SPLAYED DIVISORS AND THEIR CHERN CLASSES

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Abstract. We obtain several new characterizations of splayedness for divisors: a Leibniz property for ideals of singularity subschemes, the vanishing of a 'splayedness' module, and the requirements that certain natural morphisms of modules and sheaves of logarithmic derivations and logarithmic differentials be isomorphisms. We also consider the effect of splayedness on the Chern classes of sheaves of differential forms with logarithmic poles along splayed divisors, as well as on the Chern-Schwartz-MacPherson classes of the complements of these divisors. A postulated relation between these different notions of Chern class leads to a conjectural identity for Chern-Schwartz-MacPherson classes of splayed divisors and subvarieties, which we are able to verify in several template situations.

1. Introduction

Two divisors in a nonsingular variety V are *splayed* at a point p if their local equations at p may be written in terms of disjoint sets of analytic coordinates. Splayed divisors are transversal in a very strong sense; indeed, splayedness may be considered a natural generalization of transversality for possibly singular divisors (and subvarieties of higher codimension). In previous work, the second-named author has obtained several characterizing properties for splayedness ([Fab]). For example, two divisors are splayed at p if and only if their *Jacobian ideals* satisfy a 'Leibniz property' ([Fab], Proposition 8); and if and only if the corresponding modules of logarithmic derivations at p span the module of ordinary derivations for V at p ([Fab], Proposition 15).

In this paper we refine some of these earlier results, and consider implications for different notions of *Chern classes* associated with divisors. Specifically, we strengthen the first result recalled above, by showing that splayedness is already characterized by the Leibniz property after restriction to the union of the divisors; equivalently, this amounts to a Leibniz property for the ideals defining the *singularity subschemes* for the divisors at p (Corollary 2.6 and 2.7). We introduce a 'splayedness module', which we describe both in terms of these ideals and in terms of modules of logarithmic derivations (Definition 2.3, Proposition 2.5), and whose vanishing is equivalent to splayedness. Thus, this module quantifies precisely the failure of splayedness of two divisors meeting at a point.

These results may be expressed in terms of the quality of certain natural morphisms associated with two divisors. For example, given two divisors D_1, D_2 meeting at a point p and without common components, there is a natural monomorphism

$$\frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log(D_1\cup D_2))} \hookrightarrow \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_1)} \oplus \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_2)}$$

involving quotients of modules of logarithmic derivations. We prove that D_1 and D_2 are splayed at p if and only if this monomorphism is an isomorphism (Theorem 2.4). We also prove an analogous statement involving sheaves of *logarithmic differentials* (Theorem 2.12) giving a partial answer to a question raised in [Fab], but only subject to the vanishing of an Ext module: D_1 and D_2 are splayed if the natural inclusion

$$\Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2) \subseteq \Omega^1_{V,p}(\log D) \tag{1}$$

is an equality and $\mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_{V,p}(\log D), \mathscr{O}) = 0$. Thus, if D is free at p, then D_1 and D_2 are splayed at p if and only if the two modules in (1) are equal. In general this condition alone does not imply splayedness, as Example 2.15 shows.

One advantage of expressing splayedness in terms of these morphisms is that the characterizing conditions globalize nicely, and give conditions on morphisms of *sheaves* of logarithmic derivations and differentials characterizing splayedness at all points of intersection of two divisors. These conditions imply identities involving Chern classes for these sheaves (Corollary 2.20). In certain situations (for example in the case of curves on surfaces) these identities actually characterize splayedness. Also, there is a different notion of 'Chern class' that can be associated with a divisor D in a nonsingular variety V, namely the Chern-Schwartz-MacPherson (c_{SM}) class of the complement $V \smallsetminus D$. (See §3.1 for a rapid reminder of this notion). In previous work, the first-named author has determined several situations where this c_{SM} class *equals* the Chern class $c(\mathrm{Der}_V(-\log D))$ of the sheaf of logarithmic differentials. It is then natural to expect that c_{SM} classes of complements of splayed divisors, and more general subvarieties, should satisfy a similar type of relations as the one obtained for ordinary Chern classes of sheaves of derivations. From this point of view we analyze three template sources of splayed subvarieties: subvarieties defined by pull-backs from factors of a product (Proposition 3.3), joins of projective varieties (Proposition 3.6), and the case of curves (Proposition 3.7). In each of the three situations we are able to verify explicitly that the corresponding expected relation of c_{SM} classes does hold. We hope to come back to the question of the validity of this relation for arbitrary splayed subvarieties in future work.

The new characterizations for splayedness are given in §2, together with the implications for Chern classes of sheaves of logarithmic derivations for splayed divisors. The conjectured expected relation for c_{SM} classes of complements, together with some necessary background material, is presented in §3.

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2. Splayedness

2.1. Let V be a smooth complex projective variety of dimension n. We say that two divisors D_1 and D_2 in V are *splayed* at a point p if there exist complex analytic coordinates $x_1, \dots, x_n$ at p such that D_1, D_2 have defining equations $g(x_1, \dots, x_k, 0, \dots, 0) =$

0, $h(0,\dots,0,x_{k+1},\dots,x_n)=0$ at p, where $1\le k<n$. Here $g,h\in\mathscr{O}^{\mathrm{an}}_{V,p}\cong\mathbb{C}\{x_1,\dots,x_n\}$. We simply say that D_1 and D_2 are *splayed* if they are splayed at p for every $p\in D_1\cap D_2$.

The *Jacobian ideal* J_f of $f\in\mathbb{C}\{x_1,\dots,x_n\}$ is the ideal generated by its partial derivatives, i.e., $J_f=(\partial_{x_1}f,\dots,\partial_{x_n}f)$. We will also consider the ideal $J'_f=J_f+(f)$. A function f is called *Euler-homogeneous at* p if $J'_f=J_f$, that is, if there exists a derivation δ such that $f=\delta f$.

Unlike the Jacobian ideal, the ideal J'_f only depends on the associate class of f: if u is a unit, then $J'_{uf}=J'_f$. This implies that the ideal J'_f globalizes, in the sense that if D is a divisor with local equation $f_p=0$ at p, the ideals $J'_{f_p}\subseteq\mathscr{O}^{\mathrm{an}}_{V,p}$ for $p\in V$ determine a subscheme JD of D, which we call the *singularity subscheme* of D.

In other words, the ideal sheaf of JD in D is the image of the natural morphism of sheaves

$$\mathrm{Der}_V(-D)\longrightarrow\mathscr{O}_D$$

defined by applying derivations to local equations for D. The kernel of the corresponding morphism $\mathrm{Der}_V\to\mathscr{O}_D(D)$ defines the sheaf of *logarithmic derivations* (or *logarithmic vector fields*) with respect to D. We briefly recall the notions of logarithmic derivations and differential forms and free divisors, following [Sai80]. Let $D\subseteq V$ be a divisor that is locally at p given by $\{f=0\}$. A derivation $\delta\in\mathrm{Der}_{V,p}$ at p is *logarithmic along* D if the germ $\delta(f)$ is contained in the ideal (f) of $\mathscr{O}_{V,p}$. The module of germs of logarithmic derivations (along D) at p is denoted by

$$\mathrm{Der}_{V,p}(-\log D)=\{\delta:\delta\in\mathrm{Der}_{V,p}\ \text{ such that }\delta f\in(f)\mathscr{O}_{V,p}\}.$$

These modules are the stalks at points p of the sheaf $\mathrm{Der}_V(-\log D)$ of $\mathscr{O}_V$-modules. Similarly we define logarithmic differential forms: a meromorphic q-form ω is *logarithmic* (along D) at p if ωf and $fd\omega$ are holomorphic (or equivalently if ωf and $df\wedge\omega$ are holomorphic) at p. We denote

$$\Omega^q_{V,p}(\log D)=\{\omega:\omega\text{ germ of a logarithmic }q\text{-form at }p\}.$$

Again, this notion globalizes and yields a coherent sheaf $\Omega^q_V(\log D)$ of $\mathscr{O}_V$-modules. One can show that $\mathrm{Der}_{V,p}(-\log D)$ and $\Omega^1_{V,p}(\log D)$ are reflexive $\mathscr{O}_{V,p}$-modules dual to each other (see [Sai80], Corollary 1.7). The germ (D,p) is called *free* if $\mathrm{Der}_{V,p}(-\log D)$ resp. $\Omega^1_{V,p}(\log D)$ is a free $\mathscr{O}_{V,p}$-module. The divisor D is called a *free divisor* if (D,p) is free at every point $p\in V$.

In terms of $\mathscr{O}=\mathscr{O}^{\mathrm{an}}_{V,p}$-modules of derivations, if D has equation $f=0$ at p, then there is an exact sequence of $\mathscr{O}$-modules

$$0\longrightarrow\mathrm{Der}_{V,p}(-\log D)\longrightarrow\mathrm{Der}_{V,p}\longrightarrow J'_f/(f)\longrightarrow 0\quad.\tag{2}$$

This sequence is the local analytic aspect of the sequence of coherent $\mathscr{O}_V$-modules

$$0\longrightarrow\mathrm{Der}_V(-\log D)\longrightarrow\mathrm{Der}_V\longrightarrow\mathscr{I}_{JD,D}(D)\longrightarrow 0\quad.\tag{3}$$

where $\mathscr{I}_{JD,D}$ is the ideal sheaf of JD in D (see e.g., [Dol07], §2).

2.2. Equivalent conditions for splayedness in terms of Jacobian ideals and modules of logarithmic derivations are explored in [Fab]. In this section we reinterpret some of the results of [Fab], and discuss other characterizations.

Let $\mathscr{O} = \mathbb{C}\{x_1, \dots, x_n\}$. We will assume throughout that D_1 and D_2 are divisors defined by $g, h \in \mathscr{O}$ respectively, where g and h are reduced and without common components. According to Proposition 8 in [Fab], D_1 and D_2 are splayed if and only if (up to possibly multiplying g and h by units) J_{gh} satisfies the Leibniz property:

$$J_{gh} = gJ_h + hJ_g \quad .$$

We will prove that this equality is equivalent to the same property for J':

$$J'_{gh} = hJ'_h + hJ'_g \quad ,$$

and interpret this equality in terms of modules of derivations.

Lemma 2.1. *If $g, h \in \mathscr{O}$ have no components in common, then there is an injective homomorphism of R-modules*

$$\varphi \colon \frac{J'_g}{(g)} \oplus \frac{J'_h}{(h)} \hookrightarrow \frac{\mathscr{O}}{(gh)}$$

given by multiplication by h on the first factor and by g on the second. The image of φ contains $J'_{gh}/(gh)$.

Proof. The homomorphism φ is given by

$$(a + (g), b + (h)) \mapsto ah + bg \mod (gh) \quad .$$

It is clear that φ is well-defined. To verify that φ is injective under the assumption that g, h do not contain a common factor, assume $\varphi(a, b) = 0 \mod (gh)$; then $ah + bg = cgh$ for some representatives $a, b \in \mathscr{O}$ and $c \in \mathscr{O}$. This implies that $ah \in (g)$, and hence $a \in (g)$ since g and h have no common factors. By the same token, $b \in (h)$. Hence a and b have to be zero in $J'_g/(g)$ and $J'_h/(h)$. Finally, $J'_{gh}/(gh)$ is generated by $\partial_{x_i}(gh) \mod (gh)$. Since $\partial_{x_i}(gh) = h\partial_{x_i} g + g\partial_{x_i} h \in hJ_g + gJ_h$, we see that $\partial_{x_i}(gh) + (gh) \in hJ'_g/(gh) + gJ'_h/(gh) = \operatorname{im} \varphi$, and hence $J'_{gh}/(gh) \subseteq \operatorname{im} \varphi$, as claimed. □

The following result expresses the splayedness condition in terms of the morphism φ of Lemma 2.1.

Theorem 2.2. *Let D_1, D_2 be divisors of V, given by $g = 0$, $h = 0$ at p, where g and h have no common components. Then D_1 and D_2 are splayed at p if and only if the morphism φ of Lemma 2.1 induces an isomorphism*

$$\frac{J'_g}{(g)} \oplus \frac{J'_h}{(h)} \cong \frac{J'_{gh}}{(gh)} \quad .$$

Proof. By Lemma 2.1, there is an injective homomorphism

$$\iota \colon \frac{J'_{gh}}{(gh)} \hookrightarrow \frac{J'_g}{(g)} \oplus \frac{J'_h}{(h)} \quad .$$

This morphism ι is the unique homomorphism such that the composition

$$\frac{J'_{gh}}{(gh)} \xhookrightarrow{\ \iota\ } \frac{J'_g}{(g)} \oplus \frac{J'_h}{(h)} \xhookrightarrow{\ \varphi\ } \frac{\mathscr{O}}{(gh)}$$

is the natural inclusion $r + (gh) \mapsto r + (gh)$. Explicitly, an element of $J'_{gh}/(gh)$ is of the form $\delta(gh) + (gh)$, for a derivation δ. Since $\delta(gh) = h\delta(g) + g\delta(h)$, and

$$h\delta(g) + g\delta(h) + (gh) = \varphi(\delta(g) + (g), \delta(h) + (h)) \quad ,$$

we see that

$$\iota(\delta(gh) + (gh)) = (\delta(g) + (g), \delta(h) + (h)) \quad .$$

In other words, ι is the morphism induced by the compatibility with the morphisms of Der modules: it is the unique homomorphism making the following diagram commute:

$$\begin{array}{ccc} \dfrac{J'_{gh}}{(gh)} & \xhookrightarrow{\qquad\qquad\iota\qquad\qquad} & \dfrac{J'_g}{(g)} \oplus \dfrac{J'_h}{(h)} \\ \uparrow\cong & & \cong\uparrow \\ \dfrac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D)} & \hookrightarrow & \dfrac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_1)} \oplus \dfrac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_2)} \end{array}$$

where $D = D_1 \cup D_2$, and the vertical isomorphisms are induced by sequence (2). The monomorphism in the bottom row is induced from the natural morphism

$$\mathrm{Der}_{V,p} \longrightarrow \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_1)} \oplus \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_2)} \quad ,$$

using the fact that the kernel of this morphism is $\mathrm{Der}_{V,p}(-\log D_1) \cap \mathrm{Der}_{V,p}(-\log D_2)$, and by Seidenberg's theorem $\mathrm{Der}_{V,p}(-\log D_1) \cap \mathrm{Der}_{V,p}(-\log D_2) = \mathrm{Der}_{V,p}(-\log D)$, see e.g., [HM93], p. 313 or [OT92], Proposition 4.8. This morphism (and hence ι) is onto if and only if $\mathrm{Der}_{V,p} = \mathrm{Der}_{V,p}(-\log D_1) + \mathrm{Der}_{V,p}(-\log D_2)$, and this condition is satisfied if and only if D_1 and D_2 are splayed at p by Proposition 15 of [Fab]. The statement follows. □

2.3. The argument given above may be recast as follows. As $\mathrm{Der}_{V,p}(-\log D_1)$ and $\mathrm{Der}_{V,p}(-\log D_2)$ are submodules of $\mathrm{Der}_{V,p}$, whose intersection is $\mathrm{Der}_{V,p}(D)$, we have an exact sequence

$$0 \longrightarrow \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D)} \longrightarrow \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_1)} \oplus \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_2)}$$
$$\longrightarrow \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_1) + \mathrm{Der}_{V,p}(-\log D_2)} \longrightarrow 0$$

The last quotient may be viewed as a measure of the 'failure of splayedness at p'.

Definition 2.3. The *splayedness module* for D_1 and D_2 at p is the quotient

$$\mathrm{Splay}_p(D_1, D_2) := \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_1) + \mathrm{Der}_{V,p}(-\log D_2)} \quad .$$

By Proposition 15 in [Fab], D_1 and D_2 are splayed at p if and only if their splayedness module at p vanishes. Equivalently:

Theorem 2.4. *Let D_1, D_2 be reduced divisors of V, without common components, and let $D = D_1 \cup D_2$. Then there is a natural monomorphism of modules*

$$\frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D)} \hookrightarrow \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_1)} \oplus \frac{\mathrm{Der}_{V,p}}{\mathrm{Der}_{V,p}(-\log D_2)} \quad ,$$

and D_1, D_2 are splayed at p if and only if this monomorphism is an isomorphism.

The splayedness module may be computed as follows:

Proposition 2.5. *With notation as above, the splayedness module is isomorphic to*

$$\frac{hJ'_g + gJ'_h}{J'_{gh}} \quad .$$

Proof. Via the identification used in the proof of Theorem 2.2, the monomorphism appearing in Theorem 2.4 is the homomorphism

$$\iota\colon \frac{J'_{gh}}{(gh)} \hookrightarrow \frac{J'_g}{(g)} \oplus \frac{J'_h}{(h)} \quad .$$

Therefore, the cokernels are isomorphic; this shows that $\operatorname{coker}\iota$ is isomorphic to the splayedness module. To determine $\operatorname{coker}\iota$, use the monomorphism φ of Lemma 2.1 to identify the direct sum with a submodule of $\mathscr{O}/(gh)$; this submodule is immediately seen to equal $(hJ'_g + gJ'_h)/(gh)$. Use this identification to view ι as acting

$$\frac{J'_{gh}}{(gh)} \hookrightarrow \frac{hJ'_g + gJ'_h}{(gh)} \quad ;$$

it is then clear that $\operatorname{coker}\iota$ is isomorphic to the module given in the statement. □

Corollary 2.6. *With notation as above, D_1 and D_2 are splayed at p if and only if $J'_{gh} = hJ'_g + gJ'_h$.*

Corollary 2.7. *With notation as above, D_1 and D_2 are splayed at p if and only if $J_{gh} + (gh) = hJ_g + gJ_h + (gh)$.*

Proof. This is a restatement of Corollary 2.6. □

2.4. As recalled above, Proposition 8 from [Fab] states that D_1 and D_2 are splayed at p if and only if $J_{gh} = hJ_g + gJ_h$ up to multiplying g and h by units. Corollary 2.7 strengthens this result, as it shows that the weaker condition that these two ideals are equal *modulo* (gh) suffices to imply splayedness. In fact, this gives an alternative proof of Proposition 8 from [Fab]: as

$$\begin{aligned} D_1 \text{ and } D_2 \text{ are splayed at } p &\implies J_{gh} = hJ_g + gJ_h \text{ (for suitable choices of } g \text{ and } h) \\ &\implies J_{gh} = hJ_g + gJ_h \mod (gh) \\ &\implies D_1 \text{ and } D_2 \text{ are splayed at } p \end{aligned}$$

(the first two implications are immediate, and the third is given by Corollary 2.7), these conditions are all equivalent. Also, note that the conditions expressed in Corollaries 2.6 and 2.7 are insensitive to multiplications by units. Indeed, if $f \in \mathscr{O}$ and u is a unit, then $J'_{fu} = J'_f$. In general, $J_{fu} \neq J_f$.

Remark 2.8. The implication

$$J_{gh} = hJ_g + gJ_h \mod (gh) \implies J_{gh} = hJ_g + gJ_h$$

is straightforward if gh is Euler homogeneous at p, and then it does not require a particular choice of g, h defining D_1, D_2. Indeed, the inclusion $J_{gh} \subseteq hJ_g + gJ_h$ always holds; to verify the other inclusion, let δ_1, δ_2 be derivations, and consider $h\delta_1 g + g\delta_2 h$. By the equality $J_{gh} = hJ_g + gJ_h \mod (gh)$, there exists a derivation δ and an element a such that

$$h\delta_1 g + g\delta_2 h = \delta(gh) + agh \ .$$

If gh is Euler homogeneous, we can find a derivation ε such that $gh = \varepsilon(gh)$; thus

$$h\delta_1 g + g\delta_2 h = (\delta + a\varepsilon)(gh) \ ,$$

and this shows $hJ_g + gJ_h \subseteq J_{gh}$ as δ_1 and δ_2 were arbitrary. ⌟

Remark 2.9. In view of Proposition 8 from [Fab], it is natural to ask whether the module $(hJ_g + gJ_h)/J_{gh}$ may be another realization of the splayedness module. This is not the case. Examples may be constructed by considering Euler-homogeneous functions g, h (i.e., assume $J'_g = J_g$, $J'_h = J_h$) such that the product is not Euler-homogeneous: concretely, one may take $g = x^3 + y^2$ and $h = x^5 + y^7$. For such functions, the splayedness module is

$$\frac{hJ'_g + gJ'_h}{J'_{gh}} = \frac{hJ_g + gJ_h}{J'_{gh}} \quad ;$$

$(hJ_g + gJ_h)/J_{gh}$ surjects onto this module, but not isomorphically since the kernel J'_{gh}/J_{gh} is nonzero if gh is not Euler-homogeneous.

On the other hand, $(hJ_g+gJ_h)/J_{gh}$ *does* equal the splayedness module if gh is Euler homogeneous. Indeed, the equality $J_{gh} = J'_{gh}$ implies that $hJ'_g + gJ'_h \subseteq hJ_g + gJ_h$ by the same argument used in Remark 2.8, and the other inclusion is always true. ⌟

2.5. It is natural to ask whether splayedness can be expressed in terms of logarithmic *differential forms,* in the style of Theorem 2.4 (cf. Question 16 in [Fab]). A partial answer to this question will be given in Theorem 2.12 below. We first give a precise (but a little obscure) translation of splayedness in terms of a morphism of Ext modules of modules of differential forms. We consider the natural epimorphism

$$\Omega^1_{V,p}(\log D_1) \oplus \Omega^1_{V,p}(\log D_2) \longrightarrow \Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2)$$

and the induced morphism

$$\text{(4)} \quad \operatorname{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D_1) + \Omega^1_V(\log D_2), \mathscr{O}) \longrightarrow \operatorname{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D_1) \oplus \Omega^1_V(\log D_2), \mathscr{O}) \quad .$$

Proposition 2.10. *Let D_1, D_2 be reduced divisors of V, without common components, and let $D = D_1 \cup D_2$. Then the morphism* (4) *is an epimorphism, and its kernel is the splayedness module: there is an exact sequence*

$$0 \longrightarrow \mathrm{Splay}_p(D_1, D_2) \longrightarrow \mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D_1) + \Omega^1_V(\log D_2), \mathscr{O}) \longrightarrow \mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D_1) \oplus \Omega^1_V(\log D_2), \mathscr{O}) \longrightarrow 0$$

Therefore, D_1 and D_2 are splayed at p if and only if the morphism (4) is an isomorphism, if and only if it is injective.

Proof. Note that $\Omega^1_{V,p}(\log D_1)$, $\Omega^1_{V,p}(\log D_2)$ both embed in $\Omega^1_{V,p}(\log D)$, and an elementary verification shows that

$$\Omega^1_{V,p}(\log D_1) \cap \Omega^1_{V,p}(\log D_2) = \Omega^1_{V,p}$$

if D_1 and D_2 have no components in common. Indeed, the poles of a form in the intersection would be on a divisor contained in both D_1 and D_2. By the previous assumption on D_1 and D_2 such a form can have no poles. We then get an exact sequence

$$0 \longrightarrow \Omega^1_{V,p} \longrightarrow \Omega^1_{V,p}(\log D_1) \oplus \Omega^1_{V,p}(\log D_2) \longrightarrow \Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2) \longrightarrow 0$$

Applying the dualization functor $\mathrm{Hom}_{\mathscr{O}}(-, \mathscr{O})$ to this sequence gives the exact sequence

$$0 \to (\Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2))^\vee \to \mathrm{Der}_{V,p}(-\log D_1) \oplus \mathrm{Der}_{V,p}(-\log D_2) \to \mathrm{Der}_{V,p}$$
$$\to \mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D_1) + \Omega^1_V(\log D_2), \mathscr{O}) \to \mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D_1) \oplus \Omega^1_V(\log D_2), \mathscr{O}) \to 0$$

(the last 0 is due to the fact that $\mathrm{Der}_{V,p}$ is free by the hypothesis of nonsingularity of V). This shows that the morphism (4) is an epimorphism, and identifies its kernel with the cokernel of the natural morphism $\mathrm{Der}_{V,p}(-\log D_1) \oplus \mathrm{Der}_{V,p}(-\log D_2) \to \mathrm{Der}_{V,p}$, which is the splayedness module introduced in Definition 2.3. □

Remark 2.11. The argument also shows that $(\Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2))^\vee$ is isomorphic to $\mathrm{Der}_{V,p}(-\log D)$ regardless of splayedness. Indeed, by the sequence obtained in the proof this dual is identified with $\mathrm{Der}_{V,p}(-\log D_1) \cap \mathrm{Der}_{V,p}(-\log D_2)$, and this equals $\mathrm{Der}_{V,p}(-\log D)$ by Seidenberg's theorem as recalled in the proof of Theorem 2.2. ⌟

While the statement of Proposition 2.10 is precise, it seems hard to apply. The following result translates this criterion in terms that are more similar to those of Theorem 2.4, but at the price of a hypothesis of freeness.

Theorem 2.12. *Let D_1, D_2 be reduced divisors of V, without common components, and let $D = D_1 \cup D_2$. Then there is a natural monomorphism of modules*

$$\frac{\Omega^1_{V,p}(\log D_1) \oplus \Omega^1_{V,p}(\log D_2)}{\Omega^1_{V,p}} \hookrightarrow \Omega^1_{V,p}(\log D) \quad .$$

If D_1, D_2 are splayed at p, then this monomorphism is an isomorphism.

If $\mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_{V,p}(\log D), \mathscr{O}) = 0$ (for example, if D is free at p), then the converse implication holds.

Remark 2.13. The condition in the statement is clearly equivalent to the condition that the inclusion $\Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2) \subseteq \Omega^1_{V,p}(\log D)$ be an equality. Example 2.15 below shows that there are non-splayed divisors for which this condition does hold. Thus the situation for logarithmic *differentials vis-a-vis* splayedness appears to be less straightforward than for logarithmic *derivations.* Also see Remark 2.16. ⌟

Proof. To show that splayedness implies the stated condition, assume that D_1 and D_2 are splayed at p and defined by $g = g(z_1, \dots, z_p)$ and $h = h(z_{p+1}, \dots, z_n)$. Since g and h are reduced, we may assume that there exist indices i, resp., j such that g and $\partial_{x_i} g$, resp., h and $\partial_{x_j} h$ have no common factors. By definition, a meromorphic differential one-form has logarithmic poles along $D = D_1 \cup D_2$ if $gh\omega$ and $d(gh) \wedge \omega$ are holomorphic. Writing

$$\omega = \frac{\sum_{i=1}^p a_i dz_i + \sum_{j=p+1}^n b_j dz_j}{gh},$$

the second condition yields that each a_i divisible by h, i.e., $a_i = h\tilde{a}_i$ (and similarly each $b_j = g\tilde{b}_j$) for some $\tilde{a}_i, \tilde{b}_j \in \mathscr{O}$. Hence ω is of the form $\omega_g + \omega_h = \frac{\sum_{i=1}^p \tilde{a}_i dz_i}{g} + \frac{\sum_{j=p+1}^n \tilde{b}_j dz_j}{h}$. It is easy to see that $\omega_g \in \Omega^1_{V,p}(\log D_1)$ and $\omega_h \in \Omega^1_{V,p}(\log D_2)$.

To prove that the stated condition implies splayedness if $\mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_{V,p}(\log D), \mathscr{O}) = 0$, we use the exact sequence in Proposition 2.10. If $\Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2) = \Omega^1_{V,p}(\log D)$, then $\mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2), \mathscr{O}) = \mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_{V,p}(\log D), \mathscr{O}) = 0$. In this case the exact sequence in Proposition 2.10 becomes

$$0 \longrightarrow \mathrm{Splay}_p(D_1, D_2) \longrightarrow 0 \longrightarrow \mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D_1) \oplus \Omega^1_V(\log D_2), \mathscr{O}) \longrightarrow 0$$

and forces the splayedness module to vanish, concluding the proof. □

Corollary 2.14. *If C_1, C_2 are curves without common components on a nonsingular surface S, then C_1 and C_2 are splayed at p if and only if*

$$\Omega_{S,p}(\log C_1) + \Omega_{S,p}(\log C_2) = \Omega_{S,p}(\log(C_1 \cup C_2)) \quad .$$

Proof. Indeed, the additional condition $\mathrm{Ext}^1_{\mathscr{O}}(\Omega_{S,p}(\log(C_1 \cup C_2), \mathscr{O}) = 0$ is automatic in this case, since the locus along which a reflexive sheaf on a nonsingular variety is not free has codimension at least 3 ([Har80], Corollary 1.4) and sheaves of logarithmic differentials are reflexive. □

Example 2.15. Let $D = D_1 \cup D_2$ be the union of the cone $D_1 = \{h_1 = x^2 + y^2 - z^2 = 0\}$ and the plane $D_2 = \{h_2 = x = 0\}$ in $(V, p) = (\mathbb{C}^3, 0)$. Then D is neither splayed nor free at the origin. Indeed, $\mathrm{Der}_{V,p}(-\log D_1)$ is generated by $x\partial_x + y\partial_y + z\partial_z, y\partial_x - x\partial_y, z\partial_x + x\partial_z, z\partial_y + y\partial_z$ and $\mathrm{Der}_{V,p}(-\log D_2)$ by $x\partial_x, \partial_y, \partial_z$; it follows that ∂_x is not contained in $\mathrm{Der}_{V,p}(-\log D_1) + \mathrm{Der}_{V,p}(-\log D_2)$. Thus $\mathrm{Der}_{V,p}(-\log D_1) + \mathrm{Der}_{V,p}(-\log D_2) \neq \mathrm{Der}_{V,p}(-\log D)$, and hence D is not splayed, by Proposition 15 of [Fab] (i.e., $\mathrm{Splay}_p(D_1, D_2) \neq 0$). On the other hand, it is easy to see that

$$\Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2) = \frac{dh_1}{h_1}\mathscr{O} + \frac{dx}{x}\mathscr{O} + dy\mathscr{O} + dz\mathscr{O}.$$

By Theorem 2.9 of [Sai80], $\Omega^1_{V,p}(\log D) = \Omega^1_{V,p}(\log D_1) + \Omega^1_{V,p}(\log D_2)$. Note that it follows that D cannot be free at p, by Theorem 2.12. A computation shows that $\mathrm{Der}_{V,p}(-\log D)$ is minimally generated by 4 derivations, confirming this. ⌟

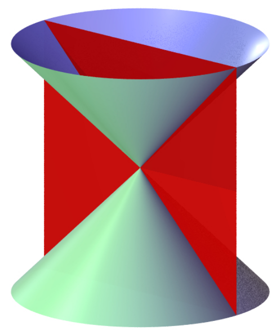

FIGURE 1. The cone D_1 and the plane D_2 are not splayed at the origin but satisfy $\Omega^1_V(\log D_1) + \Omega^1_V(\log D_2) = \Omega^1_V(\log D)$.

Remark 2.16. Example 2.15 shows that in general the first condition listed in Theorem 2.12 does not suffice to imply splayedness. One can construct similar examples using Saito's theorem (Theorem 2.9 of [Sai80]): let $D = \cup_{i=1}^m D_i$ at a point $p \in V$ be a divisor, where the D_i are the irreducible components of D. Then Saito's theorem says that if each D_i is normal, D_i intersects D_j $(i \neq j)$ transversally outside a codimension 2 set and the triple intersections $D_i \cap D_j \cap D_k$ have codimension ≥ 3 (for $i \neq j \neq k$) then

$$\Omega^1_{V,p}(\log D) = \sum_{i=1}^m \frac{dh_i}{h_i}\mathscr{O} + \Omega^1_{V,p}.$$

This is easily seen to be equal to $\sum_{i=1}^m \Omega^1_{V,p}(\log D_i)$. For non-splayed D_i this gives examples where $\mathrm{Ext}^1_{\mathscr{O}}(\Omega^1_V(\log D), \mathscr{O}) \neq 0$. ⌟

Remark 2.17. Mathias Schulze pointed out to us that a situation similar to Theorem 2.12 occurs by considering modules $\omega^\bullet_D$ of regular differential forms on D. If f defines a reduced divisor D, then the dual of the module $J'_f/(f)$ is the module $\mathcal{R}_D = \omega^0_D$ ([GS], Proposition 3.2). Dualizing the map $\iota\colon J'_{gh}/(gh) \hookrightarrow J'_g/(g) \oplus J'_h/(h)$ used in the proof of Theorem 2.2 gives a natural inclusion

$$\iota^\vee\colon \omega^0_{D_1} \oplus \omega^0_{D_2} \hookrightarrow \omega^0_D$$

where $D = D_1 \cup D_2$. If D_1 and D_2 are splayed, then ι is an isomorphism, and so is $\iota^\vee$. If D is free, then the converse holds: if D is free and $\iota^\vee$ is an isomorphism, then ω^0_D, resp. $\omega^0_{D_1}$, $\omega^0_{D_2}$ are reflexive ([GS], Corollary 3.5), so dualizing $\iota^\vee$ shows that ι is also an isomorphism; it follows that D is splayed, by Theorem 2.2. One advantage of this observation over Theorem 2.12 is that the module ω^0_D only depends on D, not on the embedding of D in a nonsingular variety V. ⌟

2.6. **Global considerations, and Chern classes.** The global version of Theorem 2.4 is the following immediate consequence at the level of sheaves of derivations.

Theorem 2.18. *Let D_1, D_2 be reduced divisors of V, without common components, and let $D = D_1 \cup D_2$. Then there is a natural monomorphism of sheaves*

$$\frac{\mathrm{Der}_V}{\mathrm{Der}_V(-\log D)} \hookrightarrow \frac{\mathrm{Der}_V}{\mathrm{Der}_V(-\log D_1)} \oplus \frac{\mathrm{Der}_V}{\mathrm{Der}_V(-\log D_2)} \quad ,$$

and D_1, D_2 are splayed if and only if this monomorphism is an isomorphism.

We can also globalize the splayedness module and introduce a 'splayedness sheaf'

$$\mathrm{Splay}_V(D_1, D_2) := \frac{\mathrm{Der}_V}{\mathrm{Der}_V(-\log D_1) + \mathrm{Der}_V(-\log D_2)} \quad .$$

so that D_1 and D_2 are splayed if and only if $\mathrm{Splay}_V(D_1, D_2)$ vanishes.

Example 2.19. For reduced curves on surfaces, splayedness is equivalent to transversality at nonsingular points; this is easily deduced from the definition of splayedness. Namely, if C_1 and C_2 are reduced curves on a surface S, then C_1 and C_2 are splayed at a point p if one can find coordinates (x, y) at p such that C_1 is locally defined by $g(x, 0)$ and C_2 by $h(0, y)$. Since C_1 and C_2 are reduced, this is only possible when $g(x, 0)$ and $h(0, y)$ are of the form ux and vy for some units $u, v \in \mathscr{O}_{S,p}$. Putting together this remark and the results proved so far, we see that if C_1, C_2 are reduced curves on a compact surface S, and we let $C = C_1 \cup C_2$, then the following are equivalent:

- The natural inclusion $\mathrm{Der}_S(-\log C_1) + \mathrm{Der}_S(-\log C_2) \hookrightarrow \mathrm{Der}_S$ is an equality.
- The natural inclusion $\Omega^1_S(\log C_1) + \Omega^1_S(\log C_2) \hookrightarrow \Omega^1_S(\log C)$ is an equality.
- C_1 and C_2 are splayed.
- C_1 and C_2 meet transversally at nonsingular points.

One more item will be added to this list in §3:

- $c_{\mathrm{SM}}(S \smallsetminus C_1) \cdot c_{\mathrm{SM}}(S \smallsetminus C_2) = c(TS) \cap c_{\mathrm{SM}}(S \smallsetminus C) \quad ,$

see Example 3.6. This will use *Chern-Schwartz-MacPherson* classes; Chern classes are our next concern. ⌟

In the next sections we will be interested in the behavior of splayedness *vis-a-vis* a conjectural statement on Chern classes of sheaves of logarithmic derivations. Recall ([Ful84], §15.1 and B.8.3) that on nonsingular varieties one can define Chern classes for any coherent sheaf, compatibly with the splitting principle: the key fact is that every coherent sheaf on a nonsingular variety admits a finite resolution by locally free sheaves. Thus, for any hypersurface D on a nonsingular variety V we may consider the class

$$c(\mathrm{Der}_V(-\log D))$$

in the Chow ring of V, or its counterpart $c(\mathrm{Der}_V(-\log D)) \cap [V]$ in the Chow group of V. (The reader will lose very little by considering these classes in the cohomology, resp. homology of V.)

In terms of Chern classes, Theorem 2.18 has the following immediate consequence:

Corollary 2.20. *Let D_1, D_2 be reduced divisors of V, without common components, and let $D = D_1 \cup D_2$. If D_1 and D_2 are splayed, then*

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(\mathrm{Der}_V(-\log D_1)) \cdot c(\mathrm{Der}_V(-\log D_2))}{c(\mathrm{Der}\, V)}$$

in the Chow ring of V.

Remark 2.21. This Chern class statement only uses the 'easy' implication in the criterion for splayedness of Theorem 2.18. It does not seem likely that splayedness can be precisely detected by a Chern class computation; of course, Corollary 2.20 may be used to prove that two divisors are *not* splayed. ⌟

Proof. If D_1 and D_2 are splayed, then by Theorem 2.18 we have an isomorphism

$$\frac{\mathrm{Der}_V}{\mathrm{Der}_V(-\log D)} \cong \frac{\mathrm{Der}_V}{\mathrm{Der}_V(-\log D_1)} \oplus \frac{\mathrm{Der}_V}{\mathrm{Der}_V(-\log D_2)}$$

of coherent sheaves, and taking Chern classes we get

$$\frac{c(\mathrm{Der}_V)}{c(\mathrm{Der}_V(-\log D))} = \frac{c(\mathrm{Der}_V)}{c(\mathrm{Der}_V(-\log D_1))} \cdot \frac{c(\mathrm{Der}_V)}{c(\mathrm{Der}_V(-\log D_2))}$$

in the Chow ring of V. The stated equality follows at once. □

Example 2.22. As an illustration of Corollary 2.20, consider a divisor D with normal crossings and nonsingular components D_i. First we note that by sequence (3), if $D = D_1$ is nonsingular, then

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(\mathrm{Der}_V)}{1+D} \quad ,$$

where $1+D_1$ is the common notation for $c(\mathscr{O}_V(D)) = 1 + c_1(\mathscr{O}_V(D))$. Indeed, $JD = \emptyset$ as D is nonsingular, so $\mathscr{I}_{JD,D}(D) = \mathscr{O}_D(D)$, and twisting the standard exact sequence for $\mathscr{O}_D$ gives the sequence

$$0 \longrightarrow \mathscr{O}_V \longrightarrow \mathscr{O}_V(D) \longrightarrow \mathscr{O}_D(D) \longrightarrow 0 \quad ,$$

showing that $c(\mathscr{O}_D(D)) = c(\mathscr{O}_V(D))/c(\mathscr{O}_V) = 1 + D$.

Now the claim is that if $D = D_1 \cup \cdots \cup D_r$ is a divisor with normal crossings and nonsingular components, then

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(\mathrm{Der}_V)}{(1+D_1)\cdots(1+D_r)} \quad .$$

This formula is in fact well-known: it may be obtained by computing explicitly the ideal of the singularity subscheme JD and taking Chern classes of the corresponding sequence (3). The point we want to make is that this formula follows immediately from Corollary 2.20, without any explicit computation of ideals. Indeed, by the normal crossings condition, $D_1 \cup \cdots \cup D_{i-1}$ and D_i are splayed for all $i > 1$; the formula holds for $r = 1$ by the explicit computation given above; and for $r > 1$ and induction, we have

$$c(\mathrm{Der}_V(-\log D)) = \frac{c(\mathrm{Der}_V)}{(1+D_1)\cdots(1+D_{r-1})}\cdot\frac{c(\mathrm{Der}_V)}{(1+D_r)}\Big/ c(\mathrm{Der}_V)$$
$$= \frac{c(\mathrm{Der}_V)}{(1+D_1)\cdots(1+D_r)}$$

as claimed. ⌟

3. Chern-Schwartz-MacPherson classes for splayed divisors and subvarieties

3.1. **CSM classes.** There is a theory of Chern classes for possibly singular, possibly noncomplete varieties, normalized so that the class for a nonsingular compact variety V equals the total Chern class $c(TV)\cap[V]$ of the tangent bundle of V, in the Chow group (or homology), and satisfying a strict functoriality requirement. This theory was developed by R. MacPherson ([Mac74]); §19.1.7 in [Ful84] contains an efficient summary of MacPherson's definition. These 'Chern classes' were found to agree with a notion defined in remarkable earlier work by M.-H. Schwartz ([Sch65a, Sch65b]) aimed at extending the Poincaré-Hopf theorem to singular varieties; they are usually called *Chern-Schwartz-MacPherson* (c_{SM}) classes. A c_{SM} class $c_{\mathrm{SM}}(\varphi)$ is defined for every constructible function φ on a variety; the key functoriality of these classes prescribes that if $f: V \to W$ is a proper morphism, and φ is a constructible function on V, then $f_* c_{\mathrm{SM}}(\varphi) = c_{\mathrm{SM}}(f_*\varphi)$. Here, $f_*\varphi$ is defined by taking topological Euler characteristics of fibers. This covariance property also determines the theory uniquely by resolution of singularity and the normalization property mentioned above.

We mention here two immediate consequences of functoriality that are useful in computations. A locally closed subset U of a variety V determines a c_{SM} class $c_{\mathrm{SM}}(U)$ in the Chow group of V: this is the c_{SM} class of the function $1\!\!1_U$ which takes the value 1 on U and 0 on its complement.

- If V is compact and $U \subseteq V$ is locally closed, then the degree $\int c_{\mathrm{SM}}(U)$ equals the topological Euler characteristic of U. Thus, c_{SM} classes satisfy a generalized version of the Poincaré-Hopf theorem, and may be viewed as a direct generalization of the topological Euler characteristic.
- Like the Euler characteristic, c_{SM} classes satisfy an inclusion-exclusion principle: if U_1, U_2 are locally closed in V, then

$$c_{\mathrm{SM}}(U_1 \cup U_2) = c_{\mathrm{SM}}(U_1) + c_{\mathrm{SM}}(U_2) - c_{\mathrm{SM}}(U_1 \cap U_2) \quad .$$

Chern classes of bundles of logarithmic derivations along a divisor with simple normal crossings may be used to provide a definition of c_{SM} classes. This approach is adopted in [Alu06b], [Alu06a]; a short summary may be found in §3.1 of [AM09]. In this section we explore the role of splayedness in more refined (and still conjectural in part) relations between c_{SM} classes and Chern classes of sheaves of logarithmic derivations.

3.2. **CSM classes of hypersurface complements.** We now consider $c_{\rm SM}$ classes of hypersurface complements. As in §2 we will assume that V is a nonsingular complex projective variety. In previous work, the first-named author has proposed a formula relating the $c_{\rm SM}$ class of the complement $U = V \smallsetminus D$ of a divisor in a nonsingular variety V with the Chern class $c(\mathrm{Der}_V(-\log D)) \cap [V]$ of the corresponding sheaf of logarithmic derivations.

Example 3.1. Using the inclusion-exclusion formula for $c_{\rm SM}$ classes given above, it is straightforward to compute the $c_{\rm SM}$ class of the complement of a divisor with simple normal crossings $D = D_1 \cup \cdots \cup D_r$ (cf. e.g., Theorem 1 in [Alu99] or Proposition 15.3 in [GP02]), and verify that in this case

$$c_{\rm SM}(V \smallsetminus D) = c(\mathrm{Der}(-\log D)) \cap [V] \tag{5}$$

by direct comparison with the class computed in Example 2.22. ⌟

It is natural to inquire whether equality (5) holds for less special divisors. It has been verified for free hyperplane arrangements ([Alu12b]) and more generally for free hypersurface arrangements that are locally analytically isomorphic to hyperplane arrangements ([Alu12a]). Xia Liao has verified it for locally quasi-homogeneous curves on a nonsingular surface ([Lia12]), and he has recently proved that the formula holds 'numerically' for all free and locally quasi-homogeneous hypersurfaces of projective varieties ([Lia]).

On the other hand, the formula is known *not* to hold in general: for example, Liao proves that the formula does not hold for curves on surfaces with singularities at which the Milnor and Tyurina numbers do not coincide.

3.3. **Enter splayedness.** Rather than focusing on the verification of (5) for cases not already covered by these results, we aim here to consider a consequence of (5) in situations where it does hold. In Corollary 2.20 we have verified that if D_1 and D_2 are splayed, then there is a simple relation between the Chern classes of the sheaves of logarithmic derivations determined by D_1, D_2, and $D = D_1 \cup D_2$. If we assume for a moment that (5) holds for these hypersurfaces, we obtain a non-trivial relation between the $c_{\rm SM}$ classes of the corresponding complements. This relation can then be probed with independent tools, and cases in which it is found to hold may be viewed as a consistency check for a general principle linking $c_{\rm SM}$ classes of hypersurface complements and Chern classes of logarithmic derivations, of which (5) is a manifestation.

Proposition 3.2. *Let D_1, D_2 be reduced divisors of V, without common components, and let $D = D_1 \cup D_2$. Assume that D_1 and D_2 are splayed, and that* (5) *holds for D_1, D_2, and D. Let U, resp. U_1, U_2 be the complement of D, resp. D_1, D_2. Then*

$$c(TV) \cap c_{SM}(U) = c_{SM}(U_1) \cdot c_{SM}(U_2) \tag{6}$$

in the Chow group of V.

Proof. This is a direct consequence of Corollary 2.20, under the assumption that (5) holds for all hypersurfaces, and noting that Der_V is the sheaf of sections of the tangent bundle. □

In the rest of this section we are going to verify (6) in a few template situations, independently of the conjectural formula (5). In fact, it can be shown that (6) *holds for all splayed divisors.* However, the proof of this fact is rather technical, and we hope to return to it in later work. The situations we will consider in this paper can be appreciated with a minimum of machinery.

3.4. **Products.** Formula (6) can in fact be stated for splayed *subvarieties,* or in fact arbitrary closed subsets, rather than just divisors. Let V_1, V_2 be nonsingular varieties, let $X_1 \subseteq V_1$, $X_2 \subseteq V_2$ be closed subsets; then $D_1 = X_1 \times V_2$ and $D_2 = V_1 \times X_2$ may be considered to be splayed at all points of the intersection $X_1 \times X_2$. In general, two closed subsets D_1, D_2 of a nonsingular variety V are splayed at $p \in D_1 \cap D_2$ if locally analytically at p, D_1, D_2 and V admit the product structure detailed above.

If this local analytic description holds globally, then (6) is a straightforward consequence of known properties of c_{SM} classes.

Proposition 3.3. *Formula* (6) *holds for* $D_1 = X_1 \times V_2$ *and* $D_2 = V_1 \times X_2$ *in* $V = V_1 \times V_2$.

Proof. Let $U_1 = V \smallsetminus D_1$, $U_2 = V \smallsetminus D_2$, $U = V \smallsetminus D$. In the special situation of this proposition,

$$\begin{aligned} U &= (V \smallsetminus (D_1 \cup D_2)) = (V \smallsetminus D_1) \cap (V \smallsetminus D_2) = ((V_1 \smallsetminus X_1) \times V_2) \cap (V_1 \times (V_2 \smallsetminus X_2)) \\ &= (V_1 \smallsetminus X_1) \times (V_2 \smallsetminus X_2) \end{aligned}$$

Now we invoke a product formula for c_{SM} classes ([Kwi92], [Alu06a]): by Théorème 4.1 in [Alu06a],

$$c_{\mathrm{SM}}((V_1 \smallsetminus X_1) \times (V_2 \smallsetminus X_2)) = c_{\mathrm{SM}}(V_1 \smallsetminus X_1) \otimes c_{\mathrm{SM}}(V_2 \smallsetminus X_2) \quad ,$$

where $\otimes$ denotes the natural morphism $A_*(V_1) \otimes A_*(V_2) \to A_*(V_1 \times V_2) = A_*(V)$ sending $\alpha_1 \otimes \alpha_2$ to $(\pi_1^*\alpha_1) \cdot (\pi_2^*\alpha_2)$, where π_1, resp. π_2 is the projection from $V_1 \times V_2$ to the first, resp. second factor. As $c(TV) = \pi_1^* c(TV_1) \cap \pi_2^* c(TV_2)$,

$$c(TV) \cap c_{\mathrm{SM}}(U) = (\pi_2^* c(TV_2) \cap \pi_1^* c_{\mathrm{SM}}(V_1 \smallsetminus X_1)) \cdot (\pi_1^* c(TV_1) \cap \pi_1^* c_{\mathrm{SM}}(V_2 \smallsetminus X_2)) \quad .$$

Finally we note that by Theorem 2.2 in [Yok99]

$$\pi_2^* c(TV_2) \cap \pi_1^* c_{\mathrm{SM}}(V_1 \smallsetminus X_1) = c_{\mathrm{SM}}((V_1 \smallsetminus X_1) \times V_2) = c_{\mathrm{SM}}(V \smallsetminus D_1)$$

and similarly for the other factor, concluding the proof. □

Remark 3.4. The fact that (6) generalizes to complements of more general closed subsets is not surprising, as it is a formal consequence of the formulas for complements of divisors. ⌟

Remark 3.5. It is natural to expect that one could now deduce the validity of (6) for arbitrary splayed divisors from Proposition 3.3 and some mechanism obtaining intersection-theoretic identities from local analytic data. Jörg Schürmann informs us that his Verdier-Riemann-Roch theorem ([Sch]) can be used for this purpose; our proof of (6) for all splayed divisors (presented elsewhere) relies on different tools. ⌟

3.5. **Joins.** The classical construction of 'joins' in projective space (see e.g., [Har92]) gives another class of examples of splayed divisors and subvarieties for which (6) can be reduced easily to known results. We deal directly with the case of subvarieties, cf. Remark 3.4. We recall that the *join* of two disjoint subvarieties of projective space is the union of the lines incident to both. For example, the ordinary cone with vertex a point p and directrix a subvariety X is the join $J(p, X)$.

Let $X_1 \subseteq \mathbb{P}^{m-1}$, $X_2 \subseteq \mathbb{P}^{n-1}$ be nonempty subvarieties, and view $\mathbb{P}^{m-1}$, $\mathbb{P}^{n-1}$ as disjoint subspaces of $V = \mathbb{P}^{m+n-1}$. Let $D_1 = J(X_1, \mathbb{P}^n)$, resp. $D_2 = J(\mathbb{P}^m, X_2)$ be the corresponding joins. The intersection $J(X_1, X_2)$ of D_1 and D_2 is the union of the set of lines in $\mathbb{P}^{m+n-1}$ connecting points of X_1 to points of X_2. The subsets D_1, D_2 are evidently splayed along their intersection $J(X_1, X_2)$; but note that V is not a product, so Proposition 3.3 does not apply in this case.

Proposition 3.6. *Formula* (6) *holds for* D_1*,* D_2*,* $V = \mathbb{P}^{m+n-1}$ *as above.*

Proof. We have to compare

$$\begin{aligned} c_{\mathrm{SM}}(V \smallsetminus D_1) \cdot c_{\mathrm{SM}}(V \smallsetminus D_2) &= (c_{\mathrm{SM}}(V) - c_{\mathrm{SM}}(D_1)) \cdot (c_{\mathrm{SM}}(V) - c_{\mathrm{SM}}(D_2)) \\ &= c_{\mathrm{SM}}(V) \cdot c_{\mathrm{SM}}(V) - c_{\mathrm{SM}}(V) \cdot (c_{\mathrm{SM}}(D_1) + c_{\mathrm{SM}}(D_2)) + c_{\mathrm{SM}}(D_1) \cdot c_{\mathrm{SM}}(D_2) \end{aligned}$$

and

$$\begin{aligned} c(TV) \cap c_{\mathrm{SM}}(V \smallsetminus (D_1 \cup D_2)) &= c(TV) \cap (c_{\mathrm{SM}}(V) - (c_{\mathrm{SM}}(D_1) + c_{\mathrm{SM}}(D_2)) + c_{\mathrm{SM}}(D_1 \cap D_2)) \\ &= c(TV) \cap c_{\mathrm{SM}}(V) - c(TV) \cap (c_{\mathrm{SM}}(D_1) + c_{\mathrm{SM}}(D_2)) + c(TV) \cap c_{\mathrm{SM}}(D_1 \cap D_2)\,. \end{aligned}$$

By the basic normalization of c_{SM} classes (cf. §3.1), capping with $c(TV)$ is the same as taking the intersection product with $c_{\mathrm{SM}}(V)$. Since $D_1 \cap D_2 = J(X_1, X_2)$, we are reduced to verifying that

$$c_{\mathrm{SM}}(D_1) \cdot c_{\mathrm{SM}}(D_2) = c(TV) \cap c_{\mathrm{SM}}(J(X_1 \cap X_2)) \quad .$$

The classes $c_{\mathrm{SM}}(X_1) \in A_*\mathbb{P}^{m-1}$, resp. $c_{\mathrm{SM}}(X_2) \in A_*\mathbb{P}^{n-1}$ may be written as polynomials α, resp. β of degree $< m$, resp. n in the hyperplane class in these subspaces. Denoting by H the hyperplane class in $V = \mathbb{P}^{m+n-1}$, we obtain formulas for $c_{\mathrm{SM}}(D_1)$, $c_{\mathrm{SM}}(D_2)$ in A_*V by applying Example 6.1 in [AM11] (and noting that $H^{n+m} = 0$ in $A_*\mathbb{P}^{m+n-1}$):

$$\begin{aligned} c_{\mathrm{SM}}(D_1) &= (1+H)^n(\alpha(H) + H^m) \cap [\mathbb{P}^{m+n-1}] \\ c_{\mathrm{SM}}(D_2) &= (1+H)^m(\beta(H) + H^n) \cap [\mathbb{P}^{m+n-1}] \quad . \end{aligned}$$

Therefore

$$\begin{aligned} c_{\mathrm{SM}}(D_1) \cdot c_{\mathrm{SM}}(D_2) &= (1+H)^{m+n}(\alpha(H) + H^m) \cdot (\beta(H) + H^{m+n}) \cap [\mathbb{P}^{m+n-1}] \\ &= c(T\mathbb{P}^{m+n-1}) \cap \big((\alpha(H) + H^m) \cdot (\beta(H) + H^{m+n}) \cap [\mathbb{P}^{m+n-1}]\big) \end{aligned}$$

This equals $c(TV) \cap c_{\mathrm{SM}}(J(X_1, X_2))$ by Theorem 3.13 in [AM11], again noting that $H^{m+n} = 0$ in $A_*\mathbb{P}^{m+n-1}$. ☐

3.6. **Splayed curves.** Finally, we deal with the case of splayed curves on surfaces. In this case, (6) is a *characterization* of splayedness (cf. Example 2.19).

Let C_1 and C_2 be reduced curves on a nonsingular compact surface S, and let $C = C_1 \cup C_2$.

Proposition 3.7. *C_1 and C_2 are splayed on S if and only if* (6) *holds, that is,* $c(TS) \cap c_{SM}(S \smallsetminus C) = c_{SM}(S \smallsetminus C_1) \cdot c_{SM}(S \smallsetminus C_2)$.

Proof. We have
$$c_{\mathrm{SM}}(S \smallsetminus C_i) = c(TS) \cap [S] - [C_i] - \chi_i$$
for $i = 1, 2$, where χ_i is a class in dimension 0 (whose degree is the topological Euler characteristic of C_i). By inclusion-exclusion,
$$\begin{aligned} c_{\mathrm{SM}}(C) &= c_{\mathrm{SM}}(C_1) + c_{\mathrm{SM}}(C_2) - c_{\mathrm{SM}}(C_1 \cap C_2) \\ &= [C_1] + [C_2] + \chi_1 + \chi_2 - [C_1 \cap C_2] \quad , \end{aligned}$$
and hence
$$c_{\mathrm{SM}}(S \smallsetminus C) = c(TS) \cap [S] - [C_1] - [C_2] - \chi_1 - \chi_2 + [C_1 \cap C_2] \quad .$$
It then follows at once that
$$c_{\mathrm{SM}}(S \smallsetminus C_1) \cdot c_{\mathrm{SM}}(S \smallsetminus C_2) - c(TS) \cap c_{\mathrm{SM}}(S \smallsetminus C) = C_1 \cdot C_2 - [C_1 \cap C_2] \quad .$$
Therefore, in this case (6) is verified if and only if $C_1 \cdot C_2 = [C_1 \cap C_2]$, that is, if and only if C_1 and C_2 meet transversally at nonsingular points. As we have recalled in §2, this condition is equivalent to the requirement that C_1 and C_2 are splayed, verifying (6) in this case and proving that this identity *characterizes* splayedness for curves on surfaces. □

Remark 3.8. In the proof of both Propositions 3.6 and 3.7 we have used the fact that (6) is equivalent to the identity

(7) $$c_{\mathrm{SM}}(D_1) \cdot c_{\mathrm{SM}}(D_2) = c(TV) \cap c_{\mathrm{SM}}(D_1 \cap D_2) \quad .$$

If D_1 and D_2 are nonsingular subvarieties of V intersecting properly and transversally (so that $D_1 \cap D_2$ is nonsingular, of the expected dimension), then (7) is precisely the expected relation between the Chern classes of the tangent bundles of D_1, D_2, and $D_1 \cap D_2$. The results of the previous sections verify (7) for several classes of splayed subvarieties (nonsingular or otherwise), and we will prove elsewhere that in fact (7) holds in general for the intersection of two splayed subvarieties. This reinforces the point of view taken by the second author in [Fab], to the effect that splayedness is an appropriate generalization of transversality for possibly singular varieties. For another situation in which the formula holds when one (but not both) of the hypersurfaces is allowed to be singular, see Theorem 3.1 in [Alu]. ⌟

References

[Alu] Paolo Aluffi. Chern classes of graph hypersurfaces and deletion-contraction. arXiv:1106.1447.

[Alu99] Paolo Aluffi. Differential forms with logarithmic poles and Chern-Schwartz-MacPherson classes of singular varieties. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(7):619–624, 1999.

[Alu06a] Paolo Aluffi. Classes de Chern des variétés singulières, revisitées. *C. R. Math. Acad. Sci. Paris*, 342(6):405–410, 2006.
[Alu06b] Paolo Aluffi. Limits of Chow groups, and a new construction of Chern-Schwartz-MacPherson classes. *Pure Appl. Math. Q.*, 2(4):915–941, 2006.
[Alu12a] Paolo Aluffi. Chern classes of free hypersurface arrangements. *J. Singul.*, 5:19–32, 2012.
[Alu12b] Paolo Aluffi. Grothendieck classes and Chern classes of hyperplane arrangements. *Int. Math. Res. Not.*, 2012.
[AM09] Paolo Aluffi and Leonardo Constantin Mihalcea. Chern classes of Schubert cells and varieties. *J. Algebraic Geom.*, 18(1):63–100, 2009.
[AM11] Paolo Aluffi and Matilde Marcolli. Algebro-geometric Feynman rules. *Int. J. Geom. Methods Mod. Phys.*, 8(1):203–237, 2011.
[Dol07] I. V. Dolgachev. Logarithmic sheaves attached to arrangements of hyperplanes. *J. Math. Kyoto Univ.*, 47(1):35–64, 2007.
[Fab] E. Faber. Towards transversality of singular varieties: splayed divisors. arXiv:1201.2186.
[Ful84] William Fulton. *Intersection theory.* Springer-Verlag, Berlin, 1984.
[GP02] Mark Goresky and William Pardon. Chern classes of automorphic vector bundles. *Invent. Math.*, 147(3):561–612, 2002.
[GS] Michel Granger and Mathias Schulze. Dual logarithmic residues and free complete intersections. arXiv:1109.2612.
[Har80] Robin Hartshorne. Stable reflexive sheaves. *Math. Ann.*, 254(2):121–176, 1980.
[Har92] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1992. A first course.
[HM93] Herwig Hauser and Gerd Müller. Affine varieties and Lie algebras of vector fields. *Manuscripta Math.*, 80(3):309–337, 1993.
[Kwi92] Michał Kwieciński. Formule du produit pour les classes caractéristiques de Chern-Schwartz-MacPherson et homologie d'intersection. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(8):625–628, 1992.
[Lia] Xia Liao. Chern classes of logarithmic vector fields for locally-homogenous free divisors. arXiv:1205.3843.
[Lia12] Xia Liao. Chern classes of logarithmic vector fields. *J. Singul.*, 5:109–114, 2012.
[Mac74] R. D. MacPherson. Chern classes for singular algebraic varieties. *Ann. of Math. (2)*, 100:423–432, 1974.
[OT92] Peter Orlik and Hiroaki Terao. *Arrangements of hyperplanes*, volume 300 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
[Sai80] Kyoji Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.
[Sch] Jörg Schürmann. A generalized Verdier-type Riemann-Roch theorem for Chern-Schwartz-MacPherson classes. arXiv:math/0202175.
[Sch65a] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. I. *C. R. Acad. Sci. Paris*, 260:3262–3264, 1965.
[Sch65b] Marie-Hélène Schwartz. Classes caractéristiques définies par une stratification d'une variété analytique complexe. II. *C. R. Acad. Sci. Paris*, 260:3535–3537, 1965.
[Yok99] Shoji Yokura. On a Verdier-type Riemann-Roch for Chern-Schwartz-MacPherson class. *Topology Appl.*, 94(1-3):315–327, 1999. Special issue in memory of B. J. Ball.

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