S^2- and P^2-category of manifolds

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Abstract

A closed topological n-manifold M^n is of S^2- (resp. $\mathbb{P}^2$)-category 2 if it can be covered by two open subsets W_1,W_2 such that the inclusions $W_i \to M^n$ factor homotopically through maps $W_i \to S^2$ (resp. $\mathbb{P}^2$). We characterize all closed n-manifolds of S^2-category 2 and of $\mathbb{P}^2$-category 2. [1] [2]

1 Introduction

While studying the minimal number of critical points of a closed smooth n-manifold M^n, denoted by $crit(M^n)$, Lusternik and Schnirelmann introduced what is now called the Lusternik-Schnirelmann category of M^n, denoted by $cat(M^n)$, which is defined to be the the smallest number of sets, open and contractible in M^n that are needed to cover M^n. They showed that $cat(M^n)$ is a homotopy type invariant with values between 2 and $n+1$ and furthermore that $cat(M^n) \leq crit(M^n)$. This invariant has been widely studied, many references can be found in [CLOT].

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In 1968 Clapp and Puppe [CP] generalized this invariant as follows: Let $\mathcal{A}$ be a class of topological spaces. For a space $A \in \mathcal{A}$ a subset B in M^n is A-*contractible* if there are maps $f : B \longrightarrow A$ and $\alpha : A \longrightarrow M^n$ such that the inclusion map $i : B \longrightarrow M^n$ is homotopic to $\alpha \cdot f$. Then $\operatorname{cat}_{\mathcal{A}} M^n$ is the smallest number m such that M^n can be covered by m open sets, each A-contractible in M^n, for some $A \in \mathcal{A}$. If $\mathcal{A} = \{A\}$ consists only of one space A write $\operatorname{cat}_A M^n$ instead of $\operatorname{cat}_{\{A\}} M^n$. Clapp and Puppe also pointed out relations between $\operatorname{cat}_{\mathcal{A}} M^n$ and the set of critical points of smooth functions of M^n to $\mathbb{R}$. For $n = 3$ Khimshiashvili and Siersma [KhS] obtained a relation between $cat_{S^1}(M^3)$ and the set of critical circles of smooth functions $M^3 \to \mathbb{R}$. In [GGH],[GGH1],[GGH2] we obtained a complete classification of the closed (topological) n-manifolds with $cat_{S^1}(M^n) = 2$.

Motivated by the work of Gromov [G] (see also [I]) we define $cat_{ame}M^n$ to be the smallest number of open and amenable sets needed to cover M^n; here a set $A \subset M$ is *amenable* if for each path-component A_k of A the image of the inclusion induced homomorphism $im(\iota_* : \pi(A_k) \to \pi(M^n))$ is an amenable group. Gromov has shown [G] that if M^n is a closed n-manifold with positive simplicial volume then $cat_{ame}(M^n) = n + 1$. Hence, by Perelman (see [MT]), if $cat_{ame}M^3 \leq 3$ then M is a graph manifold. If $\mathcal{A}$ is the class of connected CW-complexes with amenable fundamental groups then $\operatorname{cat}_{ame} M^n \leq \operatorname{cat}_{\mathcal{A}} M^n \leq cat_K M^n \leq n+1$ for any K in $\mathcal{A}$. Examples of such K are P (a point), S^1, S^2, $\mathbb{P}^2$, $S^1 \tilde{\times} S^1$ (an S^1-bundle over S^1).

In the present paper we consider the cases of $cat_{S^2}(M^n)$ and $cat_{\mathbb{P}^2}(M^n)$. The main results are **Theorem 1** which gives a classification of (topological) n-manifolds with $cat_{S^2}(M^n) = 2$, and **Theorem 2** which exhibits a complete list of the fundamental groups of all (topological) n-manifolds with $cat_{\mathbb{P}^2}(M^n) = 2$. In particular, for $n = 3$ we obtain in **Corollary 2** a complete list of all 3-manifolds of $cat_{\mathbb{P}^2}(M^3) = 2$.

The paper is organized as follows: In section 2 we point out that if $cat_K(M^n) = 2$ for a CW-complex K then M^n can be covered by two compact K-contractible submanifolds that meet only along their boundaries and we show how to pull back K-contractible subsets of M to covering spaces of M. In section 3 we associate to a decomposition of M into two K-contractible submanifolds (where $K = S^2$ or $\mathbb{P}^2$) a graph of groups and compute the fundamental group of this graph of groups. This, together with information about the homology of M^n developed in section 4 is used to prove Theorem 1 in section 5 and Theorem 2 in section 6.

2 K-contractible subsets

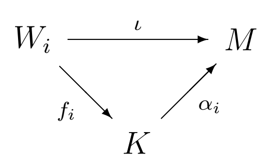
In this section we assume that $M = M^n$ is a closed connected n-manifold and K is a CW-complex.

A subset W of M is K-*contractible* (in M) if there are maps $f : W \to K$ and $\alpha : K \to M$ such that the inclusion $\iota : W \to M$ is homotopic to $\alpha \cdot f$.

$cat_K(M)$ is the smallest number m such that M can be covered by m open K-contractible subsets.

Note that a subset of a K-contractible set is also K-contractible. It is easy to show that cat_K is a homotopy type invariant.

In particular, if $\text{cat}_K(M) = 2$ then M is covered by two open sets W_0, W_1 and for $i = 0, 1$, there are maps f_i and α_i such that the diagram below is homotopy commutative:

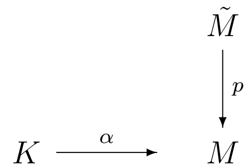


The following proposition allows us to replace the open sets W_i by compact submanifolds that meet only along their boundaries.

Proposition 1. *If* $\text{cat}_K M = 2$ *then* M *can be expressed as a union of two compact* K*-contractible* n*-submanifolds* W_0, W_1 *such that* $W_0 \cap W_1 = \partial W_0 = \partial W_1$.

This was proved in [GGH] for $K = S^1$ using topological transversality (see [KS] and [Q]). The same proof applies for any finite complex K.

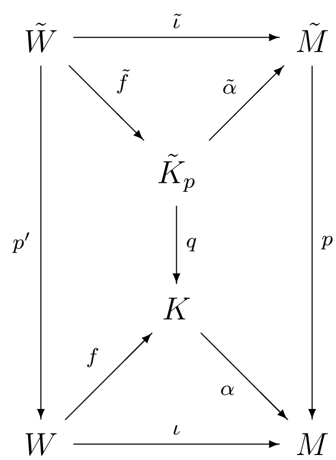
Now suppose $p : \tilde{M} \to M$ is a covering map. For $\alpha : K \to M$ let $\tilde{K}_p$ be the pullback of



i.e. $\tilde{K}_p = \{\, (x, y) \in K \times \tilde{M} \mid \alpha(x) = p(y) \,\}$ and let $q : \tilde{K}_p \to K$, $\tilde{\alpha} : \tilde{K}_p \to \tilde{M}$ be the maps induced by the projections $q(x, y) = x$, $\tilde{\alpha}(x, y) = y$.

Lemma 1. *Let $W \hookrightarrow M$ be K-contractible in M with $\iota \simeq \alpha \cdot f$ and let $p : \tilde{M} \to M$ be a covering map. Then $\tilde{W} := p^{-1}(W)$ is $\tilde{K}_p$-contractible in $\tilde{M}$.*

Proof. We have a diagram



where p' is the restriction of p and $\tilde{\iota}$ is the inclusion. The homotopy $\iota \simeq \alpha \cdot f$ lifts to a homotopy $\tilde{\iota} \simeq \tilde{h}$ for some map $\tilde{h} : \tilde{W} \to \tilde{M}$ such that $(\alpha \cdot f) \cdot p' = p \cdot \tilde{h}$. Now define $\tilde{f}$ by $\tilde{f}(z) = (fp'(z), \tilde{h}(z))$ to get $q \cdot \tilde{f} = f \cdot p'$ and $(\tilde{\alpha} \cdot \tilde{f}) = \tilde{h} \simeq \tilde{\iota}$. $\square$

3 Fundamental group

In this section we consider the structure of $\pi_1(M^n)$ for a closed n-manifold M^n with $cat_{S^2}(M^n) = 2$ or $cat_{\mathbb{P}^2}(M^n) = 2$ by using the theory of graphs of groups ([S]).

Since clearly $cat_{S^2}(S^1) = cat_{\mathbb{P}^2}(S^1) = 2$ we assume from now on that $n > 1$.

By Proposition 1 we may assume that

- $M^n = W_0 \cup W_1$ such that $F := W_0 \cap W_1 = \partial W_0 = \partial W_1$. Here $W_i = W_i^n$ are K-contractible n-submanifolds of M where $K = S^2$ or $K = \mathbb{P}^2$.

Consider the graph G of (M, F) whose vertices (resp. edges) are in one-to-one correspondence with the components W_i^j of W_i, $i = 0, 1$ (resp. with the components $F_{jk} = W_0^j \cap W_1^k$ of F). Vertices of G corresponding to W_0^j and W_1^k are joined by the edges corresponding to the components of $W_0^j \cap W_1^k$. For the associated graph $\mathcal{G}$ of groups the group G_v associated to a vertex v corresponding to a component W_i^j of W_i is $im(\pi_1(W_i^j) \to \pi_1(M))$ and the group G_e associated to an edge e corresponding to a component F_k of F is $im(\pi_1(F_k) \to \pi_1(M))$. In our case these groups are either Z_2 or trivial. For the vertices v,v' of e the monomorphisms $G_e \to G_v$ and $G_e \to G_{v'}$ are induced by inclusions.

The fundamental group of M is isomorphic to the fundamental group $\pi\mathcal{G}$ of $\mathcal{G}$ (see for example [SW]).

For the computation of $\pi\mathcal{G}$ we follow [S]: Pick an orientation of each edge of G. For each (oriented) edge e from a vertex v to a vertex v' the corresponding element in $\pi\mathcal{G}$ is denoted by g_e. The monomorphism $G_e \to G_v$ (resp. $G_e \to G_{v'}$) sends a generator a_e of G_e to a generator b_v of G_v (resp. to a generator $b_{v'}$ of $G_{v'}$). Let T be a maximal tree T in G. Then $\pi\mathcal{G}$ is generated by the g_e for each (oriented) edge e in $G - T$ and the generators b_v of G_v and defining relations are $g_e b_v g_e^{-1} = b_{v'}$ for $e \in G - T$ and $b_v = b_{v'}$ for $e \in T$.

From this presentation of $\pi\mathcal{G}$ it follows that if all vertex groups of $\mathcal{G}$ are trivial then $\pi\mathcal{G} \cong \mathrm{F}$, for some free group F, hence

Lemma 2. *If* $cat_{S^2}(M^n) = 2$ *then* $\pi\mathcal{G}$ *is a free group (possibly trivial).*

So the only closed 2-manifold of S^2-category 2 is S^2.

So assume now that $cat_{\mathbb{P}^2}(M^n) = 2$. If the group associated to a vertex v (resp. edge e) is Z_2 we say that v (resp. e) is a Z_2-vertex (resp. a Z_2-edge). An edge-path in G consisting of Z_2-vertices and Z_2-edges will be called a Z_2-path.

Lemma 3. *Assume there are more than two* Z_2*-vertices in* $\mathcal{G}$*. Then the subgraph of* G *consisting of the* Z_2*-vertices and* Z_2*-edges is connected.*

Proof. G is a bipartite graph with vertices colored by the components of W_0 and W_1. We may assume that there are at least two Z_2-vertices v, v' corresponding to different components W_0^0, W_0^k of W_0. We claim that there is a Z_2-path in G from v to v'.

To see this note that we have a homotopy-commutative diagram

$$\begin{array}{ccc} W_0^0 \cup W_0^k & \longrightarrow & M \\ & {\scriptstyle f_0} \searrow \quad \nearrow {\scriptstyle \alpha_0} & \\ & \mathbb{P}^2 & \end{array}$$

and since v and v' are Z_2-vertices, there are loops β and γ in $int(W_0^0)$ and $int(W_0^k)$ which are not trivial in M. Both are homotopic to a loop representing the non trivial element of the image of $\alpha_{0*} : \pi_1(\mathbb{P}^2) \to \pi_1(M)$. Hence β and γ are homotopic in M.

Let $H : S^1 \times I \to M$ be a homotopy between β and γ. By general position we may assume that $H^{-1}(F)$ is a union of disjoint simple closed curves in $int(S^1 \times I)$. Let $s_0 = S^1 \times \{0\}$ and let $s_1, s_2, \ldots s_{r-1}$ be the essential components of $H^{-1}(F)$ (those which do not bound disks in $S^1 \times I$) indexed in such a way that s_i separates s_0 from s_{i+1} $(i = 1, ..., r-2)$. Let $s_r = S^1 \times \{1\}$ (r is odd ≥ 3). For any i, H restricted to s_i defines a loop homotopic to β

and therefore Hs_i is nontrivial in M.

There is a path $\omega : [0,1] \to S^1 \times I$, joining $S^1 \times \{0\}$ to $S^1 \times \{1\}$, which does not intersect inessential components of $H^{-1}(F)$ and such that, for $j = 0, ..., r-1$,
i) $H\omega([j/r, (j+1)/r])$ is contained in a component W_i^j of W_i, where i is j mod 2 and
ii) $H\omega(j/r)$ is in a component F^j of F for $0 < j < r$.

v (resp. v') is the vertex associated to W_0^0 (resp. $W_0^{r-1} = W_0^k$) and the edges corresponding to the sequence $F^1, F^2, \ldots, F^{r-1}$ define a Z_2-path from v to v'.

This proves the claim.

By the same proof we see that any Z_2-vertex associated to a component of W_1 can be joined by a Z_2-path in G to the vertex corresponding to W_1^1. Hence the subgraph of G consisting of the Z_2-vertices and Z_2- edges is connected. □

Now we can describe the structure of $\pi\mathcal{G}$:

Lemma 4. *If* $cat_{\mathbb{P}^2}(M^n) = 2$ *then* $\pi\mathcal{G}$ *is one of the following groups:*
F, $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathrm{F}$, $(\mathbb{Z}_2 \times \mathrm{F}') * \mathrm{F}$
where F *and* F' *are free groups (possibly trivial).*

Proof. $\pi\mathcal{G}$ is generated by the g_e for each (oriented) edge e in $G - T$ and the generators b_v of $G_v \cong \mathbb{Z}_2$ for each Z_2-vertex v.

If all vertex groups are trivial then $\pi\mathcal{G} \cong \mathrm{F}$, for some free group F.

If all vertex groups but one is trivial then $\pi\mathcal{G} \cong \mathbb{Z}_2 * \mathrm{F} = (\mathbb{Z}_2 \times \mathrm{F}') * \mathrm{F}$ for $\mathrm{F}' = 1$.

Assume that all vertex groups but two are trivial. If all edge groups are trivial then $\pi\mathcal{G} \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathrm{F}$. If there is at least one nontrivial edge group then $\pi\mathcal{G} \cong (\mathbb{Z}_2 \times \mathrm{F}) * \mathrm{F}'$ for some free groups F, F'.

If there are more than two non-trivial ($\mathbb{Z}_2$-)vertex groups then by Lemma 3 the subgraph G' of G consisting of all non-trivial vertex and edge groups is connected and we may choose a maximal tree T' in G' with $T' \subset T$. Then $\pi\mathcal{G} \cong \pi\mathcal{G}''$, where $G'' = G/T'$ is obtained by collapsing T' to a vertex and $\mathcal{G}''$ is as in the previous paragraph. $\square$

4 Homology groups

In this section we compute the homology groups of a closed n-manifold M^n with $cat_K(M^n) = 2$ for certain CW-complexes K.

We assume that

- $M^n = W_0 \cup W_1$ such that $F := W_0 \cap W_1 = \partial W_0 = \partial W_1$.

Let $\mathcal{R}$ be a ring for which M is orientable over $\mathcal{R}$. The exact cohomology sequence of (M, W_i) is isomorphic via Lefschetz-Duality to the exact homology sequence of (M, W_{i-1}), $(i = 0, 1)$ and we obtain a commutative diagram

$$\begin{array}{ccccc} H^{n-j}(M^n, W_{1-i}; \mathcal{R}) & \longrightarrow & H^{n-j}(M^n; \mathcal{R}) & \xrightarrow{\iota^*_{n-j}} & H^{n-j}(W_{1-i}; \mathcal{R}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_j(W_i; \mathcal{R}) & \xrightarrow{\iota^j_*} & H_j(M^n; \mathcal{R}) & \longrightarrow & H_j(M^n, W_i; \mathcal{R}) \end{array}$$

where ι^*_{n-j} and ι^j_* are induced by inclusion. Thus we have an exact sequence

$$(*) \qquad 0 \to im\, \iota^j_* \to H_j(M^n; \mathcal{R}) \to im\, \iota^*_{n-j} \to 0$$

If K_i is a CW-complex and W_i is K_i-contractible $(i = 0, 1)$ with inclusions $\iota_i \simeq \alpha_i \cdot f_i$, then i_* and i^* can be factored as

$$\iota^j_* \,:\, H_j(W_i;\mathcal{R}) \xrightarrow{f_{i*}} H_j(K_i;\mathcal{R}) \xrightarrow{\alpha_{i*}} H_j(M^n;\mathcal{R})$$

$$\iota^*_{n-j} \,:\, H^{n-j}(M^n;\mathcal{R}) \xrightarrow{\alpha^*_i} H^{n-j}(K_{1-i};\mathcal{R}) \xrightarrow{f^*_i} H^{n-j}(W_{1-j};\mathcal{R})$$

Example 1. W_i is K_i-contractible and $K_0 = K_1 = S^2$.
For $0 < j < n$ and $\mathcal{R} = \mathbb{Z}$ or $\mathbb{Z}_2$ the images $im\,\iota^j_*$, $im\,\iota^*_{n-j}$ are cyclic (possibly trivial) for $j = 2$, $j = n-2$, respectively, and 0 otherwise. In particular for $n \neq 4$ it follows that $H_j(M^n;\mathbb{Z}_2) = 0$ for $j \neq 0, 2, n-2, n$ and $H_j(M^n;\mathbb{Z}_2)$ is 0 or $\mathbb{Z}_2$ for $j = 2, n-2$.
If $n = 3$ we obtain $H_1(M^3;\mathbb{Z}_2) = 0$ or $\cong \mathbb{Z}_2$. Since $\pi_1(M^3)$ is free (by Lemma 2) it follows that $\pi_1(M^3) = 1$ or $\mathbb{Z}$ and so M^3 is either S^3 or an S^2-bundle over S^1.
If $n > 3$ then $H_1(M^n;\mathbb{Z}_2) = 0$ so M^n is orientable and, by Lemma 2, $\pi(M^n) = 1$. We can therefore apply $(*)$ with $\mathcal{R} = \mathbb{Z}$.

Example 2. W_i is K_i-contractible and $K_0 = K_1 = \mathbb{P}^2$.
For $0 < j < n$ and $\mathcal{R} = \mathbb{Z}$ or $\mathbb{Z}_2$ the images $im\,\iota^j_*$; $im\,\iota^*_{n-j}$ are cyclic (possibly trivial) for $j = 1, 2$; $j = n-1, n-2$, respectively, and 0 otherwise. In particular
$H_1(M^n;\mathbb{Z}_2)$ has order ≤ 4 for $n = 3$
$H_1(M^n;\mathbb{Z}_2)$ has order ≤ 2 for $n > 3$
If M^n is orientable then $H_1(M^n;\mathbb{Z})$ is finite (of order at most 4 for $n = 3$ and order at most 2 for $n > 3$).

Example 3. W_i is K_i-contractible and $K_0 = K_1 = 2\mathbb{P}^2$ (the disjoint union of two projective planes).
If M^n is orientable then $H_1(M^n;\mathbb{Z})$ is finite (of order at most 16 for $n = 3$ and order at most 4 for $n > 3$).

5 $cat_{S^2}(M^n) = 2$

E. Turner [T] shows that for $n > 5$, a smooth closed n-manifold of type $(n, k, 1)$ admits a decomposition as a union of two D^{n-2}-bundles over S^k along their boundaries. Hence these manifolds have $cat_{S^k}(M^n) = 2$. We use Turners definition without the assumption that M is smooth:

Definition 1. A topological n-manifold M is of type (n, k, r) if M is simply-connected, $3 < 2k+1 < n$, and $H_k(M) = H_{n-k}(M) = \mathbb{Z}^r$ the only nontrivial homology groups in positive dimensions less than n.

Now assume that M is a topological n-manifold M with $cat_{S^2}(M) = 2$

For $n > 3$ we know from Example 1 that M is simply-connected and furthermore for $n > 4$ possibly nontrivial homology groups (in positive dimensions less than n) occur at most for dimensions 2 and $n-2$, in which case the homology groups are cyclic.

If $n > 4$ and $H_2(M) = 0$ then $H_{n-2}(M) = 0$ by Poincaré Duality and since M is simply connected, $\pi_j(M) = H_j(M) = 0$ for $j < n$, and $\pi_n(M) = H_j(M) = Z$ for $j = n$ by Hurewicz. Let $f : S^n \to M$ represent a generator of $\pi_n(M)$. Then f induces isomorphisms $f_* : H_*(S^n) \to H_*(M)$, so f is a homotopy equivalence by Whitehead. Hence M is homeomorphic to S^n.

If $n > 5$ and $H_2(M) \neq 0$ then from Poincaré and Universal Coefficients the torsion subgroups $tor(H_2(M)) = tor(H^{n-2}(M)) = tor(H_{n-3}(M)) = 0$ and so $H_2(M) = \mathbb{Z}$ and $H_{n-2}(M) = H^2(M) = Hom(H_2(M); \mathbb{Z}) = \mathbb{Z}$. Hence M has type $(n, 2, 1)$.

If $n = 4$ then $tor(H_2(M)) = tor(H^2(M)) = tor(H_1(M)) = 0$ and it follows from Example 1 and $(*)$ with $\mathcal{R} = \mathbb{Z}$ that $H_2(M)$ is either 0, $\mathbb{Z}$, or $\mathbb{Z}^2$. These simply-connected 4-manifolds have been classified by Friedman [F]: M^4 is one of the following:
S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# (-\mathbb{C}P^2)$,
$*\mathbb{C}P^2$, $*(\mathbb{C}P^2 \# \mathbb{C}P^2)$, $*(\mathbb{C}P^2 \# (-\mathbb{C}P^2))$.
Here $*M$ denotes a (nonsmoothable) manifold homotopy equivalent to M with nonzero Kirby-Siebenmann invariant.
Conversely each of these is homotopy equivalent to a manifold which is a union of two submanifolds each homeomorphic to D^4 or a D^2-bundle over S^2 so they are of $cat_{S^2} = 2$.

If $n = 5$ then $H^4(M^5; \mathbb{Z}_2) = H_1(M^5; \mathbb{Z}_2) = 0$, so the Kirby-Siebenmann invariant in $H^4(M^5; \mathbb{Z}_2)$ is zero and therefore M^5 is smoothable (Thm. 5.4 p.318 of [KS]). The simply-connected smooth 5-manifolds M with $H_2(M)$ cyclic have been classified by Barden (Theorem 2.3 in [B]). M^5 is one of the

following (with the notations from [B]):
$X_0 = S^5$, X_{-1}, X_∞ or $M_\infty = S^3 \times S^2$.
Here X_{-1} is the Wu-manifold, the only simply-connected 5-manifold with second homology a nontrivial finite cyclic group and X_∞ is the nontrivial S^3-bundle over S^2. Since this is the double of the nontrivial D^3-bundle over S^2 it has $cat_{S^2} = 2$.

We sum up these results in

Theorem 1. *Let M^n be a closed topological n-manifold with $cat_{S^2}(M^n) = 2$. Then M^n is one of the following:*

$$M^n \approx \begin{cases} S^2 & \text{if } n = 2 \\ S^3, \text{ an } S^2\text{-bundle over } S^1 \text{ (there are two)} & \text{if } n = 3 \\ S^4,\ S^2 \times S^2,\ \mathbb{C}P^2,\ \mathbb{C}P^2\#\mathbb{C}P^2,\ \mathbb{C}P^2\#(-\mathbb{C}P^2), & \\ \qquad *\mathbb{C}P^2,\ *(\mathbb{C}P^2\#\mathbb{C}P^2),\ *(\mathbb{C}P^2\#(-\mathbb{C}P^2)) & \text{if } n = 4 \\ S^5, \text{ an } S^3\text{-bundle over } S^2 \text{ (there are two), Wu' s manifold} & \text{if } n = 5 \\ S^n \text{ or of type } (n, 2, 1) & \text{if } n > 5 \end{cases}$$

Let us say that M^n is a *twisted double* over a D^{n-2}-bundle over S^2 if $M = V_0 \cup V_1$ with $V_0 \cap V_1 = \partial V_0 = \partial V_1$ and $V_0 \approx V_1$ homeomorphic to either the trivial or nontrivial D^{n-2}-bundle over S^2.

We now show that for $n > 5$ all the manifolds other than S^n in this Theorem are such twisted doubles, so all the manifolds M^n in the Theorem do have $cat_{S^2}(M^n) = 2$.

Corollary 1. *For $n > 5$ a closed (topological) n-manifold M has $cat_{S^2}(M^n) = 2$ if and only if M^n is S^n or a twisted double over a D^{n-2}-bundle over S^2.*

Proof. If $n = 6$ the (not necessarily smooth) manifolds of type $(6, 2, 1)$ have been classified by P. E. Jupp ([J], Proposition 1). They are obtained as a union of two D^4-bundles over S^2 along their boundaries.

If M has type $(n, 2, 1)$ for $n > 6$ then $H^4(M; \mathbb{Z}_2) = 0$ and M has a PL-structure since the Kirby-Siebenmann obstruction is 0. Now a generator of $H_2(M)$ can be represented by a Pl-embedded locally flat 2-sphere in M with normal bundle V_0 the trivial or nontrivial D^{n-2} bundle over S^2. Let $V_1 = \overline{M - V_0}$. From the homology and cohomology sequences of (M, V_1) it follows (compare e.g. [GGH2], proof of Prop. 3) that V_1 has the homology of S^2. Furthermore V_1 is 1-connected and smoothable. Embedding a smooth S^2, representing the generator of $H_2(V_1)$, in the interior of V_1 it follows from Theorem 4.1 of [SM] that V_1 is a D^{n-2}-bundle over S^2. Note that V_1 is homeomorphic to V_0 since $\partial V_1 = \partial V_0$ and the boundary of the nontrivial D^{n-2}-bundle is not homeomorphic to $S^{n-3} \times S^2$ (since its second Stiefel Whitney class is non zero). □

The twisted doubles of $D^{n-2} \times S^2$ are classified by Levine [L] (page 40 section 5.4).

6 $cat_{\mathbb{P}^2}(M^n) = 2$

In this section we classify the fundamental groups of all closed n-manifolds of $\mathbb{P}^2$-category 2. Recall that $\pi_1(M^n)$ is isomorphic to the fundamental group $\pi\mathcal{G}$ of $\mathcal{G}$ as in Lemma 4.

Theorem 2. *Let M^n be a closed n-manifold with $cat_{\mathbb{P}^2}(M^n) = 2$. Then $\pi_1(M^n)$ is one of the following groups:*

$$\pi_1(M^n) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \\ 1 & \text{if } n = 2 \\ 1,\ \mathbb{Z}_2,\ \mathbb{Z}_2 * \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z} & \text{if } n = 3 \\ 1,\ \mathbb{Z}_2 & \text{if } n > 3 \end{cases}$$

Proof. For $n = 1$ write S^1 as a union of two intervals. For $n = 2$ note that $cat_{\mathbb{P}^2}(\mathbb{P}^2) = 1$ and the fundamental group of any other non simply-connected M^2 is not in the list of Lemma 4. So suppose from now on that $n > 2$.

If M^n is orientable then $H_1(M^n; \mathbb{Z})$ is finite by Example 2 and the only possibilities for the groups in Lemma 4 are 1, $\mathbb{Z}_2$, $\mathbb{Z}_2 * \mathbb{Z}_2$. Furthermore if $n > 3$ Example 2 shows that $H_1(M^n; \mathbb{Z}_2)$ has order ≤ 2 so the only possibilities in this case are 1, $\mathbb{Z}_2$.

Thus assume that M^n is non-orientable. We check the groups in Lemma 4.

We claim that $\pi_1(M^n)$ can not be free. Otherwise, if $p : \tilde{M} \to M$ is the two-fold orientable cover then $\pi_1(M)$ and $\pi_1(\tilde{M})$ are free of rank ≥ 1 and we have a homotopy commutative diagram

$$(**)\qquad \begin{array}{ccc} W_i & \longrightarrow & M \\ & {\scriptstyle f_i}\searrow \quad \nearrow{\scriptstyle \alpha_i} & \\ & \mathbb{P}^2 & \end{array}$$

such that the image $\alpha_{i*}\pi_1(\mathbb{P}^2)$ is trivial in $\pi_1(M)$. Hence α_i has two lifts to $\tilde{M}$ and it follows that the pullback $\tilde{\mathbb{P}}^2$ of

$$(***)\qquad \begin{array}{ccc} & & \tilde{M} \\ & & \big\downarrow {\scriptstyle p} \\ \mathbb{P}^2 & \xrightarrow{\ \alpha_i\ } & M \end{array}$$

is homeomorphic to $2\mathbb{P}^2$, the disjoint union of two copies of $\mathbb{P}^2$. By Lemma 1 $\tilde{W}_i := p^{-1}(W_i)$ is $\tilde{\mathbb{P}}^2$-contractible in $\tilde{M}$. Hence $cat_{\tilde{\mathbb{P}}^2}(\tilde{M}) = 2$ and $H_1(\tilde{M};\mathbb{Z})$ is finite by Example 3, a contradiction.

If $\pi_1(M^n) = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathrm{F}$ or $\pi_1(M^n) = (\mathbb{Z}_2 \times \mathrm{F}') * \mathrm{F}$ then since by Example 2, $H_1(M^n;\mathbb{Z}_2)$ has order ≤ 2, (≤ 4) for $n > 3$, $(n = 3$, respectively), it follows that $\pi_1(M^n) = \mathbb{Z}_2$ for $n > 3$ and for $n = 3$ we have the possibilities $\pi_1(M^3) = \mathbb{Z}_2 * \mathbb{Z}_2$, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}$, or $\mathbb{Z}_2 * \mathbb{Z}$.

To complete the proof of the Theorem we show that for $n = 3$, $\pi_1(M^3) \ncong \mathbb{Z}_2 * \mathbb{Z}$.

Assuming that $\pi_1(M^3) = \mathbb{Z}_2 * \mathbb{Z}$ it follows from Kneser's Conjecture (proved by Stallings [ST]) and Perelman that $M = \mathbb{P}^3\#(S^2 \tilde{\times} S^1)$ and for the two-fold orientable cover $p : \tilde{M} \to M$ we have $\tilde{M} = \mathbb{P}^3\#(S^2 \times S^1)\#\mathbb{P}^3$. Note that $p_*\pi_1(\tilde{M})$ is a normal subgproup of $\pi_1(M)$ that contains elements of order 2. Since in $\mathbb{Z}_2 * \mathbb{Z}$ all elements of order 2 are conjugate, $p_*\pi_1(\tilde{M})$ contains *all* elements of order 2 and therefore (referring to diagram $(**)$), $\alpha_{i*}(\pi_1\mathbb{P}^2) \subset \pi_1(\tilde{M})$. As before this implies that the pullback $\tilde{\mathbb{P}}^2$ of $(***)$ is homeomorphic to $2\mathbb{P}^2$, $\tilde{W}_i = p^{-1}(W_i)$ is $\tilde{\mathbb{P}}^2$-contractible in $\tilde{M}$, $cat_{\tilde{\mathbb{P}}^2}(\tilde{M}) = 2$ and $H_1(\tilde{M};\mathbb{Z})$ is finite, which is not the case.

□

In particular for closed 3-manifolds M^3 we obtain

Corollary 2. *$cat_{\mathbb{P}^2}(M^3) = 2$ if and only if M^3 is one of the following* 3-*manifolds: S^3, $\mathbb{P}^3$, $\mathbb{P}^3 \# \mathbb{P}^3$, $\mathbb{P}^2 \times S^1$.*

Proof. By Perelman [MT] the closed 3-manifolds with fundamental groups 1 and $\mathbb{Z}_2$ are S^3 and $\mathbb{P}^3$. By Kneser's conjecture, if $\pi_1(M) = \mathbb{Z}_2 * \mathbb{Z}_2$ then $M = M_1 \# M_2$ for some closed M_i with $\pi_1(M_i) = \mathbb{Z}_2$ and so $M = \mathbb{P}^3 \# \mathbb{P}^3$. If $\pi_1(M) = \mathbb{Z}_2 \times \mathbb{Z}$ then by Epstein's Theorem [E] the element of finite order is carried by a 2-sided projective plane in M and so the orientable double cover $\tilde{M}$ of M has fundamental group $\mathbb{Z}$. By Perelman $M = S^2 \times S^1$ and it follows from Tao [TA]) that $M = \mathbb{P}^2 \times S^1$.

Conversely every M^3 in the list has $cat_{\mathbb{P}^2} = 2$, since M^3 is a union of two 3-submanifolds along their boundary, each a 3-ball or I-bundle over $\mathbb{P}^2$. □

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