LONG TIME STABILITY OF A CLASSICAL EFFICIENT SCHEME FOR TWO DIMENSIONAL NAVIER–STOKES EQUATIONS

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Abstract. This paper considers the long-time stability property of a popular semi-implicit scheme for the 2D incompressible Navier–Stokes equations in a periodic box that treats the viscous term implicitly and the nonlinear advection term explicitly. We consider both the semi-discrete (discrete in time but continuous in space) and fully discrete schemes with either Fourier Galerkin spectral or Fourier pseudospectral (collocation) methods. We prove that, in all cases, the scheme is long time stable provided that the timestep is sufficiently small. The long time stability in the L^2 and H^1 norms further leads to the convergence of the global attractors and invariant measures of the scheme to those of the Navier–Stokes equations at vanishing timestep.

Key words. 2d Navier–Stokes equations, semi-implicit schemes, global attractor, invariant measures, spectral and collocation

AMS subject classifications. 65M12, 65M70, 76D06, 37L40

1. Introduction.

The celebrated two-dimensional Navier–Stokes system for homogeneous incompressible Newtonian fluids in the vorticity–streamfunction formulation takes the form

$$\begin{aligned} \frac{\partial \omega}{\partial t} + \nabla^{\perp}\psi \cdot \nabla\omega - \nu\Delta\omega = f, \\ -\Delta\psi = \omega, \end{aligned} \tag{1.1}$$

NSE

where ω denotes the vorticity, ψ is the streamfunction $\nabla^{\perp} = (\partial_y, -\partial_x)$ and f represents (given) external forcing. For simplicity we will assume periodic boundary condition, i.e. the domain is a two dimensional torus $\mathbb{T}^2$, and that all functions have mean zero over the torus.

It is well-known that two dimensional incompressible flows can be extremely complicated with possible chaos and turbulent behavior [7,13,15,30,32,41]. Although some of the features of this turbulent or chaotic behavior may be deduced via analytic means, it is widely believed that numerical methods are indispensable for obtaining a better understanding of these complicated phenomena. For analytic forcing, it is known that the solution is analytic in space (in fact Gevrey class regular [14]), and hence a spectral approach is the obvious choice for spatial discretization. As for time discretization, one of the popular schemes [1, 4, 33] is the following semi-implicit algorithm, which treats the viscous term implicitly and the nonlinear advection term explicitly,

$$\frac{\omega^{n+1} - \omega^n}{k} + \nabla^{\perp}\psi^n \cdot \nabla\omega^n - \nu\Delta\omega^{n+1} = f^n. \tag{1.2}$$

scheme

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Here k is the timestep, and ω^n, ω^{n+1} are the approximations of the vorticity at the discrete times $nk, (n+1)k$, respectively. The convergence of this scheme on any fixed time interval is standard and well-known [18–21, 23, 37]. There are many off-the-shelf efficient solvers for (1.2), since it essentially reduces to a Poisson solver at each timestep.

It is also well-known that the NSE (1.1) is long time enstrophy stable in the sense that the enstrophy $\left(\frac{1}{2}\|\omega\|_{L^2}^2\right)$ is bounded uniformly in time, and it possesses a global attractor $\mathcal{A}$ and invariant measures [7, 13, 41]. In fact, it is the long time dynamics characterized by the global attractor and invariant measure that are central to the understanding of turbulence. Therefore a natural question is whether numerical schemes such as (1.2) can capture the long time dynamics of the NSE (1.1) in the sense of convergence of global attractors and invariant measures. At least, we would require that the scheme inherit the long time stability of the NSE.

There is a long list of works on time discretization of the NSE and related dissipative systems that preserve the dissipativity in various forms [11,12,24,25,34–36,42,43]. It has also been discovered recently that if the dissipativity of a dissipative system is preserved appropriately, then the numerical scheme would be able to capture the long time statistical property of the underlying dissipative system asymptotically, in the sense that the invariant measures of the scheme would converge to those of the continuous-in-time system [47]. The main purpose of this article is to show that the classical scheme (1.2) is long time stable in L^2 and H^1, and that the global attractor, as well as the invariant measures of the scheme, converge to those of the NSE at vanishing timestep.

2. Long time behavior of the semi-discrete scheme. We first recall the well-known periodic Sobolev spaces on $\Omega = (0, 2\pi) \times (0, 2\pi)$ with average zero:

$$\dot{H}^m_{per}(\Omega) := \left\{ \phi \in H^m_{loc}(\mathbf{R}^2) \,\middle|\, \int_\Omega \phi = 0 \text{ and } \phi \text{ is } 2\pi\text{-periodic in each direction} \right\}. \quad (2.1)$$

$\dot{H}^{-m}_{per}$ is defined as the dual space of $\dot{H}^m_{per}$ with the duality induced by the L^2 inner product. The adoption of $\dot{H}^m_{per}$ is well-known [7,40] since this space is invariant under the Navier–Stokes dynamics (1.1), provided that the initial data and the forcing term belong to the same space. We also define $\|\cdot\|_{H^s} := \|\cdot\|_{H^s(\Omega)}$ and $\dot{L}^2 := \dot{H}^0_{per}$ with the norm $\|\cdot\|_2 = \|\cdot\|_{L^2}$.

2.1. Long time stability of the scheme. We first prove that the scheme (1.2) is stable for all time.

dynsyst

LEMMA 2.1. *The scheme* (1.2) *forms a dynamical system on* $\dot{L}^2$*.*

Proof. It is easy to see that for $\omega^n \in \dot{L}^2$, we have $\psi^n \in \dot{H}^2_{per}$. Hence $\nabla^\perp\psi^n \cdot \nabla\omega^n \in \dot{H}^{-1-\alpha}_{per}$ for all $\alpha \in (0,1)$. Therefore, the classical scheme (1.2), which can be viewed as a Poisson type problem $\omega^{n+1}/k - \nu\Delta\omega^{n+1} = f^n - \nabla^\perp\psi^n \cdot \nabla\omega^n + \omega^n/k \in \dot{H}^{-1-\alpha}_{per}$, possesses a unique solution in $\dot{L}^2$ (in fact in $\dot{H}^{1-\alpha}_{per}$) and the solution depends continuously on the data. Therefore it defines a (discrete) semi-group on $\dot{L}^2$.

Using the Wente estimate (A.1), we have $\nabla^\perp\psi^n \cdot \nabla\omega^n \in \dot{H}^{-1}$, which allows us to take $\alpha = 0$ in the above. ∎

Now we derive the long time stability of the scheme (1.2) both in L^2 and in H^1. Our proof relies on a Wente type estimate on the nonlinear term (see Appendix A, which may be of independent interest.

We first show that the scheme (1.2) is uniformly bounded in L^2, provided that the timestep is sufficiently small. To this end, we take the scalar product of (1.2) with $2k\,\omega^{n+1}$ and, using the relation

$$2(\varphi-\psi,\varphi)_{L^2} = \|\varphi\|_2^2 - \|\psi\|_2^2 + \|\varphi-\psi\|_2^2 \tag{2.2}$$

where $\|\cdot\|_2$ denotes the L^2 norm, we obtain

$$\begin{aligned}\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \|\omega^{n+1}-\omega^n\|_2^2 + 2\nu k\|\omega^{n+1}\|_{H^1}^2 + 2k\,b(\psi^n,\omega^n,\omega^{n+1})&\\ = 2k\,(f^n,\omega^{n+1})_{L^2}&\end{aligned} \tag{2.3}$$ 1

where

$$b(\psi,\omega,\tilde\omega) := (\nabla^\perp\psi\cdot\nabla\omega,\tilde\omega)_{L^2} = -b(\psi,\tilde\omega,\omega), \tag{2.4}$$

the last equality obtaining upon integration by parts. Using the Cauchy–Schwarz and the Poincaré inequalities, we majorize the right-hand side of (2.3) by

$$2k\|f^n\|_2\|\omega^{n+1}\|_2 \le 2c_pk\|f^n\|_2\|\omega^{n+1}\|_{H^1} \le \nu k\|\omega^{n+1}\|_{H^1}^2 + \frac{c_p^2}{\nu}k\|f^n\|_2^2. \tag{2.5}$$ 2

Using the Wente type estimate (A.2), we bound the nonlinear term as

$$\begin{aligned}2k\,b(\psi^n,\omega^n,\omega^{n+1}) &= 2k\,b(\psi^n,\omega^{n+1},\omega^{n+1}-\omega^n)\\ &\le 2c_wk\|\nabla^\perp\psi^n\|_{H^1}\|\omega^{n+1}\|_{H^1}\|\omega^n-\omega^{n+1}\|_2\\ &\le \tfrac12\|\omega^{n+1}-\omega^n\|_2^2 + 2c_w^2k^2\|\nabla^\perp\psi^n\|_{H^1}^2\|\omega^{n+1}\|_{H^1}^2\\ &\le \tfrac12\|\omega^{n+1}-\omega^n\|_2^2 + 2c_w^2k^2\|\omega^n\|_2^2\|\omega^{n+1}\|_{H^1}^2.\end{aligned} \tag{2.6}$$ 3

Relations (2.3)–(2.6) imply

$$\begin{aligned}\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \tfrac12\|\omega^{n+1}-\omega^n\|_2^2 + (\nu-2c_w^2k\|\omega^n\|_2^2)k\,\|\omega^{n+1}\|_{H^1}^2&\\ \le \frac{c_p^2}{\nu}k\|f^n\|_2^2.&\end{aligned} \tag{2.7}$$ 4

Here and in what follows, C and c denote generic constants whose value may not be the same each time they appear. Numbered constants, e.g., c_{42}, have fixed values; c_p is the Poincaré constant, $\|w\|_2 \le c_p\|w\|_{H^1}$, and c_w is the constant from Wente's inequalities (A.1)–(A.3).

We are now able to prove the following:

t:bdh **Lemma 2.2.** *Let $\omega_0 \in \dot L^2$ and let ω^n be the solution of the numerical scheme (1.2). Also, let $f \in L^\infty(\mathbf{R}_+;\dot L^2)$ and set $\|f\|_\infty := \|f\|_{L^\infty(\mathbf{R}_+;L^2)}$. Then there exists $M_0 = M_0(\|\omega_0\|_2,\nu,\|f\|_\infty)$ such that if*

$$k \le \frac{\nu}{4c_w^2M_0^2}, \tag{2.8}$$ 5a

then

$$\|\omega^n\|_2^2 \le \Bigl(1+\frac{\nu k}{2c_p^2}\Bigr)^{-n}\|\omega_0\|_2^2 + \frac{2c_p^4}{\nu^2}\|f\|_\infty^2\Bigl[1-\Bigl(1+\frac{\nu k}{2c_p^2}\Bigr)^{-n}\Bigr],\ \forall n\ge 0 \tag{2.9}$$ q:bdv

and

$$\frac{\nu}{2}k\sum_{n=i}^{m}\|\omega^n\|_{H^1}^2 \le \|\omega^{i-1}\|_2^2 + \frac{c_p^2}{\nu}\|f\|_\infty^2(m-i+1)k, \quad \forall\, i=1,\cdots,m. \tag{2.10}$$ 6

We note that (2.9) implies

$$\|\omega^n\|_2^2 \le M_0^2(\|\omega_0\|_2,\nu,\|f\|_\infty) := \|\omega_0\|_2^2 + \frac{2c_p^4}{\nu^2}\|f\|_\infty^2 \qquad \text{for } n=0,\cdots. \tag{2.11}$$ q:bdinh

Proof. We will first prove (2.9) by induction on n. It is clear that (2.9) holds for $n=0$. Assuming that (2.9) holds for $n=0,\cdots,m$, we then have (2.11) for $n=0,\cdots,m$. Then (2.7) and (2.8) yield

$$\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \frac{1}{2}\|\omega^{n+1}-\omega^n\|_2^2 + \frac{\nu}{2}k\|\omega^{n+1}\|_{H^1}^2 \le \frac{c_p^2}{\nu}k\|f^n\|_2^2 \tag{2.12}$$ 7

for all $n=0,\cdots,m$. Using again the Poincaré inequality, the above inequality implies

$$\|\omega^{n+1}\|_2^2 \le \frac{1}{\alpha}\|\omega^n\|_2^2 + \frac{c_p^2}{\alpha\nu}k\|f^n\|_2^2, \tag{2.13}$$ 8

where

$$\alpha = 1 + \frac{\nu}{2c_p^2}k. \tag{2.14}$$ 9

Using recursively (2.13), we find

$$\begin{aligned}\|\omega^{m+1}\|_2^2 &\le \frac{1}{\alpha^{m+1}}\|\omega^0\|_2^2 + \frac{c_p^2k}{\nu}\sum_{i=1}^{m+1}\frac{1}{\alpha^i}\|f^{m+1-i}\|_2^2 \\ &\le \left(1+\frac{\nu k}{2c_p^2}\right)^{-m-1}\|\omega_0\|_2^2 + \frac{2c_p^4}{\nu^2}\|f\|_\infty^2\left[1-\left(1+\frac{\nu k}{2c_p^2}\right)^{-m-1}\right],\end{aligned} \tag{2.15}$$ 10

and thus (2.9) holds for $n=m+1$. We therefore have that (2.9) holds for all $n\ge 0$, as does (2.11).

Now summing inequality (2.12) with n from i to m and dropping some positive terms, we find

$$\begin{aligned}\frac{\nu}{2}k\sum_{n=i}^{m}\|\omega^{n+1}\|_{H^1}^2 &\le \|\omega^i\|_2^2 + \frac{c_p^2}{\nu}k\sum_{n=i}^{m}\|f^n\|_2^2 \\ &\le \|\omega^i\|_2^2 + \frac{c_p^2}{\nu}\|f\|_\infty^2(m-i+1)k,\end{aligned} \tag{2.16}$$ 11

which is exactly (2.10). This completes the proof of Lemma 2.2. □

C1 Corollary 1. *If*

$$0 < k \le \min\left\{\frac{\nu}{4c_w^2M_0^2}, \frac{2c_p^2}{\nu}\right\} =: k_0, \tag{2.17}$$ q:k0

then

$$\|\omega^n\|_2^2 \le 2\rho_0^2, \quad \forall\, nk \ge T_0(\|\omega_0\|_2, \|f\|_\infty) := \frac{8c_p^2}{\nu} \ln\left(\frac{\|\omega_0\|_2}{\rho_0}\right), \tag{2.18}$$ q:tabs

where $\rho_0 := (\sqrt{2}c_p^2/\nu)\|f\|_\infty$.

Proof. From the bound (2.9) on $\|\omega^n\|_2^2$, we infer that

$$\|\omega^n\|_2^2 \le \Big(1 + \frac{\nu k}{2c_p^2}\Big)^{-n} \|\omega_0\|_2^2 + \rho_0^2,$$

and using assumption (2.17) on k and the fact that $1 + x \ge \exp(x/2)$ if $x \in (0, 1)$, we obtain

$$\|\omega^n\|_2^2 \le \exp\Big(-nk\frac{\nu}{4c_p^2}\Big)\|\omega_0\|_2^2 + \rho_0^2.$$

For $nk \ge T_0$, the last inequality implies the conclusion (2.18) of the Corollary. □

Now we show that the H^1 norm is also bounded uniformly in time under the same kind of constraint as for the L^2 estimate. To this end, we first prove in Lemma 2.3 that with H^1 initial data, ω^n is bounded for $n \le N$ for some N. We then show in Lemma 2.5, with the aid of a version of the discrete uniform Gronwall lemma, that with L^2 initial data, ω^n is bounded for all $n \ge N$.

finite LEMMA 2.3. *Let* $\omega_0 \in \dot{H}^1$ *and let* ω^n *be the solution of the numerical scheme* (1.2). *Also, let* $k \le k_0$*, with* k_0 *as in Corollary 1, and let* $r \ge 8c_p^2/\nu$ *be arbitrarily fixed. Then, for* $n = 1, \cdots, N_0 + N_r - 1$,

$$\|\omega^n\|_{H^1}^2 \le 4^{(2c_w^2/\nu)M_0^2(T_0+r)}\Big(\|\omega_0\|_{H^1}^2 + \frac{1}{c_w^2 M_0^2}\|f\|_\infty^2\Big) \tag{2.19}$$ 12

where $N_0 = \lfloor T_0/k \rfloor$*, with* $N_r = \lfloor r/k \rfloor$ *and* T_0 *that in Corollary 1.*

Proof. The proof relies on induction on n.

Taking the scalar product of (1.2) with $-2k\,\Delta\omega^{n+1}$, we obtain

$$\begin{aligned} \|\omega^{n+1}\|_{H^1}^2 - \|\omega^n\|_{H^1}^2 + \|\omega^{n+1} - \omega^n\|_{H^1}^2 + 2\nu k\|\Delta\omega^{n+1}\|_2^2 \\ - 2k\, b(\psi^n, \omega^n, \Delta\omega^{n+1}) = -2k(f^n, \Delta\omega^{n+1})_{L^2}. \end{aligned} \tag{2.20}$$ 13

We bound the right-hand side of (2.20) using the Cauchy–Schwarz inequality,

$$-2k(f^n, \Delta\omega^{n+1})_{L^2} \le 2k\|f^n\|_2\|\Delta\omega^{n+1}\|_2 \le \frac{\nu k}{2}\|\Delta\omega^{n+1}\|_2^2 + \frac{2k}{\nu}\|f^n\|_2^2. \tag{2.21}$$ 14

Using the Wente type estimate (A.2), we bound the nonlinear term as

$$\begin{aligned} 2k\, b(\psi^n, \omega^n, \Delta\omega^{n+1}) &= 2k\, b(\psi^n, \omega^n - \omega^{n+1}, \Delta\omega^{n+1}) \\ &\qquad + 2k\, b(\psi^n, \omega^{n+1}, \Delta\omega^{n+1}) \\ &\le 2c_w k\|\nabla^\perp\psi^n\|_{H^1}\|\omega^{n+1} - \omega^n\|_{H^1}\|\Delta\omega^{n+1}\|_2 \\ &\qquad + 2c_w k\|\nabla^\perp\psi^n\|_{H^1}\|\omega^{n+1}\|_{H^1}\|\Delta\omega^{n+1}\|_2 \\ &\le \frac{1}{2}\|\omega^{n+1} - \omega^n\|_{H^1}^2 + 2c_w^2 k^2\|\nabla^\perp\psi^n\|_{H^1}^2\|\Delta\omega^{n+1}\|_2^2 \\ &\qquad + \frac{\nu}{2}k\|\Delta\omega^{n+1}\|_2^2 + \frac{2c_w^2}{\nu}k\|\nabla^\perp\psi^n\|_{H^1}^2\|\omega^{n+1}\|_{H^1}^2 \end{aligned}$$

$$\leq \frac{1}{2}\|\omega^{n+1}-\omega^n\|_{H^1}^2 + 2c_w^2k^2\|\omega^n\|_2^2\|\Delta\omega^{n+1}\|_2^2 + \frac{\nu}{2}k\|\Delta\omega^{n+1}\|_2^2 + \frac{2c_w^2}{\nu}k\|\omega^n\|_2^2\|\omega^{n+1}\|_{H^1}^2. \tag{2.22}$$ 15

Relations (2.20)–(2.22) imply

$$\Big(1-\frac{2c_w^2}{\nu}M_0^2k\Big)\|\omega^{n+1}\|_{H^1}^2 - \|\omega^n\|_{H^1}^2 + \frac{1}{2}\,\|\omega^{n+1}-\omega^n\|_{H^1}^2 + \big(\nu - 2c_w^2kM_0^2\big)\,k\,\|\Delta\omega^{n+1}\|_2^2 \leq \frac{2k}{\nu}\,\|f^n\|_2^2, \tag{2.23}$$ 16

from which we find

$$\|\omega^{n+1}\|_{H^1}^2 \leq \frac{1}{\alpha}\|\omega^n\|_{H^1}^2 + \frac{2k}{\alpha\nu}\|f\|_\infty^2, \tag{2.24}$$ 17

where

$$\alpha = 1 - \frac{2c_w^2}{\nu}kM_0^2 > 0. \tag{2.25}$$ 18

Using recursively (2.24), we find

$$\begin{aligned}\|\omega^{n+1}\|_{H^1}^2 &\leq \frac{1}{\alpha^{n+1}}\|\omega_0\|_{H^1}^2 + \frac{2}{\nu}k\|f\|_\infty^2\sum_{i=1}^{n+1}\frac{1}{\alpha^i}\\ &\leq \left(1-\frac{2c_w^2}{\nu}kM_0^2\right)^{-1-n}\left[\|\omega_0\|_{H^1}^2 + \frac{1}{c_w^2M_0^2}\|f\|_\infty^2\right].\end{aligned} \tag{2.26}$$

Since $2c_w^2M_0^2k/\nu \leq 1/2$ by hypothesis (2.17) and

$$1-x \geq 4^{-x} \text{ if } x \in (0,1/2),$$

relation (2.26) gives conclusion (2.19) of Lemma 2.3. Thus, the lemma is proved. ∎

In order to obtain a uniform bound valid for $n \geq N_0 + N_r$, we need the following discrete uniform Gronwall lemma, which has been proved in [42] and we repeat here for convenience.

dugronwall LEMMA 2.4. *We are given $k > 0$, positive integers n_0, n_1, and positive sequences ξ_n, η_n, ζ_n such that*

$$k\,\eta_{n+1} < \tfrac{1}{2}, \qquad \forall n \geq n_0, \tag{2.27}$$ time

$$(1-k\,\eta_{n+1})\,\xi_{n+1} \leq \xi_n + k\,\zeta_{n+1}, \qquad \forall n \geq n_0. \tag{2.28}$$ gronseq2

Assume also that

$$k\sum_{n=n_2}^{n_2+n_1+1}\eta_n \leq A_\eta, \qquad k\sum_{n=n_2}^{n_2+n_1+1}\zeta_n \leq A_\zeta \qquad \text{and} \qquad k\sum_{n=n_2}^{n_2+n_1+1}\xi_n \leq A_\xi, \tag{2.29}$$ groncond

for all $n_2 \geq n_0$. We then have,

$$\xi_{n+1} \leq \Big(\frac{A_\xi}{kn_1} + A_\zeta\Big)\,\mathrm{e}^{4A_\eta}, \quad \forall n > n_0 + n_1. \tag{2.30}$$ ugronest

Proof. Let m_1 and m_2 be such that $n_0 < m_1 \le m_2 \le m_1 + n_1$. Using recursively (2.28), we derive

$$\xi_{m_1+n_1+1} \le \prod_{n=m_2}^{m_1+n_1+1} \frac{1}{1-k\eta_n}\xi_{m_2-1} + k\sum_{n=m_2}^{m_1+n_1+1} \zeta_n \prod_{j=n}^{m_1+n_1+1} \frac{1}{1-k\eta_j}. \tag{2.31}$$

Using the fact that $1 - x \ge \mathrm{e}^{-4x}$ for all $x \in (0, 1/2)$, and recalling assumptions (2.27) and (2.29), we obtain

$$\xi_{m_1+n_1+1} \le (\xi_{m_2-1} + A_\zeta)e^{4A_\eta}.$$

Multiplying this inequality by k, summing m_2 from m_1 to m_1+n_1 and using the third assumption (2.29) gives conclusion (2.30) of the lemma. □

We are now able to derive a uniform bound for $\|\omega^n\|_{H^1}$ valid for sufficiently large n. More precisely, we have the following:

5 LEMMA 2.5. *Let $\omega_0 \in \dot{L}^2$ and let ω^n be the solution of the numerical scheme (1.2). Also, let $k \le k_0$, with k_0 that in Corollary 1. Then there exist constants $M_1 = M_1(\nu, \|f\|_\infty)$ and $N = N(\|\omega_0\|_2, \nu, \|f\|_\infty)$ such that*

$$\|\omega^n\|_{H^1} \le M_1 \quad \text{for all } n \ge N. \tag{2.32}$$ 20

Proof. Let k be as in the hypothesis, T_0 be as in Corollary 1, r as in Lemma 2.3 and set $N_0 := \lfloor T_0/k \rfloor$. We will apply Lemma 2.4 to (2.23), with $\xi_n = \|\omega^n\|^2_{H^1}$, $\eta_n = 2c_w^2 M_0^2/\nu$, $\zeta_n = 2\|f\|^2_\infty/\nu$, $n_0 = N_0 + 2$, $n_1 = N_r - 2$. For $n_2 \ge n_0$, we compute (taking into account that, by (2.18), $\|\omega^n\|_2^2 \le 2\rho_0^2$ for $n \ge N_0$)

$$k\sum_{n=n_2}^{n_2+n_1+1} \eta_n = k\sum_{n=n_2}^{n_2+n_1+1} \frac{2c_w^2}{\nu}M_0^2 \le \frac{4c_w^2}{\nu}\rho_0^2 r =: A_\eta, \tag{2.33}$$

$$k\sum_{n=n_2}^{n_2+n_1+1} \zeta_n = k\sum_{n=n_2}^{n_2+n_1+1} \frac{2}{\nu}\|f\|^2_\infty \le \frac{2}{\nu}\|f\|^2_\infty r =: A_\zeta, \tag{2.34}$$

$$\begin{aligned} k\sum_{n=n_2}^{n_2+n_1+1} \xi_n &= k\sum_{n=n_2}^{n_2+n_1+1} \|\omega^n\|^2_{H^1} && \text{[by (2.10)]} \\ &\le \frac{2}{\nu}\left(\|\omega^{n_2-1}\|_2^2 + \frac{c_p^2}{\nu}\|f\|^2_\infty (n_1+2)k\right) && \text{[by (2.18)]} \\ &\le \frac{2}{\nu}\left[2\rho_0^2 + \frac{c_p^2}{\nu}\|f\|^2_\infty r\right] =: A_\xi. \end{aligned} \tag{2.35}$$

By (2.30), we obtain

$$\begin{aligned} \|\omega^n\|^2_{H^1} &\le \left[\frac{4}{\nu}\left(\frac{2\rho_0^2}{r} + \frac{1}{\nu\lambda_1}\|f\|^2_\infty\right) + \frac{2}{\nu}\|f\|^2_\infty r\right] \exp\left(\frac{16c_w^2}{\nu}\rho_0^2 r\right) && (2.36) \\ &=: M_1^2(\nu, \|f\|_\infty), \quad \forall n \ge N_0 + N_r. && (2.37) \end{aligned}$$

Taking $N = N_0 + N_r$, we obtain the conclusion (2.32) of Lemma 2.5. □

We summarize the above results in the following:

stability

THEOREM 2.6. *The classical scheme* (1.2) *defines a discrete dynamical system on* $\dot{L}^2$ *that is long time stable in both* L^2 *and* H^1 *norms. More precisely, for any* $\omega_0 \in \dot{L}^2$*, there exist constants* $k_0 = k_0(\|\omega_0\|_2, \nu, \|f\|_\infty)$*,* $M_0 = M_0(\|\omega_0\|_2, \nu, \|f\|_\infty)$*,* $M_1 = M_1(\nu, \|f\|_\infty)$ *and* $N = N(\|\omega_0\|_2, \nu, \|f\|_\infty)$ *such that*

$$\|\omega^n\|_2 \le M_0, \qquad \forall n \ge 0,\ \forall k \in (0, k_0), \tag{2.38}$$

$$\|\omega^n\|_{H^1} \le M_1, \qquad \forall n \ge N,\ \forall k \in (0, k_0). \tag{2.39}$$

2.2. Convergence of long time statistics. In this section, we assume that the forcing f is time independent. It is easy to see that the scheme (1.2), when restricted to the invariant ball $B(0, \rho_0)$ and with small enough time-step specified in the previous section, possesses a global attractor $\mathcal{A}_k$. Here we show that the long time statistical properties as well as $\mathcal{A}_k$ of the scheme (1.2) converge to the long time statistical properties and the attractor $\mathcal{A}$ of the Navier–Stokes sytem (1.1) at vanishing timestep size. This is a straightforward application of the abstract convergence result (Prop. 2) in Appendix B, which itself is a slight modification of the results presented in [47].

conv_stat

THEOREM 2.7. *Let* $\partial_t f = 0$*. The global attractor and the long time statistical properties of the classical scheme* (1.2)*, defined as a dynamical system on* $B(0, \rho_0)$*, converge to those of the Navier–Stokes system* (1.1) *at vanishing timestep.*

Proof. We use the abstract convergence result Prop. 2, taking $X = B(0, \rho_0)$, i.e. a ball in $\dot{L}^2$ centered at the origin with radius ρ_0. (The size of the ball needs to be adjusted depending on the absorbing property of the scheme.) Such a ball is appropriate for long time behavior since it is absorbing for the 2D NSE [7, 41].

The uniform continuity (H5) of the Navier–Stokes system (1.1) is a classical result [7, 40]. The uniform dissipativity (H3) of the scheme (1.2) for small enough timestep with the choice of the phase space X follows from Theorem 2.6. The uniform convergence on finite time interval (H4) is proved in Lemma 2.8 below. □

t:error

LEMMA 2.8. *Let* ω *be the solution of the continuous system* (1.1) *with* $\omega(0) = \omega_0 \in \mathcal{A}$ *and* ω^n *that of* (1.2) *with* $\omega^0 = \omega_0$*. Assume that* f *is sufficiently smooth so that*

$$M_V := \sup_{\omega \in \mathcal{A}} \big(\|\partial_{tt}\omega\|_{H^{-1}}^2 + \|\omega\|_{L^2}^2 \|\partial_t \omega\|_{L^2}^2 \big) < \infty, \tag{2.40}$$

and that Theorem 2.6 holds. Then for $k < k_0$ *one has*

$$\|\omega^n - \omega(nk)\|_2^2 \le k\, C(M_0, M_V; \nu) \tag{2.41}$$

for all $nk \in [0, 1]$.

Proof. We follow the approach in [31, §17] and take $\partial_t f = 0$. For notational convenience, we write $t_n := nk$ and $\omega_n := \omega(nk)$. Using the identity

$$\int_{nk}^{(n+1)k} (t - nk)\, \partial_{tt}\omega(t)\, \mathrm{d}t = k\, \partial_t \omega\big|_{(n+1)k} - \omega_{n+1} + \omega_n\,, \tag{2.42}$$

we have

$$\frac{\omega_{n+1}-\omega_n}{k}+\nabla^\perp\psi_n\cdot\nabla\omega_n-\nu\Delta\omega_{n+1}=f+R_{n+1}\,. \tag{2.43}$$

Here $-\Delta\psi_n:=\omega_n$ and the local truncation error is

$$-R_{n+1}:=\nabla^\perp\delta\psi_{n+1}\cdot\nabla\omega_n-\nabla^\perp\psi_{n+1}\cdot\nabla\delta\omega_{n+1}+\frac{1}{k}\int_{nk}^{(n+1)k}(t-nk)\,\partial_{tt}\omega(t)\;\mathrm{d}t \tag{2.44}$$

q:rndef

with

$$\delta\omega_{n+1}:=\omega_{n+1}-\omega_n=\int_{nk}^{(n+1)k}\partial_t\omega(t)\;\mathrm{d}t \quad\text{and}\quad -\Delta\delta\psi_{n+1}:=\delta\omega_{n+1}. \tag{2.45}$$

We now consider the error $e^n:=\omega_n-\omega^n$, which satisfies

$$\begin{aligned}\frac{e^{n+1}-e^n}{k}-\nu\Delta e^{n+1}&=\nabla^\perp\psi^n\cdot\nabla\omega^n-\nabla^\perp\psi_n\cdot\nabla\omega_n+R_{n+1}\\&=-\nabla^\perp\psi_n\cdot\nabla e^n-\nabla^\perp\phi^n\cdot\nabla\omega^n+R_{n+1}\end{aligned} \tag{2.46}$$

with $e^0=0$ and $-\Delta\phi^n:=e^n$. Multiplying by $2k\,e^{n+1}$, we find

$$\begin{aligned}&\|e^{n+1}\|_2^2-\|e^n\|_2^2+\|e^{n+1}-e^n\|_2^2+2\nu k\|e^{n+1}\|_{H^1}^2\\&\qquad+2k\,b(\psi_n,e^{n+1},e^{n+1}-e^n)+2k\,b(\phi^n,\omega^n,e^{n+1})\\&\qquad=2k\,(R_{n+1},e^{n+1}).\end{aligned} \tag{2.47}$$

Bounding the nonlinear terms as

$$\begin{aligned}2k\,(\nabla^\perp\psi_n\cdot\nabla e^{n+1},e^{n+1}-e^n)&\le\|e^{n+1}-e^n\|_2^2+k^2\|\nabla^\perp\psi_n\cdot\nabla e^{n+1}\|_2^2\\&\le\|e^{n+1}-e^n\|_2^2+c_w^2k^2\|\omega_n\|_2^2\,\|\nabla e^{n+1}\|_2^2\end{aligned} \tag{2.48}$$

where (A.2) has been used for the second inequality, and

$$\begin{aligned}2k\,(\nabla^\perp\phi^n\cdot\nabla\omega^n,e^{n+1})&\le2k\,\|\nabla^\perp\phi^n\cdot\nabla e^{n+1}\|_2\|\omega^n\|_2\\&\le2c_wk\,\|e^n\|_2\|e^{n+1}\|_{H^1}\|\omega^n\|_2\\&\le\nu k\,\|e^{n+1}\|_{H^1}^2+\frac{c_w^2k}{\nu}\,\|\omega^n\|_2^2\,\|e^n\|_2^2\,,\end{aligned} \tag{2.49}$$

we obtain,

$$\begin{aligned}&\|e^{n+1}\|_2^2+k\,\big(\nu-C_w^2k\,\|\omega_n\|_2^2\big)\,\|e^{n+1}\|_{H^1}^2\\&\qquad\le\Big(1+\frac{c_w^2k}{\nu}\,\|\omega^n\|_2^2\Big)\,\|e^n\|_2^2+ck\,\|R_{n+1}\|_{H^{-1}}^2\,.\end{aligned} \tag{2.50}$$

For the last step, we have used the readily verified facts that $\|\omega(t)\|_2\le M_0$ for all $t\ge0$, and that $k\le k_0$ then implies $\nu-c_w^2k\,\|\omega(t)\|_2^2\ge\nu/2>0$.

It remains to bound R_{n+1} in H^{-1}, so for the second term in (2.44) we compute, for any fixed $\varphi\in\dot H^1$,

$$\begin{aligned}\big|b\big(\psi_{n+1},\partial_t\omega,\varphi\big)\big|&=\big|\big(\nabla^\perp\varphi\cdot\nabla\psi_{n+1},\partial_t\omega\big)_{L^2}\big|\\&\le c_w\,\|\varphi\|_{H^1}\|\psi_{n+1}\|_{H^2}\|\partial_t\omega\|_{L^2}\end{aligned} \tag{2.51}$$

where (A.2) and the identity $b(p,q,r) = b(q,r,p) = b(r,p,q)$ have been used. Similarly, for the first term,

$$\begin{aligned} \big|b\big(\omega_n, \partial_t\psi, \varphi\big)\big| &= \big|\big(\nabla^\perp\varphi\cdot\nabla\partial_t\psi, \omega_n\big)_{L^2}\big| \\ &\le c_w\,\|\varphi\|_{H^1}\|\omega_n\|_{L^2}\|\partial_t\psi\|_{H^2}. \end{aligned} \tag{2.52}$$

The last term in (2.44) is readily bounded, and we have by Cauchy–Schwarz,

$$\begin{aligned} \|R_{n+1}\|^2_{H^{-1}} \le c\,k \sup_{t\in[nk,(n+1)k]} \|\omega(t)\|^2_{L^2} \int_{nk}^{(n+1)k} \|\partial_t\omega(t)\|^2_{L^2}\,\mathrm{d}t \\ + k\int_{nk}^{(n+1)k} \|\partial_{tt}\omega(t)\|^2_{H^{-1}}\,\mathrm{d}t. \end{aligned} \tag{2.53}$$

The following bound then follows easily

$$\begin{aligned} \|\omega_{n+1} - \omega^{n+1}\|_2^2 = \|e^{n+1}\|_2^2 &\le c\Big(1 + \frac{ck}{\nu}M_0^2\Big)^{n+1}\sum_{j=0}^{n} k\,\|R_{j+1}\|^2_{H^{-1}} \\ &\le c\,k^2\,\exp\Big(\frac{c\,(n+1)k}{\nu}\,M_0^2\Big)M_2((n+1)k) \end{aligned} \tag{2.54}$$

where

$$M_2(t) := \int_0^t \|\partial_{tt}\omega(t')\|^2_{H^{-1}}\,\mathrm{d}t' + \sup_{t'\in[0,t]}\|\omega(t')\|^2_{L^2}\int_0^t \|\partial_t\omega(t')\|^2_{L^2}\,\mathrm{d}t', \tag{2.55}$$

and with it the lemma. ☐

3. Galerkin Fourier spectral approximation. The results of the last section carry over essentially word-for-word to the following Galerkin Fourier spectral approximation of (1.1),

$$\frac{\omega_N^{n+1} - \omega_N^n}{k} + P_N(\nabla^\perp\psi_N^n\cdot\nabla\omega_N^n) - \nu\Delta\omega_N^{n+1} = P_N f^n. \tag{3.1}$$ q:scheme-fg

where $N\in\mathbb{N}$, ω_N^n, $\psi_N^n \in \mathcal{P}_N := \big\{$trigonometric polynomials in Ω with frequency in either direction at most $N\big\}$ and P_N is the orthogonal projection from $\dot{L}^2(\Omega)$ onto $\mathcal{P}_N$.

More precisely, we have the following:

t:bdh-fg LEMMA 3.1. *Let ω_N^n be the solution of the numerical scheme* (3.1) *with $\omega_N^0 \in \dot{L}^2$. Also, let $f \in L^\infty(\mathbf{R}_+;\dot{L}^2)$ and set $\|f\|_\infty := \|f\|_{L^\infty(\mathbf{R}_+;L^2)}$. Then there exists $M_0 = M_0(\|\omega^0\|_2, \nu, \|f\|_\infty)$ such that if*

$$k \le \frac{\nu}{4c_w^2 M_0^2}, \tag{3.2}$$ q:hypk-fg

then

$$\|\omega_N^n\|_2^2 \le \Big(1 + \frac{\nu k}{2c_p^2}\Big)^{-n}\|\omega_0\|_2^2 + \frac{2c_p^4}{\nu^2}\|f\|_\infty^2\left[1 - \Big(1 + \frac{\nu k}{2c_p^2}\Big)^{-n}\right],\ \forall n\ge 0 \tag{3.3}$$ q:bdv-fg

and

$$\frac{\nu}{2}k\sum_{n=i}^{m}\|\omega_N^n\|_{H^1}^2 \le \|\omega_N^{i-1}\|_2^2 + \frac{c_p^2}{\nu}\|f\|_\infty^2(m-i+1)k, \quad \forall\, i=1,\cdots,m. \tag{3.4}$$ q:sumv-fg

t:bdv-fg LEMMA 3.2. *Let ω_N^n be the solution of the numerical scheme (3.1) with $\omega_N^0 \in \dot{L}^2$. Also, let $k \le k_0$, with k_0 that in Corollary 1. Then there exist constants $M_1 = M_1(\nu, \|f\|_\infty)$ and $N_M = N_M(\|\omega^0\|_2, \nu, \|f\|_\infty)$ such that*

$$\|\omega_N^n\|_{H^1} \le M_1 \quad \text{for all } n \ge N_M. \tag{3.5}$$ q:20-fg

We note that the constants are the same as those in Lemmas 2.3 and 2.5 (i.e. they can be taken independently of N). The proofs are essentially identical, so we shall not repeat them here.

4. Collocation Fourier spectral approximation. Here we consider the collocation Fourier spectral spatial approximation of the scheme (1.2). In order to maintain the long time stability of the fully discretized scheme, a common technique of using a modified form of the nonlinear term is utilized (see for instance [39]). Moreover, we will use an alternative approach for the nonlinear analysis: instead of applying the Wente type estimate, we will use $\|\nabla\psi\|_{L^\infty}$, which is in turn bounded by $\|\psi\|_{H^3}^{\epsilon}\|\psi\|_{H^2}^{1-\epsilon}, \forall\epsilon \in (0,1)$. This alternative approach leads to a slightly more restrictive timestep for stability, but has the advantage of being easily adapted to the fully discrete collocation Fourier approximation.

4.1. Fourier collocation spectral differentiation. Consider a 2-D domain $\Omega = (0, L_x) \times (0, L_y)$. For simplicity of presentation we assume that $L_x = L_y = 1$ and $L_x = N_x \cdot h_x$, $L_y = N_y \cdot h_y$ for some mesh sizes $h_x = h_y = h > 0$ and some positive integers $N_x = N_y = 2N+1$. All variables are evaluated at the regular numerical grid (x_i, y_j), with $x_i = ih$, $y_j = jh$, $0 \le i, j \le N$.

For a periodic function f over the given 2-D numerical grid, its discrete Fourier expansion is given by

$$f_{i,j} = \sum_{k_1,l_1=-[N/2]}^{[N/2]} (\hat{f}_c^N)_{k_1,l_1} \mathrm{e}^{2\pi \mathrm{i}(k_1 x_i + l_1 y_j)}, \tag{4.1}$$ spectral-coll-1

where the collocation coefficients $\hat{f}_c^N$ are computed by the requirement that $f_{i,j}$ are the interpolation values of a continuous function $\boldsymbol{f}$ at the numerical grid points. Note that $\hat{f}_c^N$ may not be the regular Fourier coefficients of $\boldsymbol{f}$, due to the aliasing error. However, the two are equivalent if $\boldsymbol{f} \in \mathcal{P}_N$. In turn, its collocation interpolation operator becomes

$$\mathcal{I}_N f(\boldsymbol{x}) = \sum_{k_1,l_1=-N}^{N} (\hat{f}_c^N)_{k_1,l_1} \mathrm{e}^{2\pi \mathrm{i}(k_1 x + l_1 y)}. \tag{4.2}$$

As a result, the collocation Fourier spectral approximations to the first and second order partial derivatives (in the x direction) of f are given by

$$\left(\mathcal{D}_{Nx} f\right)_{i,j} = \sum_{k_1,l_1=-N}^{N} (2k_1\pi \mathrm{i})\, (\hat{f}_c^N)_{k_1,l_1} \mathrm{e}^{2\pi \mathrm{i}(k_1 x_i + l_1 y_j)}, \tag{4.3}$$

$$\left(\mathcal{D}_{Nx}^2 f\right)_{i,j} = \sum_{k_1,l_1=-[N/2]}^{[N/2]} \left(-4\pi^2 k_1^2\right) (\hat{f}_c^N)_{k_1,l_1} \mathrm{e}^{2\pi \mathrm{i}(k_1 x_i + l_1 y_j)}. \tag{4.4}$$

The corresponding collocation spectral differentiations in the y direction can be defined in the same way. In turn, the discrete Laplacian, gradient and divergence can be denoted as

$$\Delta_N f = \left(\mathcal{D}_{Nx}^2 + \mathcal{D}_{Ny}^2\right) f, \quad \nabla_N f = \begin{pmatrix} \mathcal{D}_{Nx} f \\ \mathcal{D}_{Ny} f \end{pmatrix},$$
$$\nabla_N \cdot \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathcal{D}_{Nx} f_1 + \mathcal{D}_{Ny} f_2, \tag{4.5}$$

at the pointwise level.

Moreover, given any periodic grid functions f and g (over the 2-D numerical grid), the spectral approximations to the L^2 inner product and L^2 norm are introduced as

$$\|f\|_2 = \sqrt{\langle f, f\rangle}, \quad \text{with} \quad \langle f, g\rangle = h^2 \sum_{i,j=0}^{2N} f_{i,j} g_{i,j}. \tag{4.6}$$

Meanwhile, such a discrete L^2 inner product can also be viewed in Fourier space instead of physical space, with the help of Parseval equality:

$$\langle f, g\rangle = \sum_{k_1,l_1=-N}^{N} (\hat{f}_c^N)_{k_1,l_1} \overline{(\hat{g}_c^N)_{k_1,l_1}} = \sum_{k_1,l_1=-N}^{N} (\hat{g}_c^N)_{k_1,l_1} \overline{(\hat{f}_c^N)_{k_1,l_1}}, \tag{4.7}$$ spectral-coll-inner product-2

in which $(\hat{f}_c^N)_{k_1,l_1}$, $(\hat{g}_c^N)_{k_1,l_1}$ are the Fourier interpolation coefficients of the grid functions f and g in the expansion as in (4.1). Furthermore, a detailed calculation shows that the following formulas of summation by parts are also valid at the discrete level:

$$\left\langle f, \nabla_N \cdot \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = -\left\langle \nabla_N f, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle, \qquad \langle f, \Delta_N g\rangle = -\langle \nabla_N f, \nabla_N g\rangle. \tag{4.8}$$

4.1.1. A preliminary estimate in Fourier collocation spectral space. It is well known that the existence of aliasing error in the nonlinear term poses a serious challenge in the numerical analysis of Fourier collocation spectral scheme. To overcome a key difficulty associated with the H^m bound of the nonlinear term obtained by collocation interpolation, the following lemma is introduced. The result is cited from a recent work [17], and the detailed proof is skipped.

-projection-4 LEMMA 4.1. *For any $\varphi \in \mathcal{P}_{2N}$ in dimension d, we have*

$$\|\mathcal{I}_N \varphi\|_{H^k} \le 2^{d/2} \|\varphi\|_{H^k}. \tag{4.9}$$

In fact, an estimate for the $k = 0$ case was reported in E's work [9, 10], with the constant given by 3^d, while this lemma sharpens the constant to $2^{d/2}$. The case with $k > d/2 = 1$ was covered in a classical approximation estimate for spectral expansions and interpolations in Sobolev spaces, reported by Canuto and Quarteroni [5]. However, due to the additional regularity requirement for interpolation operator analysis, the case of $k = 1$ was not covered in any existing literature, which we require for the H^1 bound of the nonlinear expansion in the global in time analysis.

4.2. The first order semi-implicit scheme. The fully discrete pseudo-spectral scheme follows the semi-implicit idea of (1.2) and (3.1):

$$\frac{\omega^{n+1} - \omega^n}{k} + \frac{1}{2}\left(\boldsymbol{u}^n \cdot \nabla_N \omega^n + \nabla_N \cdot (\boldsymbol{u}^n \omega^n)\right) = \nu \Delta_N \omega^{n+1} + \boldsymbol{f}^n, \tag{4.10}$$

$$-\Delta_N \psi^{n+1} = \omega^{n+1}, \tag{4.11}$$

$$\boldsymbol{u}^{n+1} = \nabla_N^{\perp} \psi^{n+1} = \left(\mathcal{D}_{Ny}\psi^{n+1}, -\mathcal{D}_{Nx}\psi^{n+1}\right). \tag{4.12}$$

It is observed that the numerical velocity $\boldsymbol{u}^{n+1} = \nabla_N^{\perp}\psi^{n+1}$ is automatically divergence-free:

$$\nabla_N \cdot \boldsymbol{u} = \mathcal{D}_{Nx} u + \mathcal{D}_{Ny} v = \mathcal{D}_{Nx}(\mathcal{D}_{Ny}\psi) - \mathcal{D}_{Ny}(\mathcal{D}_{Nx}\psi) = 0, \tag{4.13}$$

at any timestep. Meanwhile, we note that the nonlinear term is a spectral approximation to $\frac{1}{2}\boldsymbol{u}\cdot\nabla\omega$ and $\frac{1}{2}\nabla\cdot(\boldsymbol{u}\omega)$ at timestep t^n. These two terms are equivalent in the spatially continuous and Galerkin spectral case because of the divergence-free property of the numerical velocity vector. However, such an equivalence is not valid for pseudo-spectral case, due to the aliasing errors involved in the nonlinear terms. The reason for this average can be observed from the following fact: a careful application of summation by parts formula (4.8) gives

$$\langle \omega, \boldsymbol{u}\cdot\nabla_N \omega + \nabla_N \cdot (\boldsymbol{u}\omega)\rangle = \langle \omega, \boldsymbol{u}\cdot\nabla_N\omega\rangle - \langle \nabla_N \omega, \boldsymbol{u}\omega\rangle = 0. \tag{4.14}$$

In other words, the nonlinear convection term appearing in the numerical scheme (4.10), so-called skew symmetric form, makes the nonlinear term orthogonal to the vorticity field in the L^2 space, without considering the temporal discretization. This property is crucial in the stability analysis for the Fourier collocation spectral scheme (4.10)–(4.12).

REMARK 1. *The skew symmetric nonlinear convection term is well-known for its ability to overcome the difficulty associated with the aliasing errors appearing in pseudo-spectral numerical simulations. The L^2 orthogonality property* (4.14) *has been observed and widely used in earlier literatures; see the relevant discussions in [3]. However, there has been no theoretical work of a global in time stability analysis for the fully discrete nonlinear scheme* (4.10)–(4.12)*. We provide such an analysis in this article; see Lemma 4.2 below.*

In addition, we note that the pseudo-spectral numerical solution of (4.10)–(4.12) are only evaluated at the grid points. To facilitate the nonlinear analysis in Sobolev space, we denote $\boldsymbol{U}^n = (U^n, V^n)$, $\boldsymbol{\omega}^n$ and $\boldsymbol{\psi}^n$ as the continuous versions of $\boldsymbol{u}^n$, ω^n and ψ^n, respectively, with the formula given by (4.2). It is clear that $\boldsymbol{U}^n, \boldsymbol{\omega}^n, \boldsymbol{\psi}^n \in \mathcal{P}_N$ and the kinematic equation $-\Delta\boldsymbol{\psi}^n = \boldsymbol{\omega}^n$, $\boldsymbol{U}^n = \nabla^{\perp}\boldsymbol{\psi}^n$ is satisfied at the continuous level. Because of these kinematic equations, an application of elliptic regularity shows that

$$\|\boldsymbol{\psi}^n\|_{H^{m+2}} \le C \|\boldsymbol{\omega}^n\|_{H^m}, \quad \|\boldsymbol{\psi}^n\|_{H^{m+2+\alpha}} \le C \|\boldsymbol{\omega}^n\|_{H^{m+\alpha}}, \tag{4.15}$$

in which we used the fact that all profiles have mean zero over the domain:

$$\overline{\boldsymbol{\psi}^n} = 0, \quad \overline{\boldsymbol{U}^n} = \left(\overline{\partial_y \boldsymbol{\psi}^n}, -\overline{\partial_x \boldsymbol{\psi}^n}\right) = 0, \quad \overline{\boldsymbol{\omega}^n} = -\overline{\Delta \boldsymbol{\psi}^n} = 0. \tag{4.16}$$

Moreover, it is clear that the Poincaré inequality and elliptic regularity can be applied because of this property.

collocation

LEMMA 4.2. *Let* $\omega_0 \in \dot{L}^2$ *and let* ω^n *be the discrete solution of the fully discrete numerical scheme* (4.10)–(4.12). *Denote* $\boldsymbol{\omega}^n$ *as the continuous interpolation of* ω^n *in space, given by* (4.2). *Also, let* $f \in L^\infty(\mathbf{R}_+; \dot{L}^2)$ *and set* $\|f\|_\infty := \|f\|_{L^\infty(\mathbf{R}+;\dot{L}^2)}$. *Then there exists* $M_0 = M_0(\|\omega_0\|_2, \nu, \|f\|_\infty)$ *such that if*

$$k \le \frac{\nu}{4c_w^2 M_0^2}, \tag{4.17}$$

constraint-coll-dt

then

$$\|\boldsymbol{\omega}^n\|_{H^1} \le M_0, \, \forall \, n \ge 0, \tag{4.18}$$

coll-est-L2-1

$$\|\boldsymbol{\omega}^n\|_{H^1}^2 \le \left(1 + \frac{\nu k}{2c_p^2}\right)^{-n} \|\boldsymbol{\omega}^0\|_{H^1}^2 + \frac{2c_p^4}{\nu^2}\|f\|_\infty^2 \left[1 - \left(1 + \frac{\nu k}{2c_p^2}\right)^{-n}\right], \forall \, n \ge 0, \tag{4.19}$$

coll-est-L2-2

and

$$\frac{\nu}{2} k \sum_{n=i}^{m} \|\boldsymbol{\omega}^n\|_{H^2}^2 \le \|\boldsymbol{\omega}^{i-1}\|_{H^1}^2 + \frac{c_p^2}{\nu}\|f\|_\infty^2 (m - i + 1)k, \qquad \forall \, i = 1, \cdots, m. \tag{4.20}$$

The proof of this lemma is organized as follows. First, an H^δ a-priori assumption for the continuous version of the numerical solution $\boldsymbol{\omega}^n$ is made. In turn, this assumption leads to a global in time L^2 bound, with a standard application of Sobolev embedding and Hölder's inequality. However, this L^2 bound is not sufficient to recover the a-priori assumption, due to the fact that the Wente type analysis is not available for the collocation spectral approximation. Instead, a global in time H^1 stability can also be derived with the help of the leading L^2 bound. Moreover, both the global in time L^2 and H^1 bound constants are independent of the a priori constant $\tilde{C}_1$. As a result, the a priori assumption can be recovered so that an induction can be applied to establish the above lemma.

4.3. Leading estimate: $L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$ **estimate for** ω**.** Assume a-priori that

$$\|\boldsymbol{\omega}^n\|_{H^\delta} \le \tilde{C}, \quad \boldsymbol{\omega}^n \text{ is the continuous version of } \omega^n, \tag{4.21}$$

est-coll-a priori

for some $\delta > 0$ at timestep t^n. Note that $\tilde{C}$ is a global constant in time. We are going to prove that such a bound for the numerical solution is also available at timestep t^{n+1}.

Taking the discrete inner product of (4.10) with $2k\omega^{n+1}$ gives

$$\begin{aligned} &\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \|\omega^{n+1} - \omega^n\|_2^2 + 2\nu k \|\nabla_N \omega^{n+1}\|_2^2 \\ &= -k\left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^n + \nabla_N \cdot (\boldsymbol{u}^n \omega^n), \omega^{n+1}\right\rangle + 2k\left\langle \boldsymbol{f}^n, \omega^{n+1}\right\rangle, \end{aligned} \tag{4.22}$$

in which the summation by parts formula (4.8) was applied to the diffusion term. A bound for the outer force term is straightforward:

$$
\begin{aligned}
2\left\langle \boldsymbol{f}^n, \omega^{n+1}\right\rangle &\le 2\left\|f^n\right\|_2 \cdot \left\|\omega^{n+1}\right\|_2 \le 2C_2\left\|f^n\right\|_2 \cdot \left\|\nabla_N \omega^{n+1}\right\|_2 \\
&\le \frac{\nu}{2}\left\|\nabla_N \omega^{n+1}\right\|_2^2 + \frac{2C_2^2}{\nu}\left\|f^n\right\|_2^2 \le \frac{\nu}{2}\left\|\nabla_N \omega^{n+1}\right\|_2^2 + \frac{2c_p^2 M^2}{\nu}, \quad (4.23)
\end{aligned}
$$

in which a Poincaré inequality

$$
\left\|\omega^{n+1}\right\|_2 \le c_p \left\|\nabla_N \omega^{n+1}\right\|_2, \tag{4.24}
$$

est-coll-L2-3

was used in the third step. For the nonlinear term, we start with the following rewritten form:

$$
\begin{aligned}
&-k\left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^n + \nabla_N \cdot \left(\boldsymbol{u}^n \omega^n\right), \omega^{n+1}\right\rangle \\
=& -k\left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1} + \nabla_N \cdot \left(\boldsymbol{u}^n \omega^{n+1}\right), \omega^{n+1}\right\rangle \\
&+k\left\langle \boldsymbol{u}^n \cdot \nabla_N (\omega^{n+1} - \omega^n) + \nabla_N \cdot \left(\boldsymbol{u}^n (\omega^{n+1} - \omega^n)\right), \omega^{n+1}\right\rangle. \qquad (4.25)
\end{aligned}
$$

The first term disappears, using a similar analysis as (4.14):

$$
\begin{aligned}
&\left\langle \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1} + \nabla_N \cdot \left(\boldsymbol{u}^n \omega^{n+1}\right), \omega^{n+1}\right\rangle \\
&\qquad = \left\langle \omega^{n+1}, \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1}\right\rangle - \left\langle \nabla_N \omega^{n+1}, \boldsymbol{u}^n \omega^{n+1}\right\rangle = 0.
\end{aligned} \tag{4.26}
$$

est-coll-L2-5

For the second term, the summation by parts formula (4.8) can be applied:

$$
\left\langle \boldsymbol{u}^n \cdot \nabla_N (\omega^{n+1} - \omega^n), \omega^{n+1}\right\rangle = -\left\langle \omega^{n+1} - \omega^n, \nabla_N \cdot \left(\boldsymbol{u}^n \omega^{n+1}\right)\right\rangle, \tag{4.27}
$$

$$
\left\langle \nabla_N \cdot \left(\boldsymbol{u}^n (\omega^{n+1} - \omega^n)\right), \omega^{n+1}\right\rangle = -\left\langle \omega^{n+1} - \omega^n, \boldsymbol{u}^n \cdot \nabla_N \omega^{n+1}\right\rangle. \tag{4.28}
$$

For the term $\nabla_N \cdot (\boldsymbol{u}^n \omega^{n+1})$, we note that it cannot be expanded as $\boldsymbol{u}^n \cdot \nabla_N \omega^{n+1}$, as in the Fourier-Galerkin approximation, even though $\boldsymbol{u}^n$ is divergence free at the discrete level (4.13). In the collocation space, we have to start from

$$
\nabla_N \cdot (\boldsymbol{u}^n \omega^{n+1}) = \mathcal{D}_{Nx}(u^n \omega^{n+1}) + \mathcal{D}_{Ny}(v^n \omega^{n+1}). \tag{4.29}
$$

est-coll-L2-6-3

To obtain an estimate of these nonlinear expansions, we recall that $\boldsymbol{U}^n = (U^n, V^n)$, $\boldsymbol{\omega}^{n+1}$ and $\boldsymbol{\psi}^{n+1}$ are the continuous versions of $\boldsymbol{u}^n$, ω^{n+1} and ψ^{n+1}, respectively. Since $\boldsymbol{U}^n, \boldsymbol{\omega}^{n+1} \in \mathcal{P}_N$, we have $\boldsymbol{U}^n \boldsymbol{\omega}^{n+1} \in \mathcal{P}_{2N}$ and an application of Lemma 4.1 indicates that

$$
\begin{aligned}
\left\|\mathcal{D}_{Nx}(u^n \omega^{n+1})\right\|_2 &= \left\|\partial_x \mathcal{I}_N (U^n \omega^{n+1})\right\|_2 \le 2\left\|\partial_x (U^n \boldsymbol{\omega}^{n+1})\right\|_2, \\
\left\|\mathcal{D}_{Ny}(v^n \omega^{n+1})\right\|_2 &= \left\|\partial_y \mathcal{I}_N (V^n \omega^{n+1})\right\|_2 \le 2\left\|\partial_y (V^n \boldsymbol{\omega}^{n+1})\right\|_2.
\end{aligned} \tag{4.30}
$$

Subsequently, a detailed expansion in the continuous space and an application of Hölder's inequality show that

$$
\begin{aligned}
\left\|\partial_x (U^n \boldsymbol{\omega}^{n+1})\right\|_2 &= \left\|U_x^n \boldsymbol{\omega}^{n+1} + U^n \boldsymbol{\omega}_x^{n+1}\right\|_2 \le \left\|U_x^n \boldsymbol{\omega}^{n+1}\right\|_2 + \left\|U^n \boldsymbol{\omega}_x^{n+1}\right\|_2 \\
&\le \left\|U_x^n\right\|_{L^{2/(1-\delta)}} \cdot \left\|\boldsymbol{\omega}^{n+1}\right\|_{L^{2/\delta}} + \left\|U^n\right\|_{L^\infty} \cdot \left\|\boldsymbol{\omega}_x^{n+1}\right\|_2.
\end{aligned} \tag{4.31}
$$

Furthermore, a 2-D Sobolev embedding gives

$$
\left\|U_x^n\right\|_{L^{2/(1-\delta)}} \left\|\boldsymbol{\omega}^{n+1}\right\|_{L^{2/\delta}} \le C\left\|U_x^n\right\|_{H^\delta} \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^1} \le C\left\|\boldsymbol{\omega}^n\right\|_{H^\delta} \left\|\nabla \boldsymbol{\omega}^{n+1}\right\|_2 \tag{4.32}
$$

in which the elliptic regularity (4.15) and the Poincaré inequality were utilized in the last step. The second part in (4.31) can be handled in a straightforward way:

$$\|U^n\|_{L^\infty} \cdot \left\|\boldsymbol{\omega}_x^{n+1}\right\|_2 \le C\,\|U^n\|_{H^{1+\delta}} \cdot \left\|\nabla \boldsymbol{\omega}^{n+1}\right\|_2 \le C\,\|\boldsymbol{\omega}^n\|_{H^\delta} \cdot \left\|\nabla \boldsymbol{\omega}^{n+1}\right\|_2, \tag{4.33}$$

with the the elliptic regularity (4.15) applied again in the second step. A combination of (4.32) and (4.33) yields

$$\left\|\partial_x(U^n \boldsymbol{\omega}^{n+1})\right\|_2 \le C\,\|\boldsymbol{\omega}^n\|_{H^\delta} \cdot \left\|\nabla \boldsymbol{\omega}^{n+1}\right\|_2. \tag{4.34}$$

Similar estimates can be derived for $\left\|\partial_y(V^n \boldsymbol{\omega}^{n+1})\right\|_2$. Going back to (4.30), we arrive at

$$\left\|\nabla_N \cdot (\boldsymbol{u}^n \omega^{n+1})\right\|_2 \le C\,\|\boldsymbol{\omega}^n\|_{H^\delta} \cdot \left\|\nabla \boldsymbol{\omega}^{n+1}\right\|_2 = C\,\|\boldsymbol{\omega}^n\|_{H^\delta} \cdot \left\|\nabla_N \omega^{n+1}\right\|_2, \tag{4.35}$$

in which the second step is based on the fact that $\boldsymbol{\omega}^n, \boldsymbol{\omega}^{n+1} \in \mathcal{P}_N$, so that the corresponding L^2 and H^δ norms are equivalent between the continuous projection and the discrete version. In addition, the nonlinear term in (4.28) can be controlled in a similar way:

$$\left\|\boldsymbol{u}^n \cdot \nabla_N \omega^{n+1}\right\|_2 \le \|\boldsymbol{u}^n\|_\infty \cdot \left\|\nabla_N \omega^{n+1}\right\|_2 = C\,\|\boldsymbol{\omega}^n\|_{H^\delta} \cdot \left\|\nabla_N \omega^{n+1}\right\|_2, \tag{4.36}$$

with a discrete Sobolev embedding inequality applied in the second step. Therefore, a substitution of (4.35)–(4.36) into (4.25), (4.26), (4.27)–(4.28) results in

$$\begin{aligned} -k\big\langle \boldsymbol{u}^n \cdot \nabla_N \omega^n + \nabla_N \cdot (\boldsymbol{u}^n \omega^n)\,, \omega^{n+1}\big\rangle & \qquad (4.37)\\ \le Ck\|\boldsymbol{\omega}^n\|_{H^\delta} \cdot \left\|\omega^{n+1} - \omega^n\right\|_2 \cdot \left\|\nabla_N \omega^{n+1}\right\|_2 & \\ \le C\tilde{C}k\left\|\omega^{n+1} - \omega^n\right\|_2 \cdot \left\|\nabla_N \omega^{n+1}\right\|_2 & \\ \le \frac{1}{2}\nu k\left\|\nabla_N \omega^{n+1}\right\|_2^2 + \frac{C\tilde{C}^2}{\nu}k\left\|\omega^{n+1} - \omega^n\right\|_2^2. & \qquad (4.38) \end{aligned}$$

est-coll-L2-7

Its combination with (4.23), (4.25), (4.26) and (4.22) leads to

$$\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \left(1 - \frac{C\tilde{C}^2}{\nu}k\right)\|\omega^{n+1} - \omega^n\|_2^2 + \nu k\|\nabla_N \omega^{n+1}\|_2^2 \le \frac{2c_p^2 M^2}{\nu}k. \tag{4.39}$$

est-coll-L2-8

Under a constraint for the timestep

$$\frac{C\tilde{C}^2}{\nu}k \le \frac{1}{2}, \quad \text{i.e.,} \quad k \le \frac{\nu}{2C\tilde{C}^2}, \tag{4.40}$$

constraint-coll-dt-1

we arrive at

$$\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \frac{1}{2}\|\omega^{n+1} - \omega^n\|_2^2 + \nu k\|\nabla_N \omega^{n+1}\|_2^2 \le \alpha k, \tag{4.41}$$

est-coll-L2-9

with $\alpha = (2C_2^2 M^2)/\nu$. Furthermore, an application of the Poincaré inequality (4.24) implies that

$$\|\omega^{n+1}\|_2^2 - \|\omega^n\|_2^2 + \beta\nu k\|\omega^{n+1}\|_2^2 \le \alpha k, \quad \text{with } \beta = \frac{1}{c_p^2}. \tag{4.42}$$

est-coll-L2-10

Applying an induction argument to the above estimate yields

$$\|\omega^{n+1}\|_2^2 \le (1+\beta\nu k)^{-(n+1)}\|\omega^0\|_2^2 + \frac{\alpha}{\beta\nu}$$
$$\Rightarrow \quad \|\omega^{n+1}\|_2 \le (1+\beta\nu k)^{-(n+1)/2}\|\omega^0\|_2 + \sqrt{\frac{\alpha}{\beta\nu}} := C_1. \tag{4.43}$$ est-coll-L2-11

Note that C_1 is a time dependent value; however, its time dependence decays exponentially so that a global in time bound is available.

In addition, we also have the $L^2(0,T;H^1)$ bound for the numerical solution:

$$\nu k \sum_{k=i+1}^{N_k} \left\|\nabla_N \omega^k\right\|_2^2 \le \|\omega^i\|_2^2 + \alpha\left(T^* - t^i\right). \tag{4.44}$$ est-coll-L2-12

However, it is observed that the a-priori estimate (4.43) is not sufficient to bound the H^δ norm (4.21) of the vorticity field. In turn, we perform a higher order energy estimate $L^\infty(0,T;H^1)\cap L^2(0,T;H^2)$ for the numerical solution of the vorticity field.

4.4. $L^\infty(0,t_1;H^1)\cap L^2(0,t_1;H^2)$ **estimate for** ω**.** Taking the inner product of (4.10) with $-2k\Delta_N\omega^{n+1}$ gives

$$\begin{aligned}&\|\nabla_N\omega^{n+1}\|_2^2 - \|\nabla_N\omega^n\|_2^2 + \|\nabla_N\left(\omega^{n+1}-\omega^n\right)\|_2^2 + 2\nu k\|\Delta_N\omega^{n+1}\|_2^2\\ &= k\left\langle \boldsymbol{u}^n\cdot\nabla_N\omega^n + \nabla_N\cdot\left(\boldsymbol{u}^n\omega^n\right), \Delta_N\omega^{n+1}\right\rangle - 2k\left\langle \boldsymbol{f}^n, \Delta_N\omega^{n+1}\right\rangle.\end{aligned} \tag{4.45}$$

The Cauchy inequality can be applied to bound the outer force term:

$$\begin{aligned}-2\left\langle \boldsymbol{f}^n, \Delta_N\omega^{n+1}\right\rangle &\le \frac{1}{2}\nu\left\|\Delta_N\omega^{n+1}\right\|_2^2 + \frac{2}{\nu}\left\|\boldsymbol{f}^n\right\|_2^2\\ &\le \frac{1}{2}\nu\left\|\Delta_N\omega^{n+1}\right\|_2^2 + \frac{2M^2}{\nu}.\end{aligned} \tag{4.46}$$ est-coll-H1-2

For the nonlinear terms, we first make the following decomposition:

$$\begin{aligned}\boldsymbol{u}^n\cdot\nabla_N\omega^n &= -\boldsymbol{u}^n\cdot\nabla_N\left(\omega^{n+1}-\omega^n\right) - \left(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\right)\cdot\nabla_N\omega^{n+1}\\ &\qquad + \boldsymbol{u}^{n+1}\cdot\nabla_N\omega^{n+1},\end{aligned} \tag{4.47}$$ est-coll-H1-3-1

$$\begin{aligned}\nabla_N\cdot\left(\boldsymbol{u}^n\omega^n\right) &= \nabla_N\cdot\big(-\boldsymbol{u}^n(\omega^{n+1}-\omega^n) - (\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\omega^{n+1}\\ &\qquad + \boldsymbol{u}^{n+1}\omega^{n+1}\big).\end{aligned} \tag{4.48}$$ est-coll-H1-3-2

For the first term, the a-priori assumption (4.21) gives

$$\begin{aligned}\left\|-\boldsymbol{u}^n\cdot\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2 &\le \|\boldsymbol{u}^n\|_\infty\cdot\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2\\ &\le C\tilde{C}\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2,\end{aligned} \tag{4.49}$$ est-coll-H1-4

in which we applied the discrete Sobolev inequality in 2-D: $\|\boldsymbol{u}^n\|_\infty \le C\|\boldsymbol{u}^n\|_{H_h^{1+\delta}} \le C\|\omega^n\|_{H_h^\delta}$. This in turn leads to

$$\begin{aligned}&k\left\langle -\boldsymbol{u}^n\cdot\nabla_N(\omega^{n+1}-\omega^n), \Delta_N\omega^{n+1}\right\rangle\\ &\qquad \le C\tilde{C}k\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2\cdot\left\|\Delta_N\omega^{n+1}\right\|_2\\ &\qquad \le \frac{1}{4}\nu k\left\|\Delta_N\omega^{n+1}\right\|_2^2 + \frac{C\tilde{C}^2}{\nu}k\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2^2.\end{aligned} \tag{4.50}$$ est-coll-H1-5

The conservative nonlinear term $\nabla_N \cdot \left(\boldsymbol{u}^n(\omega^{n+1}-\omega^n)\right)$ can be analyzed as in (4.29)–(4.36):

$$\begin{aligned}
&\left\|\nabla_N \cdot \left(\boldsymbol{u}^n(\omega^{n+1}-\omega^n)\right)\right\|_2 \le \left\|\mathcal{D}_{Nx}\left(u^n(\omega^{n+1}-\omega^n)\right)\right\|_2 + \left\|\mathcal{D}_{Ny}\left(v^n(\omega^{n+1}-\omega^n)\right)\right\|_2 \\
&\qquad \le 2\left(\left\|\partial_x\left(U^n(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n)\right)\right\|_2 + \left\|\partial_y\left(V^n(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n)\right)\right\|_2\right), \qquad (4.51) \\
&\left\|\partial_x\left(U^n(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n)\right)\right\|_2 = \left\|U^n_x(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n) + U^n(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n)_x\right\|_2 \\
&\qquad \le \|U^n_x\|_{L^{2/(1-\delta)}} \cdot \left\|\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right\|_{L^{2/\delta}} + \|U^n\|_{L^\infty} \cdot \left\|(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n)_x\right\|_2 \\
&\qquad \le C\|\boldsymbol{\omega}^n\|_{H^\delta} \cdot \left\|\nabla(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n)\right\|_2 \le C\tilde{C}\left\|\nabla_N(\omega^{n+1}-\omega^n)\right\|_2, \qquad (4.52) \\
&\left\|\partial_y\left(V^n(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n)\right)\right\|_2 \le C\tilde{C}\left\|\nabla_N(\omega^{n+1}-\omega^n)\right\|_2, \qquad (4.53)
\end{aligned}$$

with the help of the elliptic regularity (4.15), Poincaré's inequality and 2-D Sobolev embedding. Consequently, we see that the first part of the nonlinear term (4.48) has the same bound as (4.49):

$$\left\|\nabla_N \cdot \left(\boldsymbol{u}^n(\omega^{n+1}-\omega^n)\right)\right\|_2 \le C\tilde{C}\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2, \qquad (4.54)$$

which in turn leads to an estimate similar to (4.50):

$$\begin{aligned}
&k\left\langle -\nabla_N \cdot \left(\boldsymbol{u}^n(\omega^{n+1}-\omega^n)\right), \Delta_N\omega^{n+1}\right\rangle \\
&\le \frac{1}{4}\nu k\left\|\Delta_N\omega^{n+1}\right\|_2^2 + \frac{C\tilde{C}^2}{\nu}k\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2^2. \qquad (4.55)
\end{aligned}$$

For the second term in (4.47), we start with the following Sobolev inequality:

$$\begin{aligned}
\left\|\nabla_N\omega^{n+1}\right\|_2 &= \left\|\nabla\boldsymbol{\omega}^{n+1}\right\|_2 \le \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^1} \le C\left\|\boldsymbol{\omega}^{n+1}\right\|_2^{1/2} \cdot \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^2}^{1/2} \\
&\le C\left\|\boldsymbol{\omega}^{n+1}\right\|_2^{1/2} \cdot \left\|\Delta\boldsymbol{\omega}^{n+1}\right\|_2^{1/2} \le CC_1^{1/2}\left\|\Delta\boldsymbol{\omega}^{n+1}\right\|_2^{1/2}, \qquad (4.56)
\end{aligned}$$

in which an elliptic regularity $\left\|\boldsymbol{\omega}^{n+1}\right\|_{H^2} \le C\left\|\Delta\boldsymbol{\omega}^{n+1}\right\|_2$ was utilized in the second step and the leading L^2 estimate (4.43) was used in the last step. Similarly, we also observe that the kinematic relationships

$$\boldsymbol{U}^{n+1} - \boldsymbol{U}^n = \nabla^\perp\left(\boldsymbol{\psi}^{n+1}-\boldsymbol{\psi}^n\right), \quad -\Delta\left(\boldsymbol{\psi}^{n+1}-\boldsymbol{\psi}^n\right) = \boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n, \qquad (4.57)$$

est-coll-H1-7

indicate the following Sobolev estimates:

$$\begin{aligned}
\left\|\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\right\|_\infty &\le \left\|\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right\|_{L^\infty} \\
&\le C\left\|\boldsymbol{U}^{n+1}-\boldsymbol{U}^n\right\|_{H^{1+\delta}} \le C\left\|\boldsymbol{\psi}^{n+1}-\boldsymbol{\psi}^n\right\|_{H^{2+\delta}} \le C\left\|\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right\|_{H^\delta} \\
&\le C\left\|\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right\|_2^{1-\delta}\left\|\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right\|_{H^1}^\delta \\
&\le C\left\|\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right\|_2^{1-\delta}\left\|\nabla\left(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right)\right\|_2^\delta \\
&\le C\left(2C_1\right)^{1-\delta}\left\|\nabla\left(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right)\right\|_2^\delta, \qquad (4.58)
\end{aligned}$$

in which estimate (4.43) was used in the last step. Consequently, a combination of (4.56) and (4.58) indicates that

$$\begin{aligned}
\left\|(\boldsymbol{u}^{n+1}-\boldsymbol{u}^n)\cdot\nabla_N\omega^{n+1}\right\|_2 &\le \left\|\boldsymbol{u}^{n+1}-\boldsymbol{u}^n\right\|_\infty \cdot \left\|\nabla_N\omega^{n+1}\right\|_2 \\
&\le CC_1^{1/2}\left(2C_1\right)^{1-\delta}\left\|\nabla\left(\boldsymbol{\omega}^{n+1}-\boldsymbol{\omega}^n\right)\right\|_2^\delta \cdot \left\|\Delta\boldsymbol{\omega}^{n+1}\right\|_2^{1/2} \\
&\le CC_1^{1/2}\left(2C_1\right)^{1-\delta}\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2^\delta \cdot \left\|\Delta_N\omega^{n+1}\right\|_2^{1/2}, \qquad (4.59)
\end{aligned}$$

est-coll-H1-9

due to the fact that $\boldsymbol{\omega} \in \mathcal{P}_N$. In turn, the following estimate is obtained

$$k \left\langle - \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) \cdot \nabla_N \omega^{n+1}, \Delta_N \omega^{n+1} \right\rangle \\ \le C C_1^{3/2} k \left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^\delta \cdot \left\| \Delta_N \omega^{n+1} \right\|_2^{3/2} . \tag{4.60}$$

est-coll-H1-10

Meanwhile, the second conservative nonlinear term in (4.48), $\nabla_N \cdot \left((\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) \omega^{n+1} \right)$, can be expanded and analyzed in a similar way:

$$\begin{aligned} &\left\| \nabla_N \cdot \left((\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) \omega^{n+1} \right) \right\|_2 \le \left\| \mathcal{D}_{Nx} \left((u^{n+1} - u^n) \omega^{n+1} \right) \right\|_2 + \left\| \mathcal{D}_{Ny} \left((v^{n+1} - v^n) \omega^{n+1} \right) \right\|_2 \\ &\quad \le 2 \left(\left\| \partial_x \left((U^{n+1} - U^n) \boldsymbol{\omega}^{n+1} \right) \right\|_2 + \left\| \partial_y \left((V^{n+1} - V^n) \boldsymbol{\omega}^{n+1} \right) \right\|_2 \right), \qquad (4.61) \\ &\left\| \partial_x \left((U^{n+1} - U^n) \boldsymbol{\omega}^{n+1} \right) \right\|_2 = \left\| (U^{n+1} - U^n)_x \boldsymbol{\omega}^{n+1} + (U^{n+1} - U^n) \boldsymbol{\omega}_x^{n+1} \right\|_2 \\ &\quad \le \left\| (U^{n+1} - U^n)_x \right\|_{L^{2/(1-\delta)}} \cdot \left\| \boldsymbol{\omega}^{n+1} \right\|_{L^{2/\delta}} + \left\| U^{n+1} - U^n \right\|_{L^\infty} \cdot \left\| \boldsymbol{\omega}_x^{n+1} \right\|_2 \\ &\quad \le C \left\| U^{n+1} - U^n \right\|_{H^{1+\delta}} \cdot \left\| \nabla \boldsymbol{\omega}^{n+1} \right\|_2 \\ &\quad \le C C_1^{3/2} \left\| \nabla_N \left(\omega^{n+1} - \omega^n \right) \right\|_2^\delta \cdot \left\| \Delta_N \omega^{n+1} \right\|_2^{1/2}, \qquad (4.62) \\ &\left\| \partial_y \left((V^{n+1} - V^n) \boldsymbol{\omega}^{n+1} \right) \right\|_2 \le C C_1^{3/2} \left\| \nabla_N \left(\omega^{n+1} - \omega^n \right) \right\|_2^\delta \cdot \left\| \Delta_N \omega^{n+1} \right\|_2^{1/2} . \qquad (4.63) \end{aligned}$$

Again, the elliptic regularity (4.15), Poincaré's inequality and 2-D Sobolev embedding were repeatedly used in the analysis. As a result, its combination with (4.60) leads to

$$k \left\langle - \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) \cdot \nabla_N \omega^{n+1} - \nabla_N \cdot \left((\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) \omega^{n+1} \right), \Delta_N \omega^{n+1} \right\rangle \\ \le C C_1^{3/2} k \left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^\delta \cdot \left\| \Delta_N \omega^{n+1} \right\|_2^{3/2} . \tag{4.64}$$

We can always choose $0 < \delta < \frac{1}{2}$, so that an application of Young's inequality ($ab \le \frac{a^p}{p} + \frac{b^q}{q}$ with $\frac{1}{p} + \frac{1}{q} = 1$) gives

$$\begin{aligned} &\left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^\delta \cdot \left\| \Delta_N \omega^{n+1} \right\|_2^{3/2} \le \gamma \left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^{4\delta} + \frac{\nu}{2 C C_1^{3/2}} \left\| \Delta_N \omega^{n+1} \right\|_2^2, \\ &\text{with} \quad \gamma = \frac{1}{4} \left(\frac{3 C C_1^{3/2}}{2\nu} \right)^3 . \qquad (4.65) \end{aligned}$$

Furthermore, since $4\delta < 2$, we can apply Young's inequality to $\left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^{4\delta}$ and obtain

$$\gamma \left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^{4\delta} \le \frac{1}{C C_6^{3/2}} \left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^2 + A, \tag{4.66}$$

in which A is a generic constant and depends on δ. As a result, substituting (4.65)–(4.66) into (4.60) gives an estimate for the second nonlinear term:

$$k \left\langle - \left(\boldsymbol{u}^{n+1} - \boldsymbol{u}^n\right) \cdot \nabla_N \omega^{n+1} - \nabla_N \cdot \left((\boldsymbol{u}^{n+1} - \boldsymbol{u}^n) \omega^{n+1} \right), \Delta_N \omega^{n+1} \right\rangle \\ \le k \left\| \nabla_N \left(\omega^{n+1} - \omega^n\right) \right\|_2^2 + \frac{1}{2} \nu k \left\| \Delta \omega^{n+1} \right\|_2^2 + Bk, \tag{4.67}$$

est-coll-H1-13

with $B = C C_1^{3/2} A$.

The third nonlinear term in (4.47), (4.48) can be analyzed in a similar way. We first look at $\boldsymbol{u}^{n+1} \cdot \nabla_N \omega^{n+1}$. A bound for $\|\boldsymbol{u}^{n+1}\|_\infty$ can be obtained in the same fashion as (4.58):

$$\begin{aligned}\left\|\boldsymbol{u}^{n+1}\right\|_{\infty} &\le C\left\|\boldsymbol{U}^{n+1}\right\|_{H^{1+\delta}} \le C\left\|\boldsymbol{\psi}^{n+1}\right\|_{H^{2+\delta}} \le C\left\|\boldsymbol{\omega}^{n+1}\right\|_{H^{\delta}} \le C\left\|\boldsymbol{\omega}^{n+1}\right\|_2^{1-\frac{\delta}{2}} \cdot \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^2}^{\frac{\delta}{2}} \\ &\le C\left\|\boldsymbol{\omega}^{n+1}\right\|_2^{1-\frac{\delta}{2}}\left\|\Delta\boldsymbol{\omega}^{n+1}\right\|_2^{\frac{\delta}{2}} \le CC_1^{1-\frac{\delta}{2}}\left\|\Delta_N\omega^{n+1}\right\|_2^{\frac{\delta}{2}}. \end{aligned} \tag{4.68}$$

Its combination with (4.56) shows that

$$\begin{aligned} k\left\langle\boldsymbol{u}^{n+1}\cdot\nabla_N\omega^{n+1}, \Delta_N\omega^{n+1}\right\rangle &\le k\left\|\boldsymbol{u}^{n+1}\right\|_{\infty}\cdot\left\|\nabla_N\omega^{n+1}\right\|_2\cdot\left\|\Delta_N\omega^{n+1}\right\|_2 \\ &\le CC_1^{3/2}k\left\|\Delta_N\omega^{n+1}\right\|_2^{\frac{3+\delta}{2}}. \end{aligned} \tag{4.69}$$

This analysis can be applied to the term $\nabla_N\cdot\left(\boldsymbol{u}^{n+1}\omega^{n+1}\right)$ in the same way:

$$\begin{aligned} &\left\|\nabla_N\cdot\left(\boldsymbol{u}^{n+1}\omega^{n+1}\right)\right\|_2 \le \left\|\mathcal{D}_{Nx}\left(u^{n+1}\omega^{n+1}\right)\right\|_2 + \left\|\mathcal{D}_{Ny}\left(v^{n+1}\omega^{n+1}\right)\right\|_2 \\ &\qquad \le 2\left(\left\|\partial_x\left(U^{n+1}\boldsymbol{\omega}^{n+1}\right)\right\|_2 + \left\|\partial_y\left(V^{n+1}\boldsymbol{\omega}^{n+1}\right)\right\|_2\right), \qquad (4.70) \\ &\left\|\partial_x\left(U^{n+1}\boldsymbol{\omega}^{n+1}\right)\right\|_2 = \left\|U_x^{n+1}\boldsymbol{\omega}^{n+1} + U^{n+1}\boldsymbol{\omega}_x^{n+1}\right\|_2 \\ &\qquad \le \left\|U_x^{n+1}\right\|_{L^{2/(1-\delta)}}\cdot\left\|\boldsymbol{\omega}^{n+1}\right\|_{L^{2/\delta}} + \left\|U^{n+1}\right\|_{L^{\infty}}\cdot\left\|\boldsymbol{\omega}_x^{n+1}\right\|_2 \\ &\qquad \le C\left\|U^{n+1}\right\|_{H^{1+\delta}}\cdot\left\|\nabla\boldsymbol{\omega}^{n+1}\right\|_2 \le CC_1^{3/2}\left\|\Delta_N\omega^{n+1}\right\|_2^{\frac{1+\delta}{2}}, \qquad (4.71) \\ &\left\|\partial_y\left(V^{n+1}\boldsymbol{\omega}^{n+1}\right)\right\|_2 \le CC_1^{3/2}\left\|\Delta_N\omega^{n+1}\right\|_2^{\frac{1+\delta}{2}}. \qquad (4.72) \end{aligned}$$

As a result, we arrive at the following estimate:

$$\begin{aligned} &k\left\langle\boldsymbol{u}^{n+1}\cdot\nabla_N\omega^{n+1} + \nabla_N\cdot\left(\boldsymbol{u}^{n+1}\omega^{n+1}\right), \Delta_N\omega^{n+1}\right\rangle \\ &\qquad\qquad \le CC_1^{3/2}k\left\|\Delta_N\omega^{n+1}\right\|_2^{\frac{3+\delta}{2}}. \end{aligned} \tag{4.73}$$

est-coll-H1-15-5

Again, since $\frac{3+\delta}{2} < 2$, we can apply Young's inequality and obtain

$$\left\|\Delta_N\omega^{n+1}\right\|_2^{\frac{3+\delta}{2}} \le \frac{\nu}{2CC_1^{3/2}}\left\|\Delta_N\omega^{n+1}\right\|_2^2 + C, \tag{4.74}$$

est-coll-H1-16

in which C is also a generic constant and depends on δ. Going back to (4.73), we have an estimate for the third nonlinear term:

$$\begin{aligned} &k\left\langle\boldsymbol{u}^{n+1}\cdot\nabla_N\omega^{n+1} + \nabla_N\cdot\left(\boldsymbol{u}^{n+1}\omega^{n+1}\right), \Delta_N\omega^{n+1}\right\rangle \\ &\qquad\qquad \le \frac{1}{2}\nu k\left\|\Delta_N\omega^{n+1}\right\|_2^2 + Ck. \end{aligned} \tag{4.75}$$

est-coll-H1-17

Finally, a combination of (4.45)–(4.48), (4.50), (4.55), (4.67) and (4.75) results in

$$\begin{aligned} &\|\nabla_N\omega^{n+1}\|_2^2 - \|\nabla_N\omega^n\|_2^2 + \left(1 - \left(1 + \frac{C'\tilde{C}^2}{\nu}\right)k\right)\|\nabla_N\left(\omega^{n+1} - \omega^n\right)\|_2^2 \\ &+\frac{1}{2}\nu k\|\Delta_N\omega^{n+1}\|_2^2 \le \left(\frac{2M^2}{\nu} + C\right)k. \end{aligned} \tag{4.76}$$

Under a constraint similar to (4.40) and a trivial constraint $k \le \frac{1}{4}$ for the timestep:

$$\frac{C'\tilde{C}^2}{\nu}k \le \frac{1}{2}, \;\; k \le \frac{1}{4}, \quad \text{i.e.,} \quad k \le \min\left(\frac{\nu}{2C'\tilde{C}^2}, \frac{1}{4}\right), \tag{4.77}$$

constraint-coll-dt-2

we have

$$\|\nabla_N\omega^{n+1}\|_2^2 - \|\nabla_N\omega^n\|_2^2 + \frac{1}{4}\left\|\nabla_N\left(\omega^{n+1}-\omega^n\right)\right\|_2^2 + \frac{1}{2}\nu k\|\Delta_N\omega^{n+1}\|_2^2 \le \tilde{\alpha}k,$$
$$\text{with } \tilde{\alpha} = \frac{2M^2}{\nu} + C. \qquad (4.78)$$

Furthermore, an application of elliptic regularity (denoted by another constant c_p'

$$\|\nabla_N\omega^{n+1}\|_2 \le c_p'\|\Delta_N\omega^{n+1}\|_2, \qquad (4.79)$$ ellip regularity-coll

implies that

$$\|\nabla_N\omega^{n+1}\|_2^2 - \|\nabla_N\omega^n\|_2^2 + \tilde{\beta}\nu k\|\nabla_N\omega^{n+1}\|_2^2 \le \tilde{\alpha}k, \quad \text{with } \tilde{\alpha} = \frac{1}{2(c_p')^2}. \qquad (4.80)$$ est-coll-H1-20

Applying an induction argument to the above estimate yields

$$\|\nabla_N\omega^{n+1}\|_2^2 \le (1+\tilde{\beta}\nu k)^{-(n+1)}\|\nabla_N\omega^0\|_2^2 + \frac{\tilde{\alpha}}{\tilde{\beta}\nu}, \quad \text{i.e.,}$$
$$\|\nabla_N\omega^{n+1}\|_2 \le (1+\tilde{\beta}\nu k)^{-\frac{n+1}{2}}\|\nabla_N\omega^0\|_2 + \sqrt{\frac{\tilde{\alpha}}{\tilde{\beta}\nu}} := C_2. \qquad (4.81)$$

Again, C_2 is a time dependent value; however, its time dependence decays exponentially so that a global in time bound is available.

In addition, we also have the $L^2(0,T;H^2)$ bound for the numerical solution:

$$\frac{1}{2}\nu k\sum_{k=i+1}^{N_k}\left\|\Delta_N\omega^k\right\|_2^2 \le \|\nabla_N\omega^i\|_2^2 + \tilde{\alpha}\left(T^*-t^i\right). \qquad (4.82)$$ est-coll-H1-22

4.5. Recovery of the a-priori H^δ assumption (4.21)**.** With the $L^\infty(0,T;L^2)$ and $L^\infty(0,T;H^1)$ estimate for the numerical vorticity solution, namely (4.43) and (4.81), we are able to recover the H^δ assumption (4.21):

$$\begin{aligned}\left\|\omega^{n+1}\right\|_{H_h^\delta} &= \left\|\boldsymbol{\omega}^{n+1}\right\|_{H^\delta} \le C\left\|\boldsymbol{\omega}^{n+1}\right\|_2^{1-\delta}\cdot\left\|\boldsymbol{\omega}^{n+1}\right\|_{H^1}^\delta \\ &\le C_\delta\left\|\boldsymbol{\omega}^{n+1}\right\|_2^{1-\delta}\left\|\nabla\boldsymbol{\omega}^{n+1}\right\|_2^\delta \le C_\delta C_1^{1-\delta}C2^\delta. \end{aligned} \qquad (4.83)$$

For simplicity, by taking $\delta = \frac{1}{2}$, we see that (4.21) is also valid at timestep t^{n+1} if we set

$$\tilde{C} = C_\delta\sqrt{C_1C_2}. \qquad (4.84)$$ a priori-coll-2

Note that C_1 and C_2 are independent of $\tilde{C}$ in the derivation. The constant $\tilde{C}$ is only used in the timestep constraint (4.40). Therefore, induction can be applied so that the a-priori H^δ assumption (4.21) is valid at any timestep under a global timestep constraint

$$k \le \frac{\nu}{4C_\delta^2C_1C_2}. \qquad (4.85)$$ a priori-coll-3

Again, note that both C_1 and C_2 contain an exponential decay in time and therefore are bounded by a given constant in time.

In other words, under (4.85), a global in time constant constraint for the timestep, the proposed semi-implicit scheme (4.10)–(4.12) is unconditionally stable (in terms of spatial grid size and final time). In addition, an asymptotic decay for the L^2 and H^1 norm for the vorticity (equivalent to H^1 and H^2 norms for the velocity) can be derived. Lemma 4.2 is proven.

s:wente

Appendix A. A Wente type estimate. The goal here is to present a Wente type estimate that is applicable to our doubly periodic setting. Original estimate of the Jacobian term (essentially H^{-1} norm) goes back to [48]. Here we need an estimate on the L^2 norm of the Jacobian. The case with homogeneous Dirichlet boundary condition can be found in [26, 27].

wente

PROPOSITION 1. *There exists an absolute constant* $c_w \geq 1$ *such that*

$$\|\nabla^\perp \psi \cdot \nabla \phi\|_{H^{-1}} \leq c_w \|\psi\|_{H^1} \|\phi\|_{H^1} \qquad \forall\, \psi \in \dot{H}^1_{per}(\Omega),\ \phi \in \dot{H}^1_{per}(\Omega) \tag{A.1}$$

q:wente1

$$\|\nabla^\perp \psi \cdot \nabla \phi\|_{2} \leq c_w \|\psi\|_{H^2} \|\phi\|_{H^1} \qquad \forall\, \psi \in \dot{H}^2_{per}(\Omega),\ \phi \in \dot{H}^1_{per}(\Omega) \tag{A.2}$$

q:wente2

$$\|\nabla^\perp \psi \cdot \nabla \phi\|_{2} \leq c_w \|\psi\|_{H^1} \|\phi\|_{H^2} \qquad \forall \psi \in \dot{H}^1_{per}(\Omega),\ \phi \in \dot{H}^2_{per}(\Omega). \tag{A.3}$$

q:wente3

Proof. Let $\Omega = (0, 2\pi)^2$ as before and $\tilde{\Omega} := (-2\pi, 4\pi)^2$. Let $\rho \in c_p^\infty(\mathbf{R}^2)$ be such that $\rho = 1$ in Ω, $\rho = 0$ in $\mathbf{R}^2 - \tilde{\Omega}$ and $\rho(x) \in [0, 1]$ for all $x \in \mathbf{R}^2$. Here ψ and ϕ are 2π-periodic functions on $\mathbf{R}^2$. The proof of (A.2) is based on Lemma 1 in [26], which states that, in our notation, for $\rho\psi \in H_0^2(\tilde{\Omega})$ and $\rho\phi \in H_0^1(\tilde{\Omega})$, one has

$$\|\nabla^\perp(\rho\psi) \cdot \nabla(\rho\phi)\|_{L^2(\tilde{\Omega})} \leq C_K(\tilde{\Omega}) \|\rho\psi\|_{H^2(\tilde{\Omega})} \|\rho\phi\|_{H^1(\tilde{\Omega})} . \tag{A.4}$$

q:wente4

Noting that

$$\begin{aligned}
\|\nabla(\rho\psi)\|_{\tilde{\Omega}} &= \|\nabla(\rho\psi)\|_{\Omega} + \|\nabla(\rho\psi)\|_{\tilde{\Omega}-\Omega} \\
&\leq \|\nabla\psi\|_{\Omega} + \|\rho\nabla\psi\|_{\tilde{\Omega}-\Omega} + \|\psi\nabla\rho\|_{\tilde{\Omega}-\Omega} \\
&\leq \|\nabla\psi\|_{\Omega} + \|\nabla\psi\|_{\tilde{\Omega}-\Omega} + \|\psi\|_{\tilde{\Omega}-\Omega} \|\nabla\rho\|_{L^\infty(\tilde{\Omega}-\Omega)} \\
&\leq \|\nabla\psi\|_{\Omega} + 8\,\|\nabla\psi\|_{\Omega} + 8\, c_p \|\nabla\psi\|_{\tilde{\Omega}-\Omega} \|\nabla\rho\|_{L^\infty(\tilde{\Omega}-\Omega)} ,
\end{aligned} \tag{A.5}$$

and a similar computation for $\|\rho\psi\|_{H^2}$, the right-hand side of (A.4) is majorised as

$$\|\rho\psi\|_{H^2(\tilde{\Omega})} \|\rho\phi\|_{H^1(\tilde{\Omega})} \leq (9 + 8c_p \|\nabla\rho\|_{L^\infty(\mathbf{R}^2)})^2 \, C_K(\tilde{\Omega})^2 \, \|\psi\|_{H^2(\Omega)} \|\phi\|_{H^1(\Omega)} . \tag{A.6}$$

Since the left-hand side of (A.4) majorises $\|\nabla^\perp\psi \cdot \nabla\phi\|_{L^2(\Omega)}$, we have (A.2). For (A.3), we use the identity $\nabla^\perp\psi \cdot \nabla\phi = -\nabla^\perp\phi \cdot \nabla\psi$ and (A.2) with ψ and ϕ interchanged.

For (A.1), we take $w \in H_0^1(\tilde{\Omega})$ and compute

$$\begin{aligned}
\|\nabla^\perp\psi \cdot \nabla\phi\|_{H^{-1}(\tilde{\Omega})} &= \sup_{\|w\|_{H^1(\tilde{\Omega})}=1} (\nabla^\perp\psi \cdot \nabla\phi, w)_{L^2(\tilde{\Omega})} \\
&\leq \sup_{\|w\|_{H^1(\tilde{\Omega})}=1} \|\nabla\phi\|_{L^2(\tilde{\Omega})} \|\nabla\psi\|_{L^2(\tilde{\Omega})} \|\nabla w\|_{L^2(\tilde{\Omega})} \\
&= \|\nabla\phi\|_{L^2(\tilde{\Omega})} \|\nabla\psi\|_{L^2(\tilde{\Omega})}
\end{aligned} \tag{A.7}$$

where the inequality follows from (3.8) in [48]. Arguing as above, (A.1) follows. ∎

s:conv **Appendix B. A convergence result on long time behaviors.** Here we present a modified version of the abstract result presented in [47], so that it is applicable to the current situation, where the phase space is only a subset of a Hilbert (or reflexive Banach) space.

bs_conv_stat PROPOSITION 2. *Let $\{S(t)\}_{t\geq 0}$ be a continuous semi-group on a complete metric space X which is a subset of a separable Hilbert space H with the inherited distance (norm) $\|\cdot\|$. Suppose that the semi-group generates a continuous dissipative dynamical system (in the sense of possessing a compact global attractor $\mathcal{A}$) on X. Let $\{S_k\}_{0<k\leq k_0}$ be a family of continuous maps on X which generates a family of discrete dissipative dynamical system (with global attractor $\mathcal{A}_k$) on X. We further assume that the following two conditions are satisfied.*

H1: *[Uniform boundedness] There exists a $k_1 \in (0, k_0]$ such that $\{S_k\}_{0<k\leq k_1}$ is uniformly bounded in the sense that*

$$K = \bigcup_{0<k\leq k_1} \mathcal{A}_k \tag{B.1}$$

u-bdd

is bounded in X.

H2: *[Finite time uniform convergence] S_k uniformly converges to S on any finite time interval (modulo any initial layer) and uniformly for initial data from the global attractor of the scheme in the sense that there exists $t_0 > 0$ such that for any $T^* > t_0 > 0$*

$$\lim_{k\to 0} \sup_{\boldsymbol{u}\in\mathcal{A}_k, nk\in[t_0,T^*]} \|S_k^n \boldsymbol{u} - S(nk)\boldsymbol{u}\| = 0. \tag{B.2}$$

u-conv-long

Then the global attractors converge in the sense of Hausdorff semi-distance, i.e.

$$\lim_{k\to 0} dist_H(\mathcal{A}_k, \mathcal{A}) = 0. \tag{B.3}$$

Moreover, if the following three more stringent conditions are satisfied:

H3: *[Uniform dissipativity] There exists a $k_1 \in (0, k_0)$ such that $\{S_k\}_{0<k\leq k_1}$ is uniformly dissipative in the sense that*

$$K = \bigcup_{0<k\leq k_1} \mathcal{A}_k \tag{B.4}$$

u-dissip

is pre-compact in X.

H4: *[Uniform convergence on the unit time interval] S_k uniformly converges to S on the unit time interval (modulo an initial layer) and uniformly for initial data from the global attractor of S_k in the sense that for any $t_0 \in (0, 1)$*

$$\lim_{k\to 0} \sup_{\boldsymbol{u}\in\mathcal{A}_k, nk\in[t_0,1]} \|S_k^n \boldsymbol{u} - S(nk)\boldsymbol{u}\| = 0. \tag{B.5}$$

u-conv

H5: *[Uniform continuity of the continuous system] $\{S(t)\}_{t\geq 0}$ is uniformly continuous on K on the unit time interval in the sense that for any $T^* \in [0, 1]$*

$$\lim_{t\to T^*} \sup_{\boldsymbol{u}\in K} \|S(t)\boldsymbol{u} - S(T^*)\boldsymbol{u}\| = 0, \tag{B.6}$$

u-cont

then the invariant measures of the discrete dynamical system $\{S_k\}_{0<k\leq k_0}$ *converge to invariant measures of the continuous dynamical system* S. *More precisely, let* $\mu_k \in \mathcal{IM}_k$ *where* $\mathcal{IM}_k$ *denotes the set of all invariant measures of* S_k. *There must exist a subsequence, still denoted* $\{\mu_k\}$, *and* $\mu \in \mathcal{IM}$ *(an invariant measure of* $S(t)$*), such that* μ_k *weakly converges to* μ, *i.e.,*

$$\mu_k \rightharpoonup \mu, \ \text{as } k \to 0. \tag{B.7}$$

Proof. The proof is exactly the same as those in [46, 47]. We leave the detail to the interested reader. □

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