On the K-category of 3-manifolds for K a wedge of spheres or projective planes

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July 27, 2012

Abstract

For a complex K, a closed 3-manifold M is of K–category $\leq m$, if it can be covered by m open subsets $W_1, \ldots, W_m$ such that the inclusions $W_i \to M^n$ factor homotopically through maps $W_i \stackrel{f_i}{\to} K \stackrel{\alpha_i}{\to} M$. We compute the K-category of closed 3-manifolds M when K is a wedge of 2-spheres and obtain some results for the K-category of M when K is a wedge of two projective planes. [1] [2]

1 Introduction

The classical Lusternik-Schnirelman category $cat(X)$ of a space X is the smallest number k such that there is an open cover $W_1, \ldots, W_k$ of X with each W_i contractible in X. (If no such finite cover exists, $cat(X) = \infty$). We

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[1]**AMS classification numbers: 57N10, 55M30, 57M27, 57N16**

[2]**Key words and phrases: Lusternik-Schnirelmann category, coverings of 3-manifolds with open K-contractible subsets**

are interested in the case when $X = M$, a manifold. We assume that manifolds are connected, but the W_i's need not be connected. For M^3 a closed 3-manifold, $2 \leq cat(M^3) \leq 4$, and it has been shown in [4] that $cat(M^3) = 2$, resp. 3, if and only if $\pi_1(M^3) = 1$, resp. free non-trivial. If the condition that W_i is contractible is replaced by the weaker condition that for every basepoint $* \in W_i$ the inclusion $\iota : W_i \to X$ induces the trivial homomorphism $\iota_* : \pi_1(W_i, *) \to \pi_1(X, *)$, then W_i is said to be π_1-contractible (in X) and $cat_{\pi_1}(X)$ is the smallest number k such that there is an open cover of X by k π_1- contractible subsets. This invariant was defined by Fox [3] and it is also shown in [4] that $cat_{\pi_1}(M^3) = 1$ (resp. 2, resp. 4), if and only if $\pi_1(M^3) = 1$ (resp. free non-trivial, resp. non-free). Thus by Perelman,

$$cat_{\pi_1}(M^3) = \begin{cases} 1 & \text{if and only if } M \text{ is the 3-sphere} \\ 2 & \text{if and only if } M \text{ is a connected sum of } S^2\text{-bundles over } S^1 \\ 4 & \text{otherwise.} \end{cases}$$

M. Clapp and D. Puppe ([2]) generalized the notion of $cat(X)$ as follows: Let $\mathcal{K}$ be a non-empty class of spaces. A subset W of X is *$\mathcal{K}$-contractible* (in X) if the inclusion $\iota : W \to X$ factors homotopically through some $K \in \mathcal{K}$, i.e. there exist maps $f : W \to K$, $\alpha : K \to X$, such that ι is homotopic to $\alpha \cdot f$. The $\mathcal{K}$-category $cat_{\mathcal{K}}(X)$ of X is the smallest number of open $\mathcal{K}$-contractible subsets of X that cover X. For closed n-manifolds, $1 \leq cat_{\mathcal{K}}(M) \leq cat(M) \leq n+1$. When the family $\mathcal{K}$ contains just one space K, one writes $cat_K(M)$ instead of $cat_{\mathcal{K}}(M)$. In particular, if K is a single point, then $cat_K(M) = cat(M)$. The cases for $K = S^1$, $K = S^2$ and $K = P^2$ (the projective plane) have been considered in [5] and [6].

There is also a generalization of the notion of $cat_{\pi_1}(M)$: For a nonempty class of groups $\mathcal{G}$ we say that a subset W of M is *$\mathcal{G}$-contractible* if, for every basepoint $* \in W$, the image $\iota_*(\pi(W, *) \subset \pi(M, *)$ belongs to $\mathcal{G}$ and we let $cat_{\mathcal{G}}(M)$ be the smallest number of open $\mathcal{G}$-contractible subsets of M that cover M. It turns out that, if $\mathcal{G}$ is closed under subgroups and quotients, then $cat_{\mathcal{G}}(M) = cat_{\mathcal{A}_{\mathcal{G}}}(M)$, where $\mathcal{A}_{\mathcal{G}} = \{X : \pi(X, *) \in \mathcal{G}, \text{ for all } * \in X\}$ (Proposition 1 of [8]). For closed 3-manifolds, the cases for $\mathcal{G} = ame$ (the class of amenable groups) and $\mathcal{G} = solv$ (the class of solvable groups), have been considered in [7] and [8].

There do not seem to be any known results for $cat_K(M^3)$ when K is a compact 2-complex that is not homotopy equivalent to a closed 2-manifold.

In view of the results for $K =$ point, S^2, P^2, one may consider for example complexes K that are wedges of these 2-manifolds. As a first result in this direction we obtain in Theorem 1 a complete classification of $cat_K(M^3)$ for $K = \bigvee_n S^2$, the wedge of n copies of S^2 (for $n > 0$).

The case when $K = \bigvee_n P^2$, the wedge of n copies of P^2, seems to be difficult, especially for $n \geq 3$, since then $\pi_1(K)$ is not a solvable group. The closed 3-manifolds M for which $cat_{P^2}(M) = 2$ are S^3, P^3, $P^3\#P^3$, and $P^2 \times S^1$ ([6]), and it is easy to see that if M_1, M_2 is any manifold in this list and $M = M_1\#M_2$, then $cat_{P^2 \vee P^2}(M) = 2$. We conjecture that these are the only closed 3-manifolds M for which $cat_{P^2 \vee P^2}(M) = 2$.

2 K-contractible subsets.

Let $M = M^n$ be a closed connected n-manifold and let K be a finite CW-complex.

Recall that a subset W of M is K-*contractible* (in M) if there are maps $f : W \to K$ and $\alpha : K \to M$ such that the inclusion $\iota : W \to M$ is homotopic to $\alpha \cdot f$. The K-category $cat_K(M)$ is the smallest number m such that M can be covered by m open K-contractible subsets.

W is π_1-contractible (in M) if for every basepoint $* \in W$ the inclusion $\iota : W \to M$ induces the trivial homomorphism $\iota_* : \pi_1(W, *) \to \pi_1(M, *)$. The π_1-category $cat_{\pi_1}(M)$ is the smallest number k such that M can be covered by k open π_1- contractible subsets.

More generally, for a nonempty class of groups $\mathcal{G}$, a subset W of M is $\mathcal{G}$-*contractible* if, for every basepoint $* \in W$, the image $\iota_*(\pi(W, *) \subset \pi(M, *)$ belongs to $\mathcal{G}$ and $cat_{\mathcal{G}}(M)$ is the smallest number of open $\mathcal{G}$-contractible subsets of M that cover M.

It is easy to see that cat_K is a homotopy type invariant.

Note that a subset of a K-contractible set is K-contractible.

If $\mathcal{G}$ is closed under subgroups, then a subset of a $\mathcal{G}$-contractible set is $\mathcal{G}$-contractible.

Lemma 1. (a) Let M be a closed 3-manifold. If a 2-complex L is a retract of a 2-complex K then

$$2 \leq cat_K(M) \leq cat_L(M) \leq cat(M) \leq 4$$

(b) If furthermore K is simply connected then

$$1 \leq cat_{\pi_1}(M) \leq cat_K(M) \leq cat_L(M) \leq cat(M) \leq 4$$

Proof. (a) Clearly a (in M) contractible set is L-contractible. For an L-contractible set there are maps $f : W \to L$ and $\alpha : L \to M$ such that $\alpha \cdot f \simeq \iota$. Let $j : L \to K$ be inclusion and $r : K \to L$ be retraction. Then $(\alpha r) \cdot (jf) \simeq \iota$. Thus an L-contractible set is K-contractible. Since $id : H_3(M) \to H_3(M)$ does not factor through $H_3(K) = 0$, $2 \leq cat_K(M)$.

(b) If $\pi_1(K) = 1$, a K-contractible set is π_1-contractible. □

For the case when $\mathrm{cat}_K(M) = 2$ and $K = S^1$ it was shown in [7] that the two open K- contractible sets can be replaced by compact submanifolds that meet only along their boundaries. The same proof applies for any finite complex K to yield

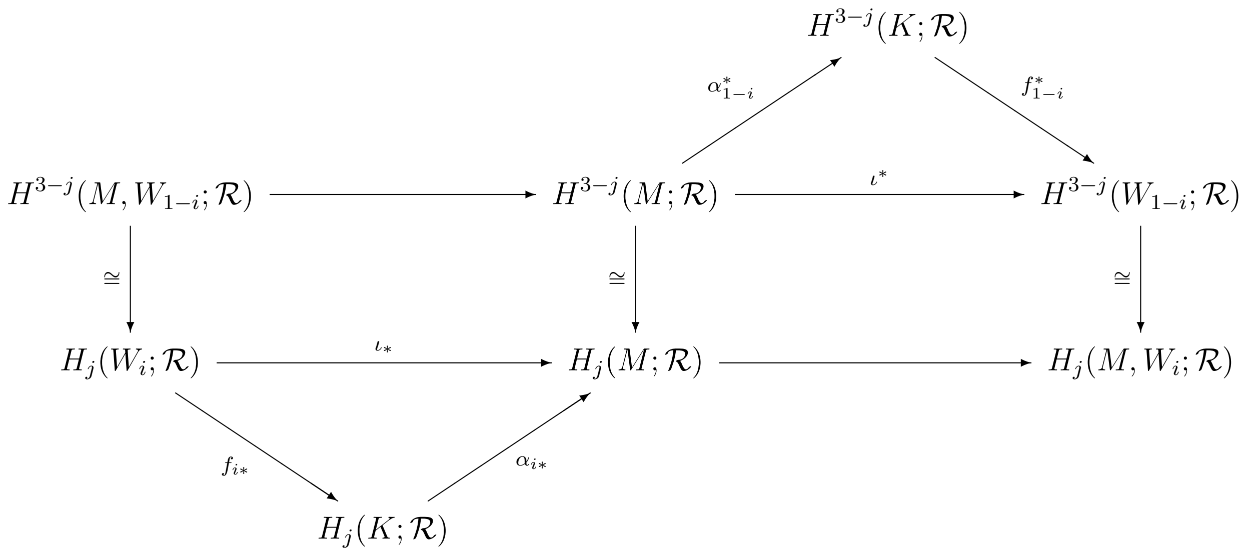
Proposition 1. *Suppose* $\mathrm{cat}_K M = 2$ *(resp.* $cat_{\mathcal{G}}(M) = 2$*). Then* M *can be expressed as a union of two compact* K*-contractible (resp.* $\mathcal{G}$*-contractible)* n*-submanifolds* W_0, W_1 *such that* $W_0 \cap W_1 = \partial W_0 = \partial W_1$.

In particular, if $\mathrm{cat}_K M = 2$, there are maps f_i and α_i (for $i = 0, 1$) such that the diagram below is homotopy commutative:

$$(*) \qquad \begin{array}{ccc} W_i & \xrightarrow{\ \iota\ } & M \\ & {\scriptstyle f_i}\searrow \quad \nearrow{\scriptstyle \alpha_i} & \\ & K & \end{array}$$

We now assume that M is a closed 3-manifold with $cat_K M = 2$ as in Proposition 1.

Let $\mathcal{R} = \mathbb{Z}$ or $\mathbb{Z}_2$ if M is orientable and $\mathbb{Z}_2$ if M is non-orientable. From the exact homology and cohomology sequences of (M, W_j), Lefschetz-Duality, and $(*)$ we obtain a commutative diagram (for $0 \leq j \leq 3$):



In particular we have an exact sequence

$$(**) \qquad 0 \to im\, \iota_* \to H_1(M;\mathcal{R}) \to im\, \iota^* \to 0$$

and obtain the following

Lemma 2. Suppose $cat_K(M) = 2$ and K is a 2-dimensional complex such that the $\mathbb{Z}_2$-rank $rk(H_1(K;\mathbb{Z}_2) \leq m$ and $rk(H^2(K;\mathbb{Z}_2)) \leq n$ $(m, n \geq 0)$. Then $rk(H_1(M;\mathbb{Z}_2) \leq m + n$.

Lemma 3. Suppose M is orientable with $cat_K(M) = 2$ and K is a 2-dimensional complex such that the rank $rk(H^1(K)) \leq m$ and $rk(H_2(K)) \leq n$ $(m, n \geq 0)$. Then $rk(H_2(M)) \leq m + n$.

3 $cat_K(M^3)$ for K a wedge of S^2's.

We use the following notation:

$\bigvee_m S^2$ denotes the wedge of m copies of S^2 (for $m > 0$).
$S^2 \tilde{\times} S^1$ denotes the trivial or non-trivial S^2-bundle over S^1.
$\#_m S^2 \tilde{\times} S^1$ denotes the connected sum of m S^2-bundles over S^1 (for $m > 0$).

Example 1. If $cat_K(M) = 2$, where $K = \bigvee_n S^2$, then $rk(H_1(M;\mathbb{Z}_2) \le n$.

This follows from the Lemma since $rk(H_1(K;\mathbb{Z}_2) = 0$ and $rk(H^2(K;\mathbb{Z}_2)) = n$.

Example 2. For $K = \bigvee_m S^2$ and $M = \#_m S^2 \tilde{\times} S^1$ we have $cat_K(M) = 2$.

For simplicity we only describe the case $m = 2$. Then $M = \hat{M}_1 \# \hat{M}_2$, where each $\hat{M}_i$ has a decomposition $S^2_i \times [-1,0] \cup S^2_i \times [0,1]$. Let M_i be obtained from $\hat{M}_i$ by deleting the interior of a small ball B_i such that ∂B_i is the union of two disks $D_{i,0}$, $D_{i,1}$ with $D_{i,0} \subset S^2_i \times [-1,0]$, $D_{i,1} \subset S^2_i \times [0,1]$. Then $M = M_1 \cup M_2 = W_1 \cup W_2$, where $W_1 = (S^2_1 \times [-1,0] \cup_{D_{1,0}=D_{2,0}} S^2_2 \times [-1,0])$, $W_2 = \cup(S^2_1 \times [0,1] \cup_{D_{1,1}=D_{2,1}} S^2_2 \times [0,1])$. Each W_i deformation retracts to $S^2 \vee S^2$ and so is $S^2 \vee S^2$-contractible.

Lemma 4. If $K_n = \bigvee_n S^2$, $M_m = \#_m S^2 \tilde{\times} S^1$, and $1 \le m \le n$, then $cat_{K_n}(M_m) = 2$.

Proof. Clearly $cat_{K_1}(M_1) = cat_{S^2}(S^2 \tilde{\times} S^1) = 2$. Since K_{n-1} is a retract of K_n, $cat_{K_n}(M_m) \le cat_{K_{n-1}}(M_m)$, so if $m \le n-1$ then $cat_{K_n}(M_m) \le 2$ by induction on n. If $m = n$ then $cat_{K_m}(M_m) = 2$ by Example 2. □

Corollary 1. $cat_{K_n}(M_m) = \begin{cases} 2 & \textit{for } 1 \le m \le n \\ 3 & \textit{for } m > n \end{cases}$

Proof. If $cat_{K_n}(M_m) = 2$, then from Example 1 it follows that $m = rk(H_1(M_m;\mathbb{Z}_2)) \le n$. On the other hand, $cat_{K_n}(M_m) \le cat(M_m) \le 3$ for all n,m, since M_m can be covered by three balls. Now the Corollary follows from Lemma 4. □

Theorem 1. *Let $K = \bigvee_n S^2$ and let M be a closed 3-manifold. Then*

$$cat_K(M) = \begin{cases} 2 & \textit{if } M = S^3 \textit{ or } M = \#_m S^2 \tilde{\times} S^1 \textit{ with } m \le n \\ 3 & \textit{if } M = \#_m S^2 \tilde{\times} S^1 \textit{ with } m > n \\ 4 & \textit{otherwise} \end{cases}$$

Proof. If $\pi_1(M)$ is not free then $4 = cat_{\pi_1}(M) \le cat_K(M) \le 4$ by [4] and Lemma 1. If $\pi_1(M)$ is free, then (by Perelman) M is S^3 or $M = M_m$. Now Corollary 1 applies. □

In particular if $n = 1$, we obtain as a special case:

Corollary 2. $cat_{S^2}(M) = \begin{cases} 2 & \textit{if } M = S^3 \textit{ or } M = S^2 \tilde{\times} S^1 \\ 3 & \textit{if } M = \#_m S^2 \tilde{\times} S^1 \textit{ with } m > 1 \\ 4 & \textit{otherwise} \end{cases}$

Note that the proof shows that in Theorem 1 the condition that $K = \bigvee_n S^2$ can be replaced by "K is a simply connected 2-complex with $rk(H^2(K;\mathbb{Z}_2)) = n \geq 1$ and K retracts onto K_n".

4 $Cat_{P^2 \vee P^2}(M) = 2$.

In this section let $K = P^2 \vee P^2$.

By $\#_m N^3$ we denote a connected sum of m copies of the 3-manifold N^3.

If W is K-contractible, then for every basepoint $* \in W$, the image $\iota_* \pi_1(W, *)$ is conjugate to a subgroup of $\alpha_* \pi_1(K, , f(*))$, which is 3-manifold group and a quotient of $\pi_1(K, f(*))$. Suppose Q is a quotient group of a group G. We say that Q is a *3-manifold quotient of G* if Q is the fundamental group of a 3-manifold. With this notation, a K-contractible set is $\mathcal{G}_K$-contractible, where $\mathcal{G}_K$ denotes the set of finitely generated subgroups of 3-manifold quotients Q of $\pi_1(K, *)$ (for any basepoint $* \in K$), and $1 \leq cat_{\mathcal{G}_K}(M) \leq cat_K(M)$.

Lemma 5. The set of finitely generated subgroups of 3-manifold quotients of $\mathbb{Z}_2 * \mathbb{Z}_2$ is $\mathcal{G}_K = \{1, \mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2 * \mathbb{Z}_2\}$.

Proof. Present $\mathbb{Z}_2 * \mathbb{Z}_2$ as the semi-direct product $G = \mathbb{Z} \rtimes \mathbb{Z}_2 = \langle r, s \,:\, s^2 = 1, srs^{-1} = r^{-1} \rangle$. Then every element of G can be uniquely written as r^m or $r^m s$, for some $m \in \mathbb{Z}$.

Let H be a finitely generated subgroup of G. If $H \subset \langle r \rangle$, then $H = 1$ or $\cong \mathbb{Z}$; if $H \cap \langle r \rangle = 1$, then $H = 1$ or $\cong \mathbb{Z}_2$; in any other case $H \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

Now let H be a subgroup of a proper quotient $Q \cong G/N$ of G. If $N \not\subset \langle r \rangle$, then $Q \cong \mathbb{Z}_2$, hence $H = 1$ or $\mathbb{Z}_2$. This is the only case that can happen: if $N = \langle r^m \rangle$ for some integer $m \neq 0$ then G/N is the finite dihedral group D_{2m} of order $2m$. In this case, if M^3 is a 3-manifold with finite fundamental group D_{2m} then M^3 can be taken to be compact ([9], Thm 8.1) and ∂M^3 is a union of 2-spheres, so M^3 can be assumed closed. Then by Milnor [10], every element of order 2 of $\pi_1(M^3)$ is central. However, for $m > 2$, this is not true for D_{2m}. If $m = 2$, then it is well known that D_{2m}, the four group, is not a 3-manifold group (see, for example, [9], Thm 9.13). □

The compact, connected 3-manifolds whose fundamental groups belong to $\mathcal{G}_K$ are well-known and are listed in the following

Proposition 2. *Let M be a compact, connected 3-manifold. Then $cat_{\mathcal{G}_K}(M) = 1$ if and only if*
$M \in \{S^3, S^2\tilde{\times}S^1, D^2\tilde{\times}S^1, P^3, P^2\times I, P^3\#P^3, P^2\times I\#P^2\times I, P^3\#P^2\times I\}$.

The following lemma was proved in [7] for a more general class $\mathcal{G}$ of groups.

Lemma 6. Let M be a closed 3-manifold with $\text{cat}_{\mathcal{G}_K}(M) \leq 2$. Then there is a closed surface F in M such that F and $\overline{M - N(F)}$ are $\mathcal{G}_K$-contractible and every component of F is a 2-sphere or incompressible.

Proof. We write $M = W_0 \cup W_1$ as in Proposition 1. For each component F' of F, $im(\pi(F') \to \pi(M))$ is contained in $im(\pi(W_i') \to \pi(M))$, where W_i' is a component of W_i, and it follows that F and $\overline{M - N(F)}$ are $\mathcal{G}_K$-contractible. Now assume that F is a closed surface in M of minimal complexity such that F and $\overline{M - N(F)}$ are $\mathcal{G}_K$-contractible.

If a non-sphere component F' of F is not incompressible , let D be a compressing disk for F'. Let $D \times I$ be a regular neighborhood such that $(D \times I) \cap F = \partial D \times I$ and $\partial D \times 0$ is an essential curve in F'. For the component F_1' of $F_1 = (F - \partial D \times I) \cup (D \times \partial I)$ that contains $D \times \{0\}$ or $D \times \{1\}$, $\text{im}(\pi(F_1') \to \pi(M))$ is a subgroup of $\text{im}(\pi(F') \to \pi(M))$. Since F' is $\mathcal{G}_K$-contractible, so is F_1.

Furthermore, if M' is the component of $\overline{M - N(F)}$ that contains F' but not D and if M_1' is the component $M' \cup D \times I$ of $\overline{M - N(F_1)}$, then $\pi(M')$ and $\pi(M_1')$ have the same image in $\pi(M)$. Since M' is $\mathcal{G}_K$-contractible, so is M_1'.

Hence F_1 and $\overline{M - N(F_1)}$ are $\mathcal{G}_K$-contractible and $c(F_1) < c(F)$, a contradiction. □

For a 3-manifold C we denote by $\hat{C}$ the manifold obtained by capping off al 2-sphere boundaries with 3-balls.

Theorem 2. *Let M be a closed 3-manifold. Then $cat_{\mathcal{G}_K}(M) \leq 2$ if and only if $M = S^3\#_m P^3\#_n(P^2\times S^1)\#_k(S^2\tilde{\times}S^1)$, for some $k, m, n \geq 0$.*

Proof. By Lemma 6, there is a closed surface F of minimal complexity in M such that F and $\overline{M - N(F)}$ are $\mathcal{G}_K$-contractible and every component F' of F is a 2-sphere or incompressible. It follows that F' and every component C of $\overline{M - F \times [0,1]}$ is π_1- injective (i.e. the inclusions into M induce injections of fundamental groups). Since F' and C are $\mathcal{G}_K$-contractible, F' and C have fundamental groups belonging to $\mathcal{G}_K$. In particular, by Lemma 5, F' is a 2-sphere or projective plane, and $\hat{C}$ is as in Proposition 2, except for $D^2\tilde{\times}S^1$, since ∂C is incompressible.

If $\hat{C}$ is is one of $P^3\#P^2\times I, P^2\times I\#P^2\times I$ or $P^3\#P^3$, let S_C be a 2-sphere splitting C into two manifolds homeomorphic to a punctured P^3 or a punctured $P^2\times I$. Now let Σ be the union of the 2-manifolds of the form S_C and the 2-sphere components of F, then split M along Σ and cap off the the 2-sphere boundary components with 3-balls. Every component of the resulting 3-manifold is homeomorphic to $S^2\tilde{\times}S^1$, P^3 or to $P^2\times S^1$. This shows that M is as in the Theorem.

Conversely, if M is of this type, let W_0 be a regular neighborhood of a disjoint collection of 2-spheres and projective planes such that every component of $W_1 := \overline{M - W_0}$ is a punctured $S^2\tilde{\times}S^1$, P^3, or $P^2\times I$. Then W_0, W_1 are $\mathcal{G}_K$-contractible (in fact $\{1, Z, Z_2\}$-contractible), and $M = W_0 \cup W_1$. □

Corollary 3. *Let M be a closed 3-manifold. If $cat_{P^2\vee P^2}(M) = 2$, then $M = S^3\#_m P^3\#_n(P^2\times S^1)\#_k(S^2\tilde{\times}S^1)$, with $0 \leq k + m + 2n \leq 4$.*

Proof. Since $1 \leq cat_{\mathcal{G}_K}(M) \leq cat_K(M)$, this follows from Theorem 2 and Lemma 2. □

If M_i is a closed 3-manifold ($i = 1, 2$) with $cat_{P^2}(M_i) = 2$, then (by Corollary 2 of [6], $M_i \in \{S^3, P^3, P^3\#P^3, P^2\times S^1\}$.

Thus Corollary 3 implies that if $cat_{P^2\vee P^2}(M) = 2$ and M has no $S^2\tilde{\times}S^2$-factors, then $M = M_1\#M_2$, where $cat_{P^2}(M_i) = 2$. The converse is given by the following

Proposition 3. *If M_i is a closed 3-manifold with $cat_{P^2}(M_i) = 2$ for $i = 1, 2$, then $cat_{P^2\vee P^2}(M_1\#M_2) = 2$.*

Proof. There is a decomposition $M_i = W_{i0} \cup W_{i1}$, with $W_{i0} \cap W_{i1} = \partial W_{i0} = \partial W_{i1}$, and where each $W_{ij} \in \{B^3, P^3_0, P^2\times I\}$ (here P^3_0 denotes the once-punctured P^3). For the connected sum $M = M_1\#M_2$ choose 3-balls $B_i \subset M_i$, so that $\partial B_i \cap W_{ij} = D_{ij}$, a disk. Then $M = W_0 \cup W_1$, with $W_j = W_{1j} \cup W_{2j}$ ($j = 0, 1$), where in the union the disks D_{1j} and D_{2j} are identified. Now $W_j \simeq W_{1j} \vee W_{2j}$ is homotopy equivalent to B^3, P^2, or $P^2 \vee P^2$, hence $P^2 \vee P^2$-contractible. □

If M is orientable with $cat_{P^2\vee P^2}(M) = 2$, then by Lemma 3, $rk(H_2(M)) = 0$, and therefore M does not contain non-separating 2-spheres. So we obtain the following

Corollary 4. *Let M be a closed orientable* 3*-manifold. Then* $cat_{P^2 \vee P^2}(M) = 2$*, if and only if* $M = S^3 \#_m P^3$*, with* $0 \leq m \leq 4$*.*

Finally we show that there can be at most one $S^2 \times S^1$-factor if M is non-orientable.

Lemma 7. Let $K = P^2 \vee P^2$ and let $M^3 = N^3 \#(S^2 \times S^1)\#(S^2 \times S^1)$, where N^3 is a closed non-orientable 3-manifold. Then $cat_K(M^3) > 2$.

Proof. Suppose $cat_K(M) \leq 2$ and let W_i be as in Proposition 1. Let $p : \tilde{M} \to M$ be the orientable 2-fold covering and $\tilde{W}_i = p^{-1}(W_i)$. Then, by Lemma 1 of [6], $\tilde{W}_i$ is $\tilde{K}_i$-contractible where $\tilde{K}_i$ is the pullback of $K \stackrel{\alpha_i}{\to} M$. Now $\tilde{K}_i$ is a (possibly disconnected) 2-fold covering of K and is homotopy equivalent to $S^2 \vee S^2 \vee S^1$, $P^2 \vee S^2 \vee P^2$ or two copies of $P^2 \vee P^2$, so $rk(H_1(\tilde{K}_i)) \leq 1$ and $rk(H^2(\tilde{K}_i)) \leq 2$. In the exact sequence (**) (before Lemma 2) $\iota_* : H_1(\tilde{W}_0) \to H_1(\tilde{M})$ factors through $H_1(\tilde{K}_0)$ and $\iota^* : H^2(\tilde{M}) \to H^2(\tilde{W}_1)$ factors through $H^2(\tilde{K}_1)$, hence $rk(H_1(\tilde{M})) \leq 1+2 = 3$. However $\tilde{M} = \tilde{N}_3 \#_4 (S^1 \times S^2)$, where $\tilde{N}_3$ is the orientable 2-fold cover of N_3, so $rk(H_1(\tilde{M})) \geq 4$, a contradiction. □

Now let $S^2 \tilde{\tilde{\times}} S^1$ denote the nontrivial S^2-bundle over S^1. Since $N^3 \#(S^2 \tilde{\tilde{\times}} S^1) = N^3 \#(S^2 \times S^1)$ for a non-orientable 3-manifold N^3, we obtain

Corollary 5. *Let M be a closed non-orientable* 3*-manifold. If* $cat_{P^2 \vee P^2}(M) = 2$*, then*

$$M = \begin{cases} (P^2 \times S^1)\#(P^2 \times S^1) & \textit{or} \\ \#_m P^3 \#(P^2 \times S^1)\#_k (S^2 \times S^1) & \textit{with } 0 \leq m+k \leq 3,\ 0 \leq k \leq 1, \textit{ or} \\ \#_m P^3 \#(S^2 \tilde{\tilde{\times}} S^1)\#_k (S^2 \times S^1) & \textit{with } 0 \leq m+k \leq 3,\ 0 \leq k \leq 1. \end{cases}$$

We conjecture that $k = 0$; in fact that there are no $(S^2 \tilde{\times} S^1)$-factors (and the last case does not occur).

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