 *A probability measure* μ *on* $\mathbb{R}$ *is selfdecomposable if, for all* $b > 1$*, there exists a probability measure* ρ_b *on* $\mathbb{R}$ *such that*

$$\widehat{\mu}(z) = \widehat{\mu}(z/b)\,\widehat{\rho_b}(z),$$

where $\widehat{\mu}(z)$ *is the characteristic function of* μ*.*

Selfdecomposable distributions on $\mathbb{R}$ are characterized by a Lévy measure whose density is of the form $k(x)/|x|$, where $k{:}\mathbb{R} \to \mathbb{R}^+$ is increasing for $x < 0$ and decreasing for $x > 0$ [23, p. 95]. Furthermore, Sato proved that given a selfdecomposable distribution, μ, on $\mathbb{R}^d$, an H-selfsimilar process, $\{X_t\}$ exists such that the characteristic function of the distribution at time t is $\widehat{\mu}\left(t^H z\right)$ [23, p.99]. Since a selfdecomposable distribution is infinitely divisible, it also generates a Lévy process. In this case the characteristic function at time t is $\widehat{\mu}^t(z)$ [23, p.35].

In the remainder of this section, three classes of additive models are introduced. The first process of each class has a time one distribution which is Variance Gamma distributed while the second has a time one distribution which is Normal Inverse Gaussian distributed. Given a process $\{X_t\}$, let Ψ, η, and Φ denote the Laplace exponent, characteristic exponent, and characteristic function of the distribution of X_t, respectively.

2.1 Lévy Processes

The three parameter Variance Gamma Process may be defined as a Brownian motion with drift which is time-changed by an independent Gamma process [18]. Let $\sigma, \nu > 0$ and $\mu \in \mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ define $X \equiv$

$\{X_t : t \geq 0\}$ to be a $\mathbb{R}$-valued Brownian motion, where X_1 is Normal(μ, σ^2) distributed, and $Z \equiv \{Z_t : t \geq 0\}$ to be an $\mathbb{R}$-valued, independent Gamma process where, for any $t > 0$, Z_t is Gamma$\left(\frac{t}{\nu}, \frac{1}{\nu}\right)$ distributed. Since for all $t > 0$ and $\xi \in \mathbb{R}$, the characteristic exponent of Z_t is given by $\eta_{Z_t\left[\frac{1}{\nu},\frac{1}{\nu}\right]}(\xi) = \log\left[(1 - i\xi\nu)^{-\frac{t}{\nu}}\right]$, it follows by the subordination theorem for the Lévy case that the characteristic function of the time t distribution of the Variance Gamma process is given by

$$\begin{aligned} E\left[e^{i\xi(VG_t)}\right] &= \exp\left\{\Psi_{Z_t}\left(\eta_{X_1}(\xi)\right)\right\} \\ &= \exp\left(\log\left(1 - i\left\{-i\eta_{X_1}(\xi)\right\}\nu\right)^{-\frac{t}{\nu}}\right) \\ &= \left(1 + \nu\left[-i\mu\xi + \frac{1}{2}\sigma^2\xi^2\right]\right)^{-\frac{t}{\nu}}. \end{aligned}$$

The Normal Inverse Gaussian process is defined as a Brownian motion with drift which is time-changed by an independent Inverse Gaussian process [1]. Fix $\delta, \gamma > 0$ and $\beta \in \mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ define $X \equiv \{X_t : t \geq 0\}$ to be a $\mathbb{R}$-valued Brownian motion, where X_1 is Normal$(\beta\delta^2, \delta^2)$ distributed, and $Z \equiv \{Z_t : t \geq 0\}$ to be a $\mathbb{R}$-valued, independent Inverse Gaussian process where, for any $t > 0$, Z_t is IG$(t, \delta\gamma)$ distributed with characteristic exponent given by $\eta_{Z_t[1,\delta\gamma]}(\xi) = -t\left\{\sqrt{-2i\xi + (\delta\gamma)^2} - \delta\gamma\right\}$, $\xi \in \mathbb{R}$. The subordination theorem for the Lévy case yields the following characteristic function of the time t distribution of the Normal Inverse Gaussian process.

$$\begin{aligned} E\left[e^{i\xi(NIG_t)}\right] &= \exp\left\{\Psi_{Z_t}\left(\eta_{X_1}(\xi)\right)\right\} \\ &= \exp\left(-t\left\{\sqrt{-2i\left\{-i\eta_{X_1}(\xi)\right\} + (\delta\gamma)^2} - \delta\gamma\right\}\right) \\ &= \exp\left(-\delta t\left\{\sqrt{\xi^2 - 2i\beta\xi + \gamma^2} - \gamma\right\}\right). \end{aligned}$$

2.2 H-selfsimilar Additive Processes

H-selfsimilar additive processes are constructed as in the proof to Sato's existence theorem [23, p.99]. It must first be verified that the time one distributions of the Variance Gamma and Normal Inverse Gaussian processes are selfdecomposable. Consequently, the following integral representation of the modified Bessel function of the second kind [11] is needed.

$$\int_0^\infty \exp\left(-\frac{\alpha^2 t}{2} - \frac{\beta^2}{2t}\right) t^{-(1+\nu)} dt = 2\left(\frac{\alpha}{\beta}\right)^\nu K_\nu\left(\alpha\beta\right)$$

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ define $X \equiv \{X_t : t \geq 0\}$ to be a $\mathbb{R}$-valued Brownian, motion, where X_1 is Normal(μ, σ^2) distributed, and $Z \equiv \{Z_t : t \geq 0\}$ to be an $\mathbb{R}$-valued, independent Gamma process where for any $t > 0$, Z_t is Gamma$\left(\frac{t}{\nu}, \frac{1}{\nu}\right)$ distributed. If for any $t > 0$, ϱ_{Z_t} is the Lévy density corresponding to Z_t, then by the subordination theorem for Lévy processes, the Lévy density corresponding to the distribution of the time t Variance Gamma random variable, VG_t, is given by

$$\begin{aligned}
\varrho_{VG_t}(x) &= \int_{(0,\infty)} \frac{d\mu_{X_s}}{dx}(x)\, \varrho_{Z_t}(s)\, ds \\
&= \int_{(0,\infty)} \frac{1}{\sqrt{2\pi\sigma^2 s}} \exp\left(-\frac{[x-\mu s]^2}{2\sigma^2 s}\right) \frac{t}{\nu s} \exp\left(-\frac{s}{\nu}\right) ds \\
&= \frac{t}{\sqrt{2\pi}\sigma\nu} \exp\left(\frac{x\mu}{\sigma^2}\right) \left[2\left(\frac{\sqrt{\frac{\mu^2}{\sigma^2} + \frac{2}{\nu}}}{|x|}\sigma\right)^{\frac{1}{2}} K_{\frac{1}{2}}\left(\frac{|x|}{\sigma}\sqrt{\frac{\mu^2}{\sigma^2} + \frac{2}{\nu}}\right)\right] \\
&= \frac{t}{\nu}\frac{1}{|x|} \exp\left(-\frac{1}{\sigma}\left[\sqrt{\frac{\mu^2}{\sigma^2} + \frac{2}{\nu}}\,|x| - \frac{\mu}{\sigma}x\right]\right).
\end{aligned}$$

The exponential is a decreasing function of x if $\nu > 0$. Consequently, the k-function given by

$$k(x) = \frac{t}{\nu} \exp\left(-\frac{1}{\sigma}\left[\sqrt{\frac{\mu^2}{\sigma^2} + \frac{2}{\nu}}\,|x| - \frac{\mu}{\sigma}x\right]\right)$$

with $t = 1$, satisfies the requirement for proving selfdecomposability of the time one distribution of the Variance Gamma process.

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ define $X \equiv \{X_t : t \geq 0\}$ to be a $\mathbb{R}$-valued Brownian motion, where X_1 is Normal$(\beta\delta^2, \delta^2)$ distributed, and $Z \equiv \{Z_t : t \geq 0\}$ to be a $\mathbb{R}$-valued, independent Inverse Gaussian process where, for any $t > 0$, Z_t is IG$(t, \delta\gamma)$ distributed. If, for any $t > 0$, ϱ_{Z_t} is the Lévy density corresponding to Z_t, then the Lévy density corresponding to the time t Normal Inverse Gaussian random variable, NIG_t, is given by

$$\begin{aligned}
\varrho_{NIG_t}(x) &= \int_{(0,\infty)} \frac{d\mu_{X_s}}{dx}(x)\,\varrho_{Z_t}(s)\,ds \\
&= \int_{(0,\infty)} \frac{1}{\sqrt{2\pi\delta^2 s}} \exp\left(-\frac{\left[x-\beta\delta^2 s\right]^2}{2\delta^2 s}\right) \frac{t}{\sqrt{2\pi}} s^{-\frac{3}{2}} \exp\left(-\frac{1}{2}\delta^2\gamma^2 s\right) ds \\
&= \frac{t}{2\pi\delta}\exp(\beta x)\left[2\left(\frac{\delta\sqrt{\beta^2+\gamma^2}}{\frac{|x|}{\delta}}\right)^1 K_1\left(\delta\sqrt{\beta^2+\gamma^2}\frac{|x|}{\delta}\right)\right] \\
&= \frac{t}{\pi}\delta\alpha\exp(\beta x)\frac{1}{|x|}K_1(\alpha|x|) \text{ where } \alpha \equiv \sqrt{\beta^2+\gamma^2}.
\end{aligned}$$

In order to check selfdecomposability, another integral representation theorem for the modified Bessel function of the second kind is needed [11]:

$$K_\nu(z) = \frac{e^{-z}}{\Gamma\left(\nu+\frac{1}{2}\right)}\sqrt{\frac{\pi}{2z}}\int_0^\infty e^{-t}t^{\nu-1/2}\left(1+\frac{t}{2z}\right)^{\nu-1/2} dt.$$

The Lévy density may now be expressed as

$$\begin{aligned}
\varrho_{NIG_t}(x) &= \frac{t}{\pi}\delta\alpha\exp(\beta x)\frac{1}{|x|}\left[\frac{e^{-\alpha|x|}}{\Gamma\left(\frac{3}{2}\right)}\sqrt{\frac{\pi}{2\alpha|x|}}\int_0^\infty e^{-t}t^{1/2}\left(1+\frac{t}{2\alpha|x|}\right)^{1/2} dt\right] \\
&= \exp(\beta x-\alpha|x|)\left[\frac{1}{|x|}\frac{t}{\pi}\delta\alpha\frac{1}{\Gamma\left(\frac{3}{2}\right)}\sqrt{\frac{\pi}{2\alpha|x|}}\int_0^\infty e^{-t}t^{1/2}\left(1+\frac{t}{2\alpha|x|}\right)^{1/2} dt\right].
\end{aligned}$$

Consequently, the product of ϱ_{NIG_t} and $|x|$ is decreasing on $\mathbb{R}^+$ and increasing on $\mathbb{R}^-$ if $\alpha > |\beta|$. It follows that the k-function given by

$$k(x) = \exp(\beta x-\alpha|x|)\left[\frac{t}{\pi}\delta\alpha\frac{1}{\Gamma\left(\frac{3}{2}\right)}\sqrt{\frac{\pi}{2\alpha|x|}}\int_0^\infty e^{-t}t^{1/2}\left(1+\frac{t}{2\alpha|x|}\right)^{1/2} dt\right]^{-1}$$

with $t = 1$, satisfies the requirement for proving selfdecomposability of the time one distribution of the Normal Inverse Gaussian process. The proof to Sato's existence theorem for selfsimilar additive processes may now be applied to construct H-selfsimilar additive versions of the Variance Gamma and Normal Inverse Gaussian processes.

For $\sigma, \nu, H > 0$ and $\mu \in \mathbb{R}$, the characteristic function of the distribution of HssVG_t is determined as follows. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $m = \mathbb{P}_{VG_1}$ be the selfdecomposable time one Variance Gamma(σ, μ, ν) distribution on $\mathbb{R}$. The characteristic function of the time t distribution corresponding to the H-selfsimilar additive process, $\{HssVG_t\}$ is given by

$$\begin{aligned} E\left[e^{i\xi(HssVG_t)}\right] &= \widehat{m}\left(t^H\xi; \sigma, \mu, \nu\right) \\ &= \left(1 + \nu\left[-i\mu t^H\xi + \frac{1}{2}\sigma^2 t^{2H}\xi^2\right]\right)^{-\frac{1}{\nu}}. \end{aligned}$$

Similarly, for the characteristic function of the distribution of HssNIG_t, let $\delta, \gamma, H > 0$, $\beta \in \mathbb{R}$, and $m = \mathbb{P}_{NIG_1}$ be the selfdecomposable Normal Inverse Gaussian(δ, β, γ) distribution on $\mathbb{R}$. The characteristic function of the time t distribution corresponding to the H-selfsimilar additive process, $\{HssNIG_t\}$ is given by

$$\begin{aligned} E\left[e^{i\xi(HssNIG_t)}\right] &= \widehat{m}\left(t^H\xi; \beta, \delta, \gamma\right) \\ &= \exp\left(-\delta\left\{\sqrt{t^{2H}\xi^2 - 2i\beta t^H\xi + \gamma^2} - \gamma\right\}\right). \end{aligned}$$

2.3 Subordinated Additive Processes

Subordinated additive processes are constructed by the extension of the subordination theorem of Lévy processes to the case of additive directing processes. Such an extension exists since, unlike the condition of independence of increments, time homogeneity is not a necessary condition for the representation of the characteristic exponent of the time t distribution of the subordinated process as a composition of the Laplace exponent corresponding to the directing process with the characteristic exponent corresponding to some Lévy process.

The first process to be constructed is a Brownian motion time changed by an independent H-selfsimilar additive Gamma process (VHG) while the second is a Brownian motion time changed by an independent H-selfsimilar additive Inverse Gaussian process (NHIG). In order to construct the selfsimilar additive versions of the directing processes, selfdecomposability of the time one distributions of the Gamma and Inverse Gaussian distributions must first be verified. The Lévy densities corresponding to the Gamma process and Inverse Gaussian process are given in Table 1 where $a, b > 0$ for both distributions [24].

Table 1: Lévy densities of Subordinator Distributions

Distribution	Lévy Density
Gamma(a,b)	$\varrho(x) = a\exp(-bx)\,x^{-1},\ \ x>0$
Inverse Gaussian(a,b)	$\varrho(x) = \frac{1}{\sqrt{2\pi}} a x^{-\frac{3}{2}} \exp\left(-\frac{1}{2}b^2 x\right),\ \ x>0$

Since the corresponding k-functions given by

$$\begin{aligned} k_\Gamma &= a\exp(-bx), \quad x>0 \quad \text{and} \\ k_{IG} &= \frac{1}{\sqrt{2\pi}} a x^{-\frac{1}{2}} \exp\left(-\frac{1}{2}b^2 x\right), \quad x>0 \end{aligned}$$

are decreasing on the positive reals, it follows that the Gamma and Inverse Gaussian distributions are selfdecomposable.

The characteristic function of the distribution of VHG_t is determined as follows. Let $\sigma, \nu, H > 0$ and $\mu \in \mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ define $X \equiv \{X_t : t \geq 0\}$ to be a $\mathbb{R}$-valued Brownian motion where X_1 is Normal(μ, σ^2) distributed. On the same probability space define $Z \equiv \{Z_t : t \geq 0\}$ to be an independent, increasing H-selfsimilar additive process such that Z_1 is Gamma$\left(\frac{1}{\nu}, \frac{1}{\nu}\right)$ distributed. Since for all $\xi \in \mathbb{R}$, the characteristic exponent of Z_1 is given by $\eta_{Z_1\left[\frac{1}{\nu},\frac{1}{\nu}\right]}(\xi) = \log\left[(1 - i\xi\nu)^{-\frac{1}{\nu}}\right]$, it follows by the extension to the subordination theorem for the case of additive directing processes that the characteristic function of the time t distribution corresponding to the subordinated process, $\{VHG_t\}$, is given by

$$\begin{aligned} E\left[e^{i\xi(VHG_t)}\right] &= \exp\left\{\Psi_{Z_t}\left(\eta_{X_1}(\xi)\right)\right\} \\ &= \exp\left\{\Psi_{Z_1}\left(t^H \eta_{X_1}(\xi)\right)\right\} \\ &= \exp\left(\log\left(1 - i\left\{-it^H\right\}\left[-\frac{1}{2}\sigma^2\xi^2 + i\mu\xi\right]\nu\right)^{-\frac{1}{\nu}}\right) \\ &= \left(1 + t^H\nu\left[-i\mu\xi + \frac{1}{2}\sigma^2\xi^2\right]\right)^{-\frac{1}{\nu}}. \end{aligned}$$

The characteristic function of the distribution of $NHIG_t$ is similarly defined. Fix $\delta, \gamma, H > 0$ and $\beta \in \mathbb{R}$. On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ define $X \equiv \{X_t : t \geq 0\}$ to be a $\mathbb{R}$-valued Brownian motion where X_1 is

Normal$(\beta\delta^2, \delta^2)$ distributed. On the same probability space define $Z \equiv \{Z_t : t \geq 0\}$ to be an independent, increasing H-selfsimilar additive process such that Z_1 is IG$(1, \delta\gamma)$ distributed. Since for all $\xi \in \mathbb{R}$, the characteristic exponent of Z_1 is given by $\eta_{Z_1[1,\delta\gamma]}(\xi) = -\left\{\sqrt{-2i\xi + (\delta\gamma)^2} - \delta\gamma\right\}$, it follows by the extension to the subordination theorem for the case of additive directing processes that the characteristic function of the time t distribution of the subordinated process, $\{NHIG_t\}$, is given by

$$\begin{aligned} E\left[e^{i\xi(NHIG_t)}\right] &= \exp\left\{\Psi_{Z_t}\left(\eta_{X_1}(\xi)\right)\right\} \\ &= \exp\left\{\Psi_{Z_1}\left(t^H\eta_{X_1}(\xi)\right)\right\} \\ &= \exp\left(-\left\{\sqrt{-2i\left\{-it^H\eta_{X_1}(\xi)\right\} + (\delta\gamma)^2} - \delta\gamma\right\}\right) \\ &= \exp\left(-\delta\left\{\sqrt{t^H\left[\xi^2 - 2i\beta\xi\right] + \gamma^2} - \gamma\right\}\right). \end{aligned}$$

3 Properties of the Increments

In order to investigate the distributional properties of the increments of the additive processes, the first four central moments of the distribution of the increment over the time interval $[s, t]$ for $0 < s < t$ are calculated for each of the Variance Gamma family of models. Given a $\mathbb{R}$-valued additive process $Y \equiv \{Y_t\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mu_{Y_{t+s}-Y_t}$ denote the distribution of $Y_{t+s} - Y_t$ for $s, t > 0$. By additivity of Y, the characteristic function of the increment is given by

$$\widehat{\mu}_{Y_{t+s}-Y_t}(\xi) = \frac{\widehat{\mu}_{Y_{t+s}}(\xi)}{\widehat{\mu}_{Y_t}(\xi)},\ 0 \leq s \leq t.$$

Since the distributions of $\{Y_t\}$ are infinitely divisible, set $\{(A_t, \nu_t, \gamma_t)\}$ as the corresponding system of unique Lévy triplets. By the previous equation the Lévy triplet of the increment is given by $(A_{t+s} - A_t, \nu_{t+s} - \nu_t, \gamma_{t+s} - \gamma_t)$. Let $\{Y_t\}$ denote the VG process or either of its two time-inhomogeneous versions. In the following tables the first four central moments of the increments, $X_{s,t} \equiv Y_{t+s} - Y_t$ for $0 \leq s < t$, are listed.

An interesting property among the three classes of models is the variance structure of the increment. For the Lévy case, the increment variance is an increasing function of maturity. If $H \in \left(0, \frac{1}{2}\right)$ both the selfsimilar and subordinated additive models have an increment variance which decreases

Table 2: Variance Gamma Process: Increment Moments

Moment	VG
$E\left[X_{s,t}\right]$	$s\mu$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^2\right]$	$s\left[\mu^2\nu+\sigma^2\right]$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^3\right]$	$s\left[2\mu^3\nu^2+3\sigma^2\nu\mu\right]$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^4\right]$	$\left([t+s]^2-t^2\right)\left[3\mu^4\nu^2+6\sigma^2\nu\mu^2+3\sigma^4\right]+s\left[6\mu^4\nu^3+12\sigma^2\nu^2\mu^2+3\sigma^4\nu\right]$

Table 3: Selfsimilar Variance Gamma Process: Increment Moments

Moment	HssVG
$E\left[X_{s,t}\right]$	$\left([t+s]^H-t^H\right)\mu$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^2\right]$	$\left([t+s]^{2H}-t^{2H}\right)\left[\mu^2\nu+\sigma^2\right]$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^3\right]$	$\left([t+s]^{3H}-t^{3H}\right)\left[2\mu^3\nu^2+3\sigma^2\nu\mu\right]$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^4\right]$	$\left([t+s]^{4H}-t^{4H}\right)\left[3\mu^4\nu^2\left(1+2\nu\right)+6\sigma^2\nu\mu^2\left(1+2\nu\right)+3\sigma^4\left(1+\nu\right)\right]$

Table 4: Variance H-Gamma Process: Increment Moments

Moment	VHG
$E\left[X_{s,t}\right]$	$\left([t+s]^H-t^H\right)\mu$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^2\right]$	$\left([t+s]^{2H}-t^{2H}\right)\left[\mu^2\nu\right]+\left([t+s]^H-t^H\right)\left[\sigma^2\right]$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^3\right]$	$\left([t+s]^{3H}-t^{3H}\right)\left[2\mu^3\nu^2\right]+\left([t+s]^{2H}-t^{2H}\right)\left[3\sigma^2\nu\mu\right]$
$E\left[\left(X_{s,t}-\overline{X}_{s,t}\right)^4\right]$	$\left([t+s]^{4H}-t^{4H}\right)\left[3\mu^4\nu^2\left(1+2\nu\right)\right]$ $+\left([t+s]^{3H}-t^{3H}\right)\left[6\sigma^2\nu\mu^2\left(1+2\nu\right)\right]+\left([t+s]^{2H}-t^{2H}\right)\left[3\sigma^4\left(1+\nu\right)\right]$

with maturity. For $H \in \left(\frac{1}{2}, 1\right)$ the increment variance of the selfsimilar distribution is increasing with maturity, while for the subordinated case, the increment variance is either decreasing or increasing depending on the choice of parameters. Finally, if $H \geq 1$ both models have an increment variance which increases with maturity.

4 Definition of the Price Process

Let $X \equiv \{X_t\}_{t\in[0,T]}$ be a $\mathbb{R}$-valued semimartingale on $(\Omega, \mathcal{F}, \mathbb{P})$ in which the characteristic function of its time t distribution with n-dimensional parameter vector, $\theta \in \mathbb{R}^n$, is denoted by $\phi^{\theta}_{X_t}$. Define $\Theta \subset \mathbb{R}^n$ as the set of parameter vectors such that for each $\theta \in \Theta$, $\phi^{\theta}_{X_t}(\xi)$ is defined for each $\xi \in \mathbb{R} \cup \{-i\}$ and each $t \in [0, T]$. In the case where $\{X_t\}$ has independent increments, let $S \equiv \left\{S^{\theta}_t\right\}_{t\in[0,T]}$ be the price process on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ used to model a risk-neutral underlier of an option with continuously compounded values of both interest rate, r, and dividend yield, q, where

$$S^{\theta}_t \stackrel{d}{=} \frac{S_0 \exp\left(X_t + [r-q]\, t\right)}{\phi^{\theta}_{X_t}(-i)} \text{ for each } t \in [0, T].$$

By definition, $\left\{e^{-(r-q)t} S^{\theta}_t\right\}$ is an $(\mathcal{F}_t - \mathbb{P})$ martingale, and the time-consistent linear pricing rule used to determine the fair price of a contingent claim, H, on $\left\{S^{\theta}_t\right\}_{t\in[0,T]}$ is given by π, where

$$\pi^{\theta}_{t,T}(H) = e^{-r(T-t)} E_{\mathbb{P}}\left[H\left(S^{\theta}_T\right) | \mathcal{F}_t\right]$$

[11].

In the case where X is a Lévy stochastic volatility model, the method of Carr, Geman, Madan and Yor (CGMY) [5] is used to obtain a martingale price process whose marginals match those of $\left\{e^{-(r-q)t} S^{\theta}_t\right\}$. The reason for doing this, as noted by CGMY, is that in the previous case, the additivity of X enabled the conditional expectation of the valuation operator to be evaluated as an unconditional one [5]. As a result, the normalization term of $\phi^{\theta}_{X_t}(-i)$ occurred in the definition of S^{θ}_t, $t \in [0, T]$. Consequently, when X lacks independent increments, one may not simply evaluate the pricing rule using the conditional expectation under $\mathbb{P}$.

In order to establish existence of a martingale whose marginals match those of $\left\{e^{-(r-q)t} S^{\theta}_t\right\}$, one needs only to assume that the option quotes are free of static arbitrage [8]. By results of CGMY [8], [19], there exists a martingale $\{M_t\}$ on some probability space $\left(\widetilde{\Omega}, \mathcal{G}, \{\mathcal{G}_t\}, \widetilde{\mathbb{P}}\right)$ for which the the following holds

$$M_t \overset{d}{=} e^{-(r-q)t} S_t^{\theta} \text{ for each } t \in [0,T] .$$

Consequently, the price process, $S \equiv \left\{S_t^{\theta}\right\}$, on $\left(\widetilde{\Omega}, \mathcal{G}, \{\mathcal{G}_t\}, \widetilde{\mathbb{P}}\right)$ is defined by

$$\left\{S_t^{\theta}\right\} \overset{d}{=} \left\{e^{(r-q)t} M_t^{\theta}\right\} .$$

Since $\left\{e^{-(r-q)t} S_t^{\theta}\right\}$ is a $\left(\mathcal{G}_t - \widetilde{\mathbb{P}}\right)$ martingale, it follows that the pricing rule is given by

$$\pi_{t,T}^{\theta}(H) = e^{-r(T-t)} E_{\widetilde{\mathbb{P}}} \left[H\left(S_T^{\theta}\right) | \mathcal{G}_t\right] .$$

Below are the characteristic functions of the time t distributions of the logarithm of the risk-neutral underlier for the models used in this study.

- risk-neutral exponential Variance Gamma

$$\Phi_{\ln S_t}^{VG}(\xi) = \frac{\exp\left\{i\xi\left(\ln S_0 + t\left[r-q\right]\right) - \frac{t}{\nu}\log\left(1+\nu\left[-i\mu\xi + \frac{1}{2}\sigma^2\xi^2\right]\right)\right\}}{\left(1-\nu\left[\mu+\frac{1}{2}\sigma^2\right]\right)^{-\frac{t}{\nu}}}$$

- risk-neutral exponential Variance $H-$Gamma

$$\Phi_{\ln S_t}^{VHG}(\xi) = \frac{\exp\left\{i\xi\left(\ln S_0 + t\left[r-q\right]\right) - \frac{1}{\nu}\log\left(1+t^H\nu\left[-i\mu\xi + \frac{1}{2}\sigma^2\xi^2\right]\right)\right\}}{\left(1-t^H\nu\left[\mu+\frac{1}{2}\sigma^2\right]\right)^{-\frac{1}{\nu}}}$$

- risk-neutral exponential H-selfsimilar Variance Gamma

$$\Phi_{\ln S_t}^{HssVG}(\xi) = \frac{\exp\left\{i\xi\left(\ln S_0 + t\left[r-q\right]\right) - \frac{1}{\nu}\log\left(1+\nu\left[-i\mu t^H\xi + \frac{1}{2}\sigma^2 t^{2H}\xi^2\right]\right)\right\}}{\left(1-\nu\left[\mu t^H+\frac{1}{2}\sigma^2 t^{2H}\right]\right)^{-\frac{1}{\nu}}}$$

- risk-neutral exponential Normal Inverse Gaussian

$$\Phi_{\ln S_t}^{NIG}(\xi) = \frac{\exp\left\{i\xi\left(\ln S_0 + t\left[r-q\right]\right) - \delta t\left[\sqrt{\xi^2 - 2i\beta\xi + \gamma^2} - \gamma\right]\right\}}{\exp\left(-\delta t\left[\sqrt{-1-2\beta+\gamma^2} - \gamma\right]\right)}$$

- risk-neutral exponential Normal H-Inverse Gaussian

$$\Phi_{\ln S_t}^{NHIG}(\xi) = \frac{\exp\left\{i\xi\left(\ln S_0 + t\left[r-q\right]\right) - \delta\left[\sqrt{t^H\left(\xi^2 - 2i\beta\xi\right) + \gamma^2} - \gamma\right]\right\}}{\exp\left(-\delta\left[\sqrt{t^H\left(-1-2\beta\right)+\gamma^2} - \gamma\right]\right)}$$

- risk-neutral exponential H-selfsimilar Normal Inverse Gaussian

$$\Phi_{\ln S_t}^{HssNIG}(\xi) = \frac{\exp\left\{i\xi\left(\ln S_0 + t\left[r-q\right]\right) - \delta\left[\sqrt{t^{2H}\xi^2 - 2i\beta t^H\xi + \gamma^2} - \gamma\right]\right\}}{\exp\left(-\delta\left[\sqrt{-t^{2H} - 2\beta t^H + \gamma^2} - \gamma\right]\right)}$$

- risk-neutral exponential Variance Gamma - Ornstein Uhlenbeck with stationary Gamma distribution

$$\Phi_{\ln S_t}^{\text{VG-OU-}\Gamma}(\xi) = \exp\left(i\xi\left(\ln S_0 + t\left[r-q\right]\right)\right)\frac{\Phi_{OU-\Gamma_t}\left(-i\log\Phi_{VG_1}\left(\xi; C, G, M\right);\ \lambda, a, b, 1\right)}{\Phi_{OU-\Gamma_t}\left(-i\log\Phi_{VG_1}\left(-i; C, G, M\right);\ \lambda, a, b, 1\right)}$$

where

$$\Phi_{VG_1}\left(\xi; C, G, M\right) = \left(\frac{GM}{GM + i\left(M-G\right)\xi + \xi^2}\right)^C$$

and

$$\begin{aligned}\varphi_{OU-\Gamma_t}\left(\xi; \lambda, a, b, y_0\right) &= \exp\{iy_0\lambda^{-1}\left[1-e^{-\lambda t}\right]\xi \\ &\quad + \frac{\lambda a}{i\xi - \lambda b}\left[b\log\left(\frac{b}{b - i\lambda^{-1}\left(1-e^{-\lambda t}\right)\xi}\right) - it\xi\right]\}\end{aligned}$$

- risk-neutral exponential Normal Inverse Gaussian - Ornstein Uhlenbeck

  with stationary Gamma distribution

$$\Phi_{\ln S_t}^{\text{NIG-OU-}\Gamma}(\xi) = \exp\left(i\xi\left(\ln S_0 + t\left[r-q\right]\right)\right)\frac{\Phi_{OU-\Gamma_t}\left(-i\log\Phi_{NIG_1}\left(\xi; \beta, \delta, \gamma\right);\ \lambda, a, b, 1\right)}{\Phi_{OU-\Gamma_t}\left(-i\log\Phi_{NIG_1}\left(-i;\ \beta, \delta, \gamma\right);\ \lambda, a, b, 1\right)}$$

where

$$\Phi_{NIG_1}\left(\xi; \beta, \delta, \gamma\right) = \exp\left\{-\delta\left[\sqrt{\xi^2 - 2i\beta\xi + \gamma^2} - \gamma\right]\right\}$$

and

$$\begin{aligned}\varphi_{OU-\Gamma_t}\left(\xi; \lambda, a, b, y_0\right) &= \exp\{iy_0\lambda^{-1}\left[1-e^{-\lambda t}\right]\xi \\ &\quad + \frac{\lambda a}{i\xi - \lambda b}\left[b\log\left(\frac{b}{b - i\lambda^{-1}\left(1-e^{-\lambda t}\right)\xi}\right) - it\xi\right]\}\end{aligned}$$

It is noted that in the Variance Gamma case, the parametrization $(\mu, \sigma, \nu) \mapsto (C, G, M)$ is used since it allows a single parameter, C, to be associated with time, unlike the original parametrization. Below is the required mapping.

$$\begin{aligned} C &= \frac{1}{\nu} \\ G &= \left(\sqrt{\frac{1}{4}\mu^2\nu^2 + \frac{1}{2}\sigma^2\nu} - \frac{1}{2}\mu\nu\right)^{-1} \\ M &= \left(\sqrt{\frac{1}{4}\mu^2\nu^2 + \frac{1}{2}\sigma^2\nu} + \frac{1}{2}\mu\nu\right)^{-1} \end{aligned}$$

The parametrization chosen for the NIG process, however, does not require such a transformation of variables, since the parameter, δ, is a multiplicative constant of time, t, and appears only once in the characteristic function given.

5 Calibration

In order to calculate European option prices, the "modified call" method of Carr and Madan [7] is used, requiring a closed form expression for the inverse Fourier transform of a damped call price in terms of the characteristic function of the distribution of the logarithm of the risk-neutral stock price. Put prices are calculated by the call price using put-call parity. For any of the eight previously defined models, let the risk-neutral underlier, $S \equiv \{S_t\}$, be defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ so that $\left\{e^{-(r-q)t}S_t\right\}$ is a $(\mathcal{F}_t - \mathbb{P})$ martingale. Let Θ denote the set of model parameter vectors which determine the distribution corresponding to the logarithm of the risk-neutral underlier, $\ln S_t$. Furthermore, let the set of N model option prices corresponding to $\theta \in \Theta$ be given by $\left\{\pi_i^\theta\right\}_{i=1\ldots N}$ and the corresponding set of weights be given by $\{w_i\}_{i=1\ldots N}$. In order to estimate the set of parameters chosen by the market, the quadratic pricing error, $\mathcal{E} : \Theta \to \mathbb{R}^+$, is minimized with

$$\mathcal{E}(\theta) = \sum\nolimits_{i=1}^{N} w_i \left[\pi_i^\theta (r, q, S_0, T_i, K_i) - P_i (T_i, K_i)\right]^2, \theta \in \Theta \qquad (1)$$

[11].

It is noteworthy to mention a few of the theoretical and computational difficulties pertaining to the calibration of a given model. First, the number of options used to calibrate a model is generally inadequate to uniquely identify the optimal model parameter vector [12]. Consequently, there may be many model parameter vectors which yield pricing errors to within a given tolerance [26]. Second, the set of values used to represent the current market values of the options, to within bid-ask, may induce numerical instability with respect to calendar time, resulting in relatively large variations in calibrated parameter vectors [26]. Finally, implementation of the least squares minimization may be difficult since the objective function is not convex in θ and may have many local minima [12].

Without resorting to a regularization scheme which includes a relative entropy penalization term [12] or an evolutionary optimization technique [15], the data sets are calibrated as posed, with the following conditions. First, the cut-off criterion for the optimization routine does not require that the gradient norm achieve a certain tolerance, but rather the relative change in the infinity norm of the minimizer (solution) achieve a certain tolerance. Second, in a manner similar to CGMY [6], deep out-of-the-money options with relatively small price to spot ratios are deleted from the data set in order to decrease noise in the error surface.

6 Results

In order to measure the pricing performance of each of the eight previously defined models, the Average Pricing Error ($A.P.E.$) is now defined. Denote N by the number of options used on a particular quote date, $\{P_i\}$, by the set of observed bid-ask averages, and $\{\pi_i\}$, by the set of corresponding calculated model prices. The Average Pricing Error as defined by Schoutens [24] is the following.

$$A.P.E. = \frac{\sum_{i=1}^{N} |P_i - \pi_i|}{\sum_{i=1}^{N} P_i}$$

The out-of-sample Average Pricing Errors were calculated using the in-sample parameter estimates and the bid-ask quotes taken 1 day and 1 week later than the in-sample quote date. Since the option prices were taken as the set of averages of the bid and ask quotes on a given quote date, each model price may deviate at most by $(ask - bid)/2$ if it is to lie between the bid and ask quotes. Setting the price difference to be $(ask - bid)/2$ and the

Table 5: Mean Average Pricing Error: Twelve Quote Date Time-average (Jan. - Dec. 2005)

Process Name	mean A.P.E. (%)		
	in-sample	1 day-ahead	1 week-ahead
VG	12.5	13.5	15.5
NIG	11.2	12.4	14.0
VHG	8.04	8.55	11.1
NHIG	6.31	7.10	9.84
HssVG	6.73	7.19	9.73
HssNIG	4.33	5.36	8.57
VG-OU-Gamma	5.81	6.66	9.09
NIG-OU-Gamma	5.21	6.29	8.82
market	4.75	5.10	4.63

observed option price to be $(ask + bid)/2$ in the previous formula yields the measure of uncertainty in market prices given below.

$$A.P.E._{market} = \frac{\sum_{i=1}^{N}(ask_i - bid_i)}{\sum_{i=1}^{N}(ask_i + bid_i)}$$

In Figures 1 - 3 are the bar graphs of the in-sample, 1 day-ahead, and 1 week-ahead out-of-sample A.P.E. for each quote date, for each model. The in-sample and out-of-sample twelve quote date time-averaged Average Pricing Errors are given in Table 5. The only model in this table with a mean Average Pricing Error which was less than the market average for the in-sample case was the HssNIG model. Furthermore, the worst performing model class, with respect to in-sample and out-of-sample average A.P.E., was the Lévy class. Not surprising is that with one additional parameter, the subordinated additive and selfsimilar models had lower A.P.E. values than their time-homogeneous counterpart. Remarkable, however, is the difference in performance between the two, four parameter additive models.

Table 6 contains the twelve quote averages of the ratio of A.P.E. of one model to that of the other for both in-sample and out-of-sample cases. In the first two rows of Table 6, the model pairs of the additive processes have identical time one distributions. As shown in this table, the four parameter

Table 6: Mean Ratio of Average Pricing Errors: Twelve Quote Date Time-average (Jan. - Dec. 2005)

	$\left\langle \frac{A.P.E._1}{A.P.E._2} \right\rangle$		
model 1 **:** *model 2*	in-sample	1 day-ahead	1 week-ahead
HssVG: VHG	0.832	0.839	0.850
HssNIG: NHIG	0.685	0.754	0.830
HssVG: VG-OU-Gamma	1.16	1.09	1.08
HssNIG: NIG-OU-Gamma	0.830	0.852	0.951

selfsimilar models outperformed their corresponding four parameter subordinated additive models for the in-sample and out-of-sample cases. In the second row the mean ratio of the pricing error for the selfsimilar NIG process to that of its corresponding subordinated additive model was 0.69 for the in-sample case and did not exceed 0.83 for the out-of-sample cases. Remarkably, the selfsimilar additive NIG model had pricing errors which were, on average, 83 to 85 percent of that corresponding to the NIG-OU-Gamma process, both for in-sample and one-day-ahead out-of-sample cases, as shown in the fourth row of Table 6. This is in contrast to the VG case where the selfsimilar process, on average, had pricing errors which were 116 and 109 percent of that corresponding to the VG-OU-Gamma process for in-sample and one-day-ahead out-of-sample A.P.E., respectively.

Below is a summary of the HssNIG pricing performance frequencies with respect to A.P.E taken from Figures 1-3.

- The fraction of quote dates in which the HssNIG model had a lower A.P.E. than those of both stochastic volatility models for the in-sample, 1 day-ahead, and 1 week-ahead out-of-sample cases were $\frac{10}{12}$,$\frac{12}{12}$, and $\frac{7}{12}$, respectively.

- The fraction of quote dates in which the HssNIG model had a lower A.P.E. than that of the market for the in-sample, 1 day-ahead, and 1 week-ahead out-of-sample cases were $\frac{8}{12}$,$\frac{5}{12}$, and $\frac{1}{12}$, respectively.

In Table 7 each pair of models lies within the same model class: Lévy, selfsimilar, subordinated additive, and stochastic volatility. For each of the

Table 7: Mean Ratio of Average Pricing Errors: Twelve Quote Date Time-average (Jan. - Dec. 2005)

	$\left\langle \frac{A.P.E._1}{A.P.E._2} \right\rangle$		
model 1 **:** *model 2*	in-sample	1 day-ahead	1 week-ahead
NIG: VG	0.892	0.916	0.887
NHIG: VHG	0.787	0.829	0.873
HssNIG: HssVG	0.641	0.746	0.852
NIG-OU-G: VG-OU-G	0.900	0.942	0.958

four model classes, the twelve month time-averaged ratio of A.P.E. of the NIG model to that of its corresponding VG model is given. Notable in row three is that the time-averaged ratio of A.P.E. for the selfsimilar NIG model to that of its VG model was less than 0.65 for the in-sample case and did not exceed 0.86 for the out-of-sample cases. The consistency with which the NIG distributions outperformed those of the VG family may be observed in the following summary statistics taken from Figures 1- 3.

- The fraction of quote dates in which each of the four models of the NIG family had a lower A.P.E. than that of the corresponding member of the VG family for in-sample, 1 day-ahead, and 1 week-ahead out-of-sample cases were $\frac{12}{12}$,$\frac{10}{12}$, and $\frac{7}{12}$, respectively.

In Figures 4 and 5 the market and calculated model option prices are plotted versus simple moneyness (K/S) for the set of options corresponding to the January 2005 quote date. The month of January was chosen since, for this quote date, the Average Pricing Error relative to that of the market was its worst for the HssNIG model. Figure 4 consists of the plots for the VG family while Figure 5 consists of the plots for the NIG family of models. In each panel of plots, the subordinated additive model showed a drastic improvement in pricing over the Lévy model with respect to long and near-term puts. Further improvement in the long and near-term puts and calls are seen as one moves from the subordinated additive case to the selfsimilar case. For the NIG family of models, the selfsimilar model usually provided better pricing of the two longest maturity set of puts than did its Lévy stochastic volatility counterpart.

Table 8: Variance Gamma Family: Twelve Quote Date Time-averaged Parameter Estimates (Jan. - Dec. 2005)

	model		
parameter	VG	VHG	HssVG
μ	-0.131 (0.034)	-0.132 (0.030)	-0.094 (0.015)
σ	0.119 (0.006)	0.107 (0.010)	0.132 (0.007)
ν	0.315 (0.079)	0.995 (0.231)	1.023 (0.161)
H		0.935 (0.084)	0.622 (0.028)

In order to illustrate the rather distinct distributional properties of the increments of the additive processes in this study, the term structures of the first four central moments of the distributions calibrated with the options available on the January 2005 quote date are included. In Figure 6 the top (resp. bottom) rows consist of the term structures of the first four central moments pertaining to the VG (resp. NIG) family of processes. Notable is the signature linear dependence of mean and variance on maturity for the Lévy processes. Furthermore, the absolute values of skewness and kurtosis decrease in a manner consistent with the well-known $\frac{1}{\sqrt{t}}$dependence for skewness and $\frac{1}{t}$ dependence for kurtosis. For the set of parameters chosen by the market, the skewness and kurtosis of the subordinated additive models are monotonic with respect to maturity with values lying in a smaller interval than that pertaining to the Lévy process.

The selfsimilar processes have a rather striking constant skewness and kurtosis. Invariance with respect to maturity is due to the fact that the characteristic function is obtained by a composition of a characteristic function at time one with the map, $\xi \mapsto t^H \xi$. When the chain rule is applied to the composition, the monomial term yields a factor of t^H. Consequently, the moment number is matched by the same number of factors of t^H, thereby allowing cancelation in the calculation of skewness and kurtosis.

The stability of the parameters for each model over time is denoted by the sample standard deviations of the corresponding time-averaged parameter estimates over the twelve quote dates. In Tables 8 and 9 the standard deviations are included in parentheses.

Table 9: Normal Inverse Gaussian Family: Twelve Quote Date Time-averaged Parameter Estimates (Jan. - Dec. 2005)

	model		
parameter	NIG	NHIG	HssNIG
β	-8.261 (2.536)	-25.938 (18.654)	-6.865 (1.808)
δ	0.188 (0.034)	0.111 (0.011)	0.138 (0.016)
γ	13.103 (1.931)	14.501 (6.223)	8.571 (1.088)
H		0.873 (0.116)	0.622 (0.027)

Table 10: Normal H-Inverse Gaussian Process: Three Quote Date Time-average of Parameter Estimates (Sept. - Nov. 2005)

parameter	NHIG
β	-52.373 (4.448)
δ	0.108 (0.010)
γ	22.663 (1.095)
H	0.729 (0.040)

In the case of the NHIG model, the standard deviations of the β and γ parameters were large compared to the same parameters for the other NIG models. This large standard deviation was due to the relatively large estimates of β and γ for the September, October, and November 2005 quote dates. The time-averaged parameter estimates and sample standard deviations pertaining to these three months are found in Table 10. Such behavior was not observed in the case of the selfsimilar NIG model, where the relative simplicity of the skewness and kurtosis term structures appears to have been beneficial with regard to parameter stability.

7 Conclusion

Two classes of time-inhomogeneous additive processes have been developed. Models based on the Variance Gamma and Normal Inverse Gaussian processes were constructed and used to model the logarithm of the risk-neutral underlier of European exercised options. Out-of-the-money SPX spot options from the year 2005 were used to obtain the risk-neutral parameters for the underlier. Not too surprisingly, the subordinated additive models usually had lower in-sample and out-of-sample Average Pricing Errors than did the corresponding Lévy models. In like manner, it was found that the selfsimilar models had lower in-sample and out-of-sample pricing errors than the corresponding subordinated additive models. The most successful model was the selfsimilar NIG process, HssNIG, which outperformed both of the six-parameter Lévy stochastic volatility models, VG-OU-Γ and NIG-OU-Γ, in addition to both of the subordinated additive models. Furthermore, the stability of the parameters of the selfsimilar NIG model, as measured by the sample standard deviation of the mean parameter estimate, was noticeably better than that of its subordinated additive counterpart. Finally, it is interesting to note that whether the form of the time one distribution was held in common between two models, as in the NHIG vs. HssNIG comparison, or the manner of time evolution was held in common, as in the HssVG vs. HssNIG comparison, the pricing error associated with the HssNIG model was significantly less than that of the other model.

8 Future Research

Although the selfsimilar models performed better than the subordinated models for SPX spot options, there may be other markets for which the subordinated additive models yield lower pricing errors. Consequently, a next step for future research is to perform the same analysis on options in other markets. Another area of research lies in the pricing of exotic options under these models in order to study their path dependent properties. It would be interesting to derive the relative entropy functions, the Kullback-Leibler distances, for these models and perform calibrations using a relative entropy regularization as discussed by Cont [11]. Such an exercise may be useful for increasing parameter stability.

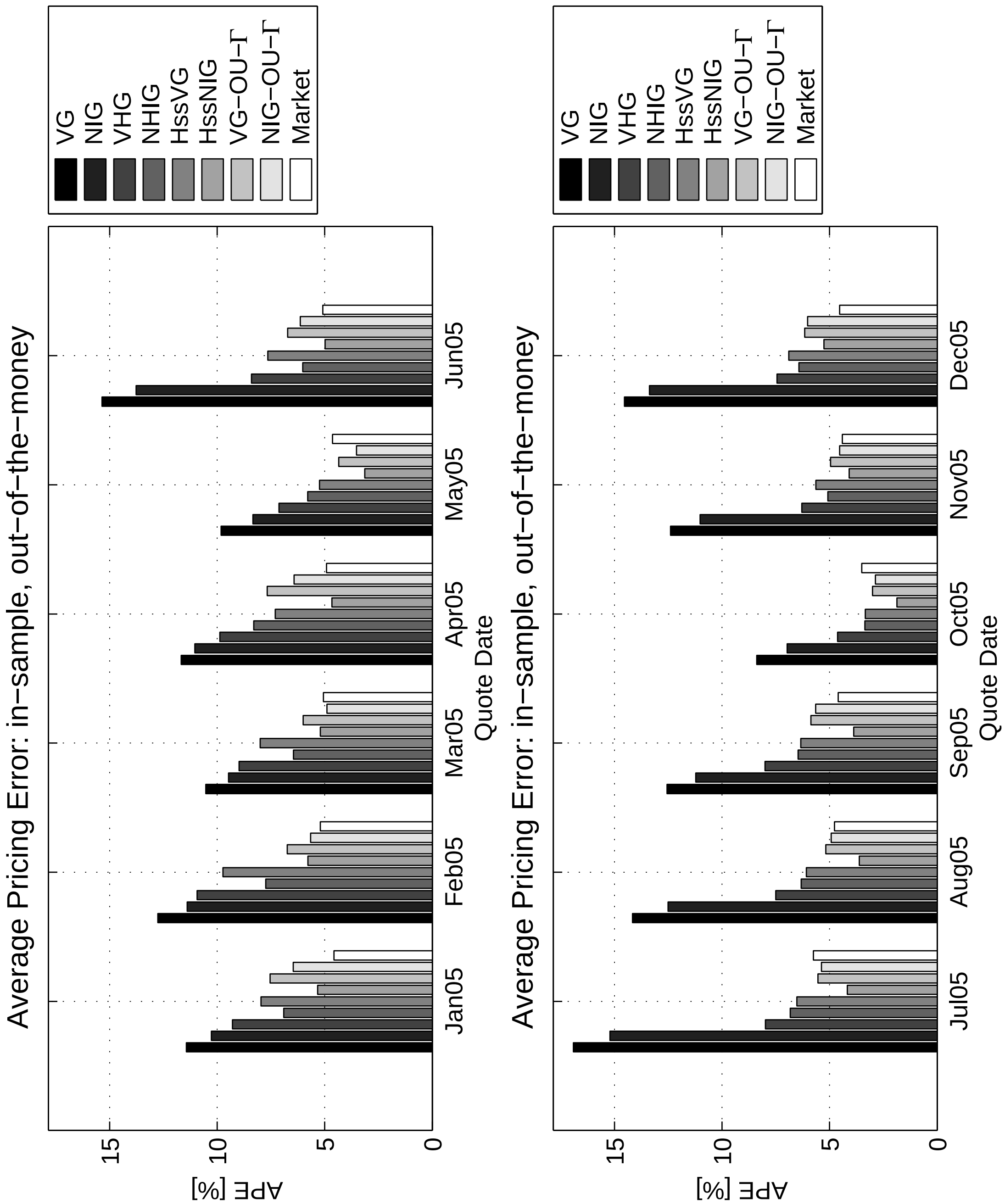

Figure 1: Average Pricing Error [in-sample, out-of-the-money options]

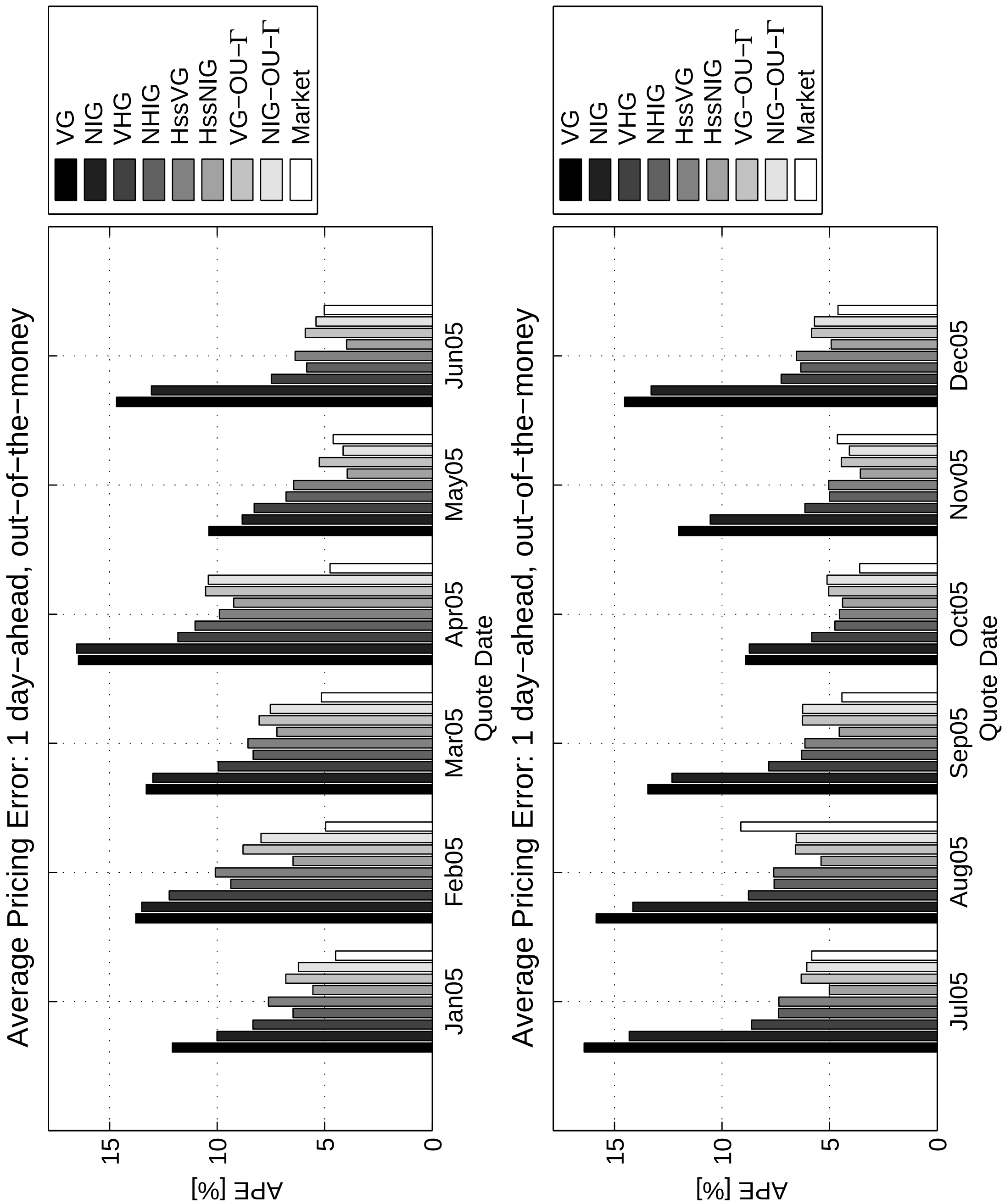

Figure 2: Average Pricing Error [1 day-ahead out-of-sample, out-of-the-money options]

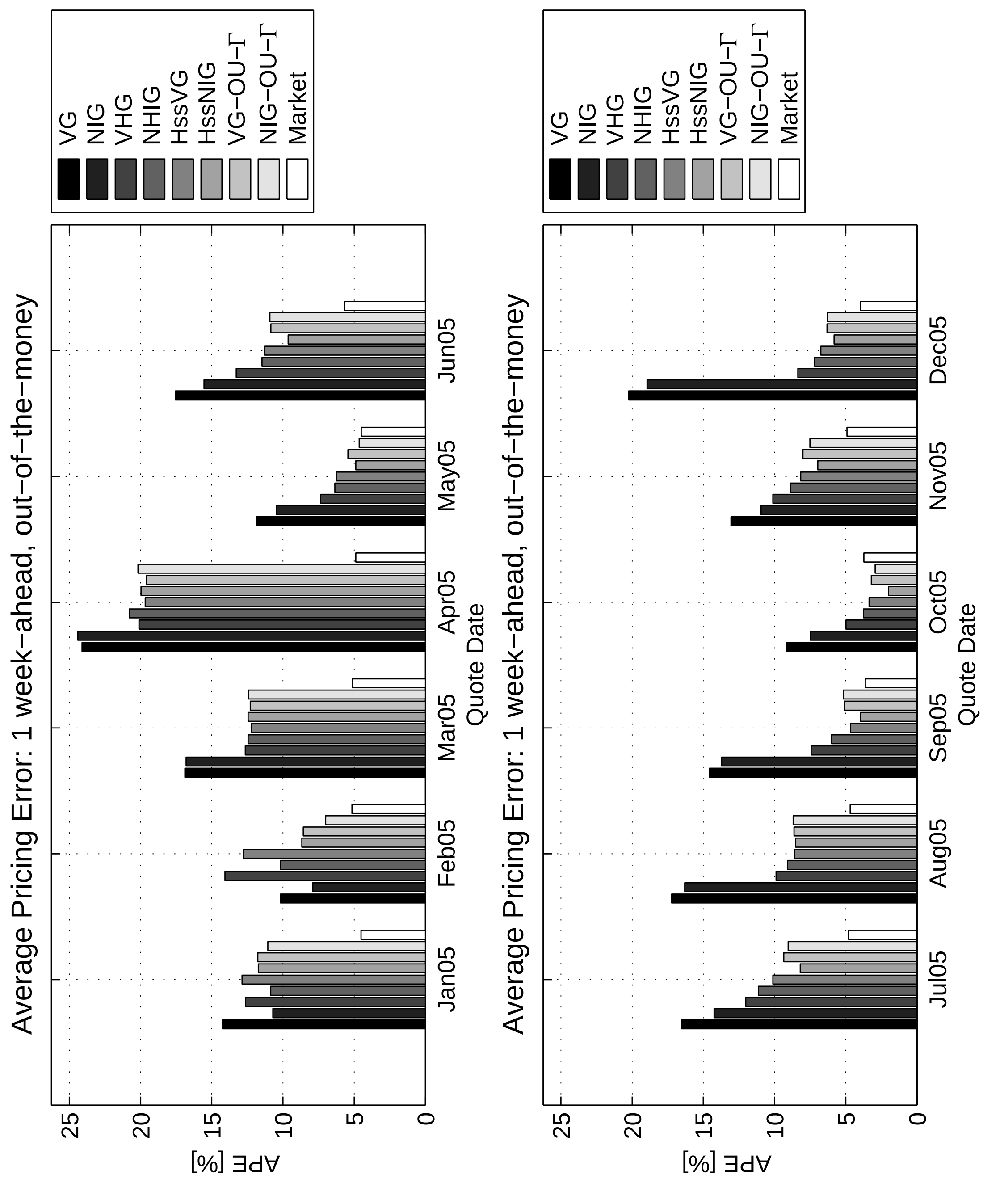

Figure 3: Average Pricing Error [1 week-ahead out-of-sample, out-of-the-money options]

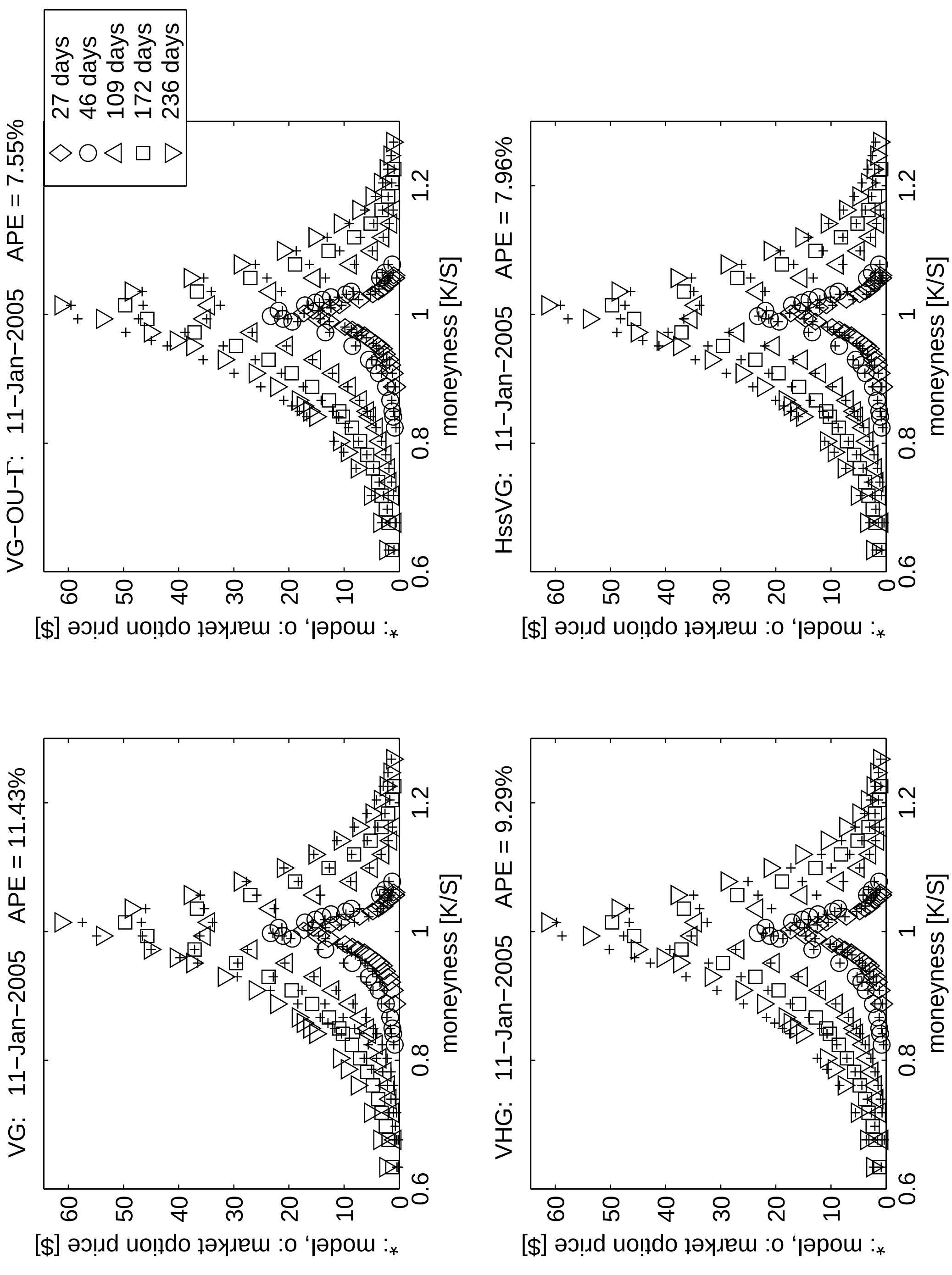

Figure 4: Market and Model Option Prices vs. Moneyness (K/S) for Variance Gamma family : January 2005 quote date, out-of-the-money options

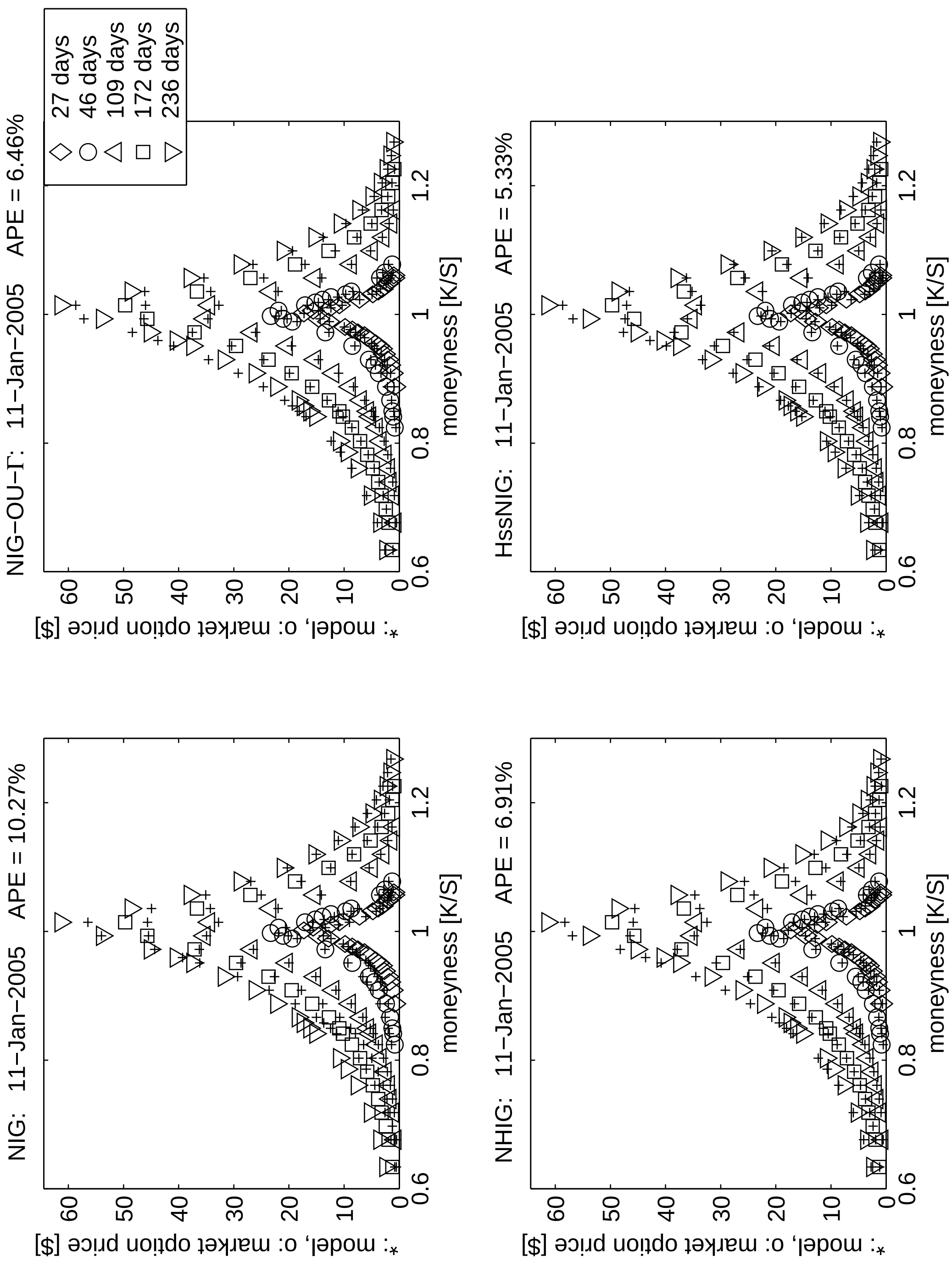

Figure 5: Market and Model Option Prices vs. Moneyness (K/S) for Normal Inverse Gaussian family : January 2005 quote date, out-of-the-money options

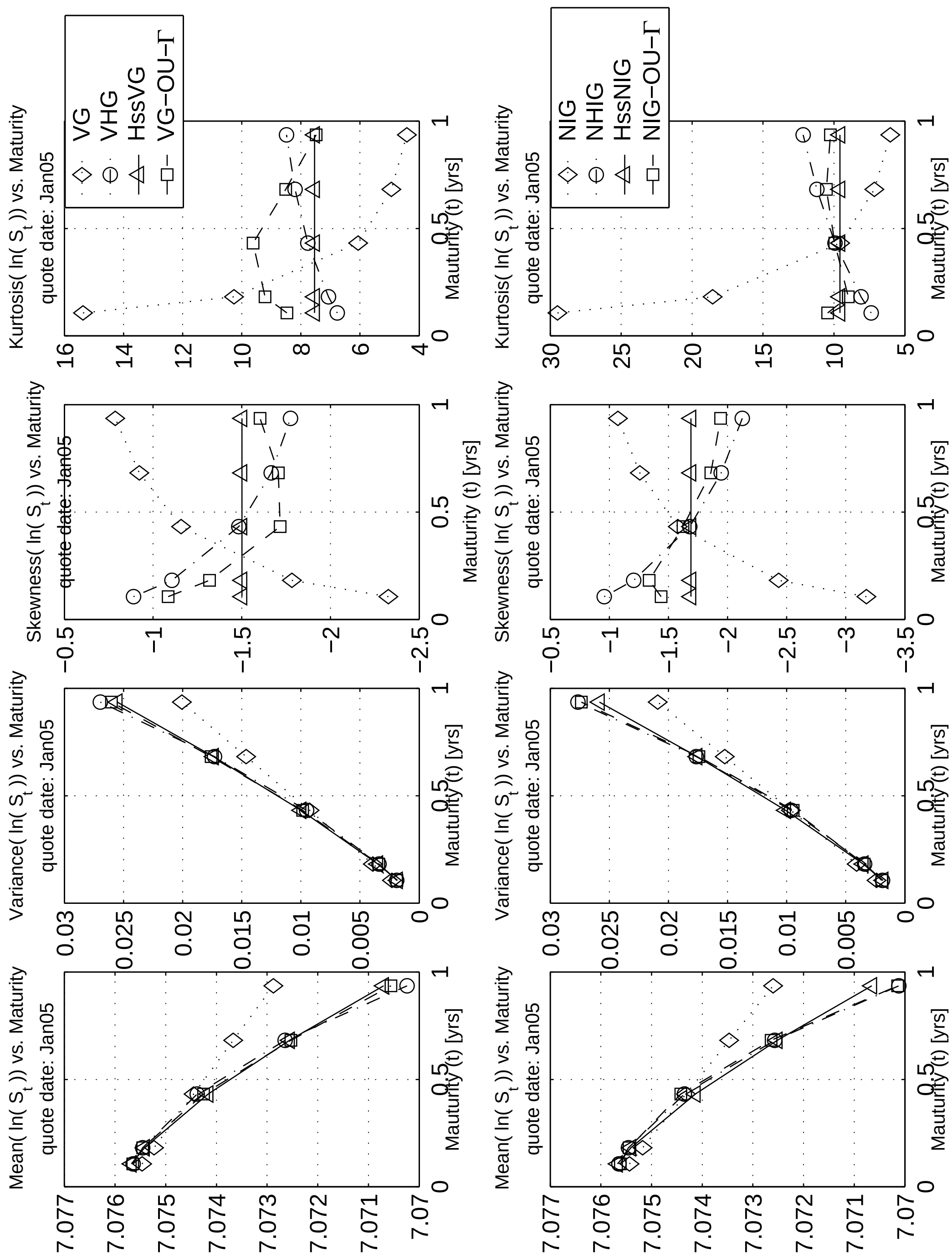

Figure 6: Risk-neutral Mean, Variance, Skewness, and Kurtosis for January 2005 quote date [VG models - top row][NIG models - bottom row]

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