

# A Riemannian Approach for Computing Geodesics in Elastic Shape Analysis\*

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**Abstract**—In the framework of elastic shape analysis, a shape is invariant to scaling, translation, rotation and reparameterization. Since this framework does not yield a closed form of geodesic between two shapes, iterative methods have been proposed. In particular, path straightening methods have been proposed and used for computing a geodesic that is invariant to curve scaling and translation. Path straightening can then be exploited within a coordinate-descent algorithm that computes the best rotation and reparameterization of the end point curves. In this paper, we propose a Riemannian quasi-Newton method to compute a geodesic invariant to scaling, translation, rotation and reparameterization and show that it is more efficient than the coordinate-descent/path-straightening approach.

## I. INTRODUCTION

Shape analysis of curves is important in various area such as computer vision, medical diagnostics, and bioinformatics. The basic idea is to obtain a boundary curve of an object in a 2D image or contours of a 3D object and analyse those curves to characterize the original object. The research on shape analysis is rich and various ideas have been proposed, e.g., point-based methods, domain-based shape representations and parameterized curve representations. One of the earliest can be traced to Kendall [Ken84], in which the representation of a shape uses landmarks. However, the choices of landmarks is subjective and may significantly influence the analysis of the original objects. For example, Figure 1 shows the geodesics given by two different landmarks. As a matter of fact, the represented points of the bottom figure in Figure 1 is chosen by using the elastic shape analysis. Specifically, unlike the landmarks approach, the elastic shape analysis also takes reparameterization of curves into account.

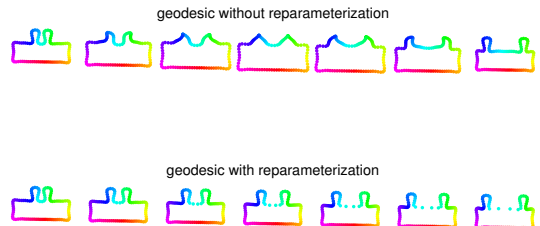


Fig. 1. Geodesics without and with reparameterization are given by the frameworks of landmark-based Kendall’s shape analysis [Ken84], [DM98] and elastic shape analysis [SKJJ11] respectively.

Many frameworks for elastic shape analysis have been proposed. Younes [You98] first introduced this kind of framework for general 2D curves. Younes et al. [YMSM08] studied on elastic analysis of closed curves using complex representations of 2D coordinates of curves. Srivastava et al. [SKJJ11] further defined a novel mathematical framework called the square root velocity functions (SRVFs), which include curves in Euclidean spaces of any dimension.

Elastic shape analysis draws more and more attention due to its superior theoretical results and effectiveness. The price for the better effectiveness is the relatively expensive in computing various objects, e.g., geodesic, mean. In this paper, the framework of SRVF defined in [SKJJ11] is considered. The advantage of SRVF framework is that it converts the complicate Riemannian metric into the standard  $\mathbb{L}^2$  metric and preserve the isometry of rotation and reparameterization group action. This allows to define the shape space in a relative simple way. In [HGSA14], a closed form of distance approximation for closed curves, which is invariant to curve scaling and translation, is used and a Riemannian approach is proposed to improve the efficiency and effectiveness for further removing rotation and reparameterization. In this paper, we explore the same idea of [HGSA14] and improve the efficiency in the computation of removing the rotation and reparameterization without using distance approximation. Also unlike the approach in [HGSA14], a geodesic can be obtained.

Two commonly used methods of computing the geodesic in elastic shape space are shooting method [KSMJ04] and path straightening method [LS85] [KS06] [JKSJ07] [SKJJ11]. The path straightening method has the advantage that all the iterates of paths connect the two points while the shooting method does not. In this paper, we focus on the path straightening method.

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This paper is organized as follows. Section II presents the SRVF framework. Section III gives the previous approach of the path straightening methods. Section IV defines the Riemannian approach and finally the experiments is demonstrated in Section V.

## II. SRVF FRAMEWORK

A shape or curve in  $\mathbb{R}^n$  is denoted by a parameterized function  $\beta(t) : \mathbb{D} \rightarrow \mathbb{R}^n$ , where  $\mathbb{D}$  is  $[0, 1]$  for open curves and  $\mathbb{D}$  is the unit circle  $\mathbb{S}^1$  for closed curves. The representation of a shape starts from its square root velocity (SRV) function,

$$q(t) = \begin{cases} \frac{\dot{\beta}(t)}{\sqrt{\|\dot{\beta}(t)\|_2}}, & \text{if } \|\dot{\beta}(t)\|_2 \neq 0; \\ 0, & \text{if } \|\dot{\beta}(t)\|_2 = 0. \end{cases}$$

where  $\|\cdot\|_2$  denote the 2-norm. The curve  $\beta$  can be recovered by  $\beta(t) = \int_0^t q(s) \|q(s)\|_2 ds$  if  $\beta(0)$  is 0. Note translation is removed since  $\dot{\beta}$  is used. Further more, rescaling can be removed by restricting curves to be of unit length. Since the length of  $\beta(t)$  is  $\int_{\mathbb{D}} \|\dot{\beta}(t)\|_2 dt = \int_{\mathbb{D}} \|q(t)\|_2^2 dt$ , the resulting space, called the preshape space  $l_n$ , for open curves is denoted as

$$l_n^o = \left\{ q \in \mathbb{L}^2([0, 1], \mathbb{R}^n) \mid \int_0^1 \|q(t)\|_2^2 dt = 1 \right\},$$

and for closed curves is denoted as

$$l_n^c = \{ q \in \mathbb{L}^2(\mathbb{S}^1, \mathbb{R}^n) \mid \int_{\mathbb{S}^1} \|q(t)\|_2^2 dt = 1, \int_{\mathbb{S}^1} q(t) \|q(t)\|_2 dt = 0 \}$$

where  $\int_{\mathbb{S}^1} q(t) \|q(t)\|_2 dt = 0$  stresses the closure condition and the super script  $o$  and  $c$  denotes open or closed curves respectively. Statements without a super-script apply to both open and closed curves. A more intuitive way to denote the preshape space  $l_n^c$  is  $\{ q \in \mathbb{L}^2(\mathbb{S}^1, \mathbb{R}^n) \mid \int_0^1 \|q(t)\|_2^2 dt = 1, \int_0^1 q(t) \|q(t)\|_2 dt = 0 \}$  and the closure condition means the difference between  $\beta(0)$  and  $\beta(1)$  is zero since  $\int_0^1 q(t) \|q(t)\|_2 dt = \beta(1) - \beta(0)$ . It can be seen that  $l_n^c$  is a submanifold of  $l_n^o$ . The metrics of the spaces are endowed from  $\mathbb{L}^2$ , i.e.,  $\langle v_1, v_2 \rangle_{\mathbb{L}^2} = \int_0^1 v_1^T v_2 dt$  for  $v_1, v_2 \in \mathbb{L}^2([0, 1], \mathbb{R}^n)$ .

In order to remove the rotation and reparameterization, we consider the rotation group

$$\text{SO}(n) = \{ O \in \mathbb{R}^{n \times n} \mid O^T O = I_n, \det(O) = 1 \}$$

and the reparameterization group

$$\Gamma = \{ \gamma : \mathbb{D} \rightarrow \mathbb{D} \mid \gamma \text{ is orientation-preserving, smooth bijections.} \}.$$

The actions of  $\text{SO}(n)$  and  $\Gamma$  on the SRV of a curve  $\beta$  are:

$$\begin{aligned} \text{SO}(n) \times l_n &\rightarrow l_n : (O, q) \rightarrow Oq, \\ l_n \times \Gamma &\rightarrow l_n : (q, \gamma) \rightarrow (q \circ \gamma) \sqrt{\dot{\gamma}} \end{aligned}$$

and it is known that the two group actions are isometric with respect to the  $\mathbb{L}^2$  metric. It follows that the orbit of the group actions is defined by

$$[q] = \left\{ O(q \circ \gamma) \sqrt{\dot{\gamma}} \mid (\gamma, O) \in \Gamma \times \text{SO}(n) \right\}$$

and the the shape space is defined as:

$$\mathfrak{L}_n = l_n / \Gamma \times \text{SO}(n) = \overline{[q]} \mid q \in l_n \},$$

where  $\overline{[q]}$  denotes the closure of  $[q]$  with respect to  $\mathbb{L}^2$ . The motivation of the closure can be found in, e.g., [SKJJ11] or [HGSA14].

Since  $\mathfrak{L}_n$  is a quotient manifold of  $l_n$  and they have the same metric, a geodesic in  $\mathfrak{L}_n$  can be represented by any geodesic in  $l_n$  that is perpendicular to any orbit that it intersects, and the distance between  $\overline{[q_0]}$  and  $\overline{[q_1]} \in \mathfrak{L}_n$  is given by

$$d_{\mathfrak{L}_n}(\overline{[q_0]}, \overline{[q_1]}) = \inf_{(\gamma, O) \in \Gamma \times \text{SO}(n)} d_{l_n}(q_0, O(q_1 \circ \gamma) \sqrt{\dot{\gamma}}).$$

## III. PATH STRAIGHTENING METHOD

### A. Path-Straightening Method in Preshape Space $l_n^c$

The preshape space of open curves is a unit sphere and its geodesic is well known. In this paper, we focus on computing a geodesic of closed curves. Throughout this paper, the use of word ‘‘geodesic’’ means a path with a constant velocity.

Let  $\mathcal{P}$  denote all the curves in  $l_n^c$ . Let the set of paths connecting two curves  $q_0, q_1$  in  $l_n^c$  be

$$\mathcal{P}_{q_1, q_2} = \{ \alpha : [0, 1] \rightarrow l_n^c \mid \alpha(0) = q_0, \alpha(1) = q_1 \}$$

We start off from an arbitrary path  $\alpha(\tau)$  in  $\mathcal{P}_{q_1, q_2}$ , and iterate until reaching a critical point of the energy function

$$E : \mathcal{P}_{q_1, q_2} \rightarrow \mathbb{R} : \alpha \mapsto \frac{1}{2} \int_0^1 \langle \dot{\alpha}(\tau), \dot{\alpha}(\tau) \rangle d\tau.$$

It has been shown in [SKJJ11, Lemma 4] that any critical of  $E$  is a geodesic of  $l_n^c$ .

A gradient method is proposed in [SKJJ11], in which the search direction is along the negative gradient and a fixed step size is used.

Consider a path  $\beta \in \mathcal{P}$  and a vector field  $v \in T_\beta \mathcal{P}$ . The covariant derivative of  $v$  along  $\beta$  is the vector field obtained by projecting  $\frac{dv}{d\tau}(\tau)$  onto the tangent space of  $T_{\beta(\tau)} l_n^c$  for all  $\tau$ . A vector field  $z \in T_\beta \mathcal{P}$  is called a covariant integral of  $v$  along  $\beta$  if the covariant derivative of  $z$  is  $v$ , i.e.,  $\frac{Dz}{d\tau} = v$ .

Let  $u$  denote the covariant integral of  $\frac{d\alpha}{d\tau}$  with zero initial value at  $\tau = 0$ . The gradient of  $E$  is given by  $w(\tau) = u(\tau) - \tau \tilde{u}(\tau)$ , where  $\tilde{u}$  is the vector field obtained by parallel translating  $u(1)$  backwards along  $\alpha$ , i.e.,  $\tilde{u}(1) = u(1)$  and  $\frac{D\tilde{u}}{d\tau}(\tau) = 0$  for all  $\tau \in [0, 1]$ .

Algorithm 1 outlines the path straightening method for computing geodesic in  $l_n^c$  of [SKJJ11].

The initial path  $\alpha$  between  $q_0$  and  $q_1$  is obtained by projecting the path  $\alpha_o$ , the geodesic between  $q_0$  and  $q_1$  in  $l_n^o$ , onto the  $l_n^c$  (see details in [SKJJ11, Item 1]). This usually offers good initial iterate and Algorithm 1 converges after only a few iterations to reach a tight stopping criterion, e.g.,  $\int_0^1 \|w(\tau)\|_{\mathbb{L}^2}^2 d\tau \leq 10^{-10}$ .

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**Algorithm 1** Path Straightening Method

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**Input:** Two curves  $\beta_0$  and  $\beta_1$ , and a step size  $t > 0$

- 1: Compute the representations  $q_0$  and  $q_1$  in  $l_n^c$ .
  - 2: Initialize a path  $\alpha$  between  $q_0$  and  $q_1$  in  $l_n^c$ .
  - 3: Compute the velocity vector field  $\frac{d\alpha(\tau)}{d\tau}$  along the path  $\alpha$ .
  - 4: Compute the covariant integral of  $\frac{d\alpha(\tau)}{d\tau}$ , denoted by  $u$
  - 5: Compute the backward parallel transport of the vector  $u(1)$  along  $\alpha$ , denoted by  $\tilde{u}$
  - 6: Compute the full gradient vector field of the energy  $E$  along the path  $\alpha$ , denoted by  $w$ , using  $w(\tau) = u(\tau) - \tau\tilde{u}(\tau)$ .
  - 7: Update  $\alpha$  along the vector field  $tw$ . If  $\int_0^1 \|w(\tau)\|_{\mathbb{L}^2}^2 d\tau$  is small, then stop. Else, goto Step 3.
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### B. Removing Orientations and Reparameterizations

In order to obtain a geodesic in shape space  $\mathcal{L}_n$ , we need to minimize the cost function  $H(O, \gamma) = d_{l_n^c}(q_0, O(q_1 \circ \gamma)\sqrt{\tilde{\gamma}})$  over the product of manifolds  $\text{SO}(n)$  and  $\Gamma$ . The algorithm in [SKJJ11] solves this optimization by alternately optimizing between  $\text{SO}(n)$  and  $\Gamma$ . This requires the computation of the gradient with respect to  $O$  and  $\gamma$ .

To this end, consider the cost function  $d_{l_n^c}(q_0, \tilde{q}_1)$  with respect to  $\tilde{q}_1$ . Let  $\alpha$  denote the geodesic between  $q_0$  and  $\tilde{q}_1$  in  $l_n^c$ . It is known from [SKJJ11] that the gradient of  $d_{l_n^c}(q_0, \tilde{q}_1)$  is  $\eta = \dot{\alpha}(1)/\|\dot{\alpha}(1)\|$ . It follows that the gradient with respect to  $O$  is

$$\text{grad}_O H(O, \gamma) = P_O \left( \int_{\mathbb{D}} \eta \sqrt{\tilde{\gamma}}(q_2 \circ \gamma)^T ds \right) \quad (\text{III.1})$$

where  $P_O(M) = (M - OM^T O)/2$ .

Note that  $\Gamma$  is an infinite dimensional manifold. The gradient with respect to  $\gamma$  is approximated by

$$\text{grad}_\gamma H(O, \gamma) \approx \sum_i^k b_i D H(O, \gamma)[b_i], \quad (\text{III.2})$$

where  $\{b_i\}_{i=1}^\infty$  is a basis of the tangent space of  $\gamma$ , the tangent space of  $\gamma$  is  $T_\gamma \Gamma = \mathbb{L}^2(\mathbb{S}^1, \mathbb{R})$ ,  $k$  is the number of the elements in the basis and  $D H(O, \gamma)[b_i]$  denotes the directional derivative of  $H(O, \gamma)$  along direction  $b_i$ . It can be shown that

$$D H(O, \gamma)[b_i] = \left\langle \eta, O \left( \sqrt{\tilde{\gamma}} \dot{q}_1(\gamma) b_i + \frac{1}{2\sqrt{\tilde{\gamma}}} \dot{b}_i q_1(\gamma) \right) \right\rangle_{\mathbb{L}^2}.$$

The suggested basis  $\{b_i\}$  is an orthonormal basis of the tangent space of  $\gamma$  under the Palais metric  $\langle v_1, v_2 \rangle_P = v_1(0)v_2(0) + \int_0^1 \dot{v}_1(\tau)\dot{v}_2(\tau)d\tau$ , i.e.,

$$\left\{ 1, \frac{\sin(nt)}{n\pi}, \frac{\cos(nt) - 1}{n\pi}, n = 1, 2, \dots \right\}. \quad (\text{III.3})$$

The algorithm of removing rotation and reparameterization is stated in Algorithm 2.

Note that in [SKJJ11] the substitution of  $l = \sqrt{\tilde{\gamma}}$  is used in  $H$ . It follows that the cost function is defined on  $\text{SO}(n)$  and the first quadrant of the unit sphere  $\mathcal{L} = \{l \in$

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**Algorithm 2** Removing rotation and reparameterization

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**Input:** Two curves  $\beta_0$  and  $\beta_1$ , and step sizes  $t_1, t_2 > 0$

- 1: Set  $\tilde{\beta}_1$  to be  $\beta_1$ ,  $O_0 = I$ ,  $\gamma_0 = \gamma_{\text{id}}$ , and  $k = 0$ .
  - 2: Compute the representations  $q_0$  of  $\beta_0$  and  $q_1$  of  $\tilde{\beta}_1$ .
  - 3: Compute the geodesic  $\alpha$  between  $q_0$  and  $q_1$  in  $l_n^c$  using Algorithm 1.
  - 4: Update the rotation by  $O_{k+1} = O_k \exp(t_1 \text{grad}_I H(I, \gamma_{\text{id}}))$ , where  $\text{grad}_I H(I, \gamma_{\text{id}})$  is (III.1).
  - 5: Update the reparameterization by  $\gamma_{k+1} = \gamma_k \circ (\gamma_{\text{id}} + t_2 \text{grad}_{\gamma_{\text{id}}} H(I, \gamma_{\text{id}}))$ , where  $\text{grad}_{\gamma_{\text{id}}} H(I, \gamma_{\text{id}})$  is (III.2). Note that  $t_2$  should be small enough such that  $\gamma_{k+1}$  is nondecreasing.
  - 6: Update  $\tilde{\beta}_1 \leftarrow O_{k+1} \beta_1 \circ \gamma_{k+1}$  and set  $q_1$  to be the SRVF of  $\tilde{\beta}_1$
  - 7: If some stopping criterion is satisfied, then stop. Else,  $k \leftarrow k + 1$  and goto Step 2.
- 

$\mathbb{L}^2([0, 1], \mathbb{R})\| \|l\|_{\mathbb{L}^2=1}$ . It is pointed out that using the basis (III.3) essentially yields the same method in [SKJJ11, Section 4.4] without the extra substitution step.

### IV. A RIEMANNIAN APPROACH

The path straightening method in the shape space can be characterized as a steepest descent method with a fixed step size. It is well-known that steepest descent method suffers from slow convergent rate, see e.g., [NW06]. In this paper, we apply a faster algorithm, a limited-memory version of Riemannian BFGS method (LRBFGS), which is introduced in [HGA14] and shown to outperform many other start-of-the-art Riemannian algorithms for many large scaled problems, e.g., [NW06], [HGSA14], [HGZ15].

Since it is observed that Algorithm 2 dominates the computational time in the sense that it needs large number of iterations, we only use LRBFGS to improve the performance of removing rotation and reparameterization.

For the closed curves, the reparameterization  $\Gamma$  can be characterized as

$$\Gamma^c = [0, 1] \times \Gamma^o$$

and its action is therefore  $l_n^c \times \Gamma^c \rightarrow l_n^c : ((q, m), \gamma) \rightarrow (q(t + m \bmod 1) \circ \gamma \bmod 1)\sqrt{\tilde{\gamma}}$ , where  $\Gamma^o$  is the reparameterization group for open curves, i.e.,

$$\Gamma^o = \{ \gamma : [0, 1] \rightarrow [0, 1] | \gamma \text{ is a diffeomorphism} \}.$$

Further setting  $l = \sqrt{\tilde{\gamma}}$ , we obtain a cost function

$$f(O, m, l) = d_{l_n^c}(q_1, O l q_2(\int_0^t l^2(s) ds + m \bmod 1)), \quad (\text{IV.1})$$

where  $(O, m, l) \in \text{SO}(n) \times \mathbb{R} \times \mathcal{L}$ . We define the metric on the tangent space of  $\text{SO}(n) \times \mathbb{R} \times \mathcal{L}$  by  $\langle (U_1, b_1, v_1), (U_2, b_2, v_2) \rangle = \text{trace}(U_1^T U_2) + b_1 b_2 + \int_0^1 v_1 v_2 ds$ . The Riemannian gradient of  $f$  with respect to this metric is given in Lemma 4.1 without proof.

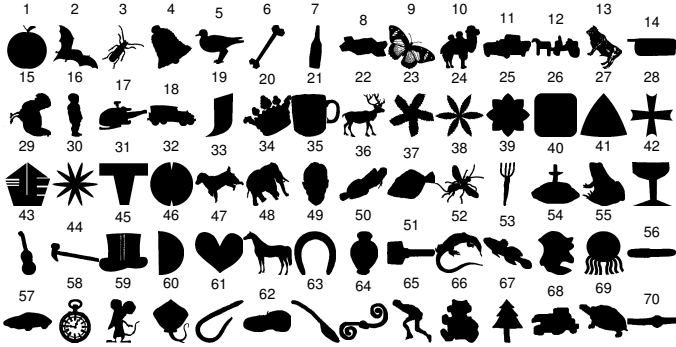


Fig. 2. Samples of curves from the MPEG-7 dataset. One sample per cluster is illustrated.

*Lemma 4.1:* The Riemannian gradient of  $f(O, m, l)$  in (IV.1) is

$$\text{grad } f(O, m, l) = (P_O(A), \int_0^1 y' ds, P_l(x - 2yl)),$$

where  $A$  denotes  $\int_0^1 \eta l q_2^T (\int_0^t l^2(s) ds + m \bmod 1) ds$ ,  $x$  denotes  $\langle \eta, O(q_2(\int_0^t l^2(s) ds + m \bmod 1)) \rangle_2$ ,  $y'$  denotes  $\langle \eta, O(l q_2'(\int_0^t l^2(s) ds + m \bmod 1)) \rangle_2$ ,  $\eta$  is the same as in (III.1) and  $P_l(v) = v - l \frac{\langle v, \tilde{l} \rangle_{\mathbb{L}^2}}{\|\tilde{l}\|_{\mathbb{L}^2}^2}$ .

In order to apply the LRBFGS algorithm in [HGA14, Algorithm 2], we also need a retraction and a vector transport. The chosen pair is the well-known exponential mapping and parallel translation for each component (see e.g., [AMS08]). They are given here for completeness. The retraction is

$$R_{(O, m, l)}(A, a, v) = \left( O \exp(O^T A), m + a, l \cos(\|v\|_{\mathbb{L}^2}) + \frac{v}{\|v\|_{\mathbb{L}^2}} \sin(\|v\|_{\mathbb{L}^2}) \right)$$

and the vector transport is

$$\mathcal{T}_{(A, a, v)}((B, b, w)) = \left( O \exp(O^T A/2) O^T B \exp(O^T A/2), b, w - \frac{2\langle w, \tilde{l} \rangle_{\mathbb{L}^2}}{\|\tilde{l}\|_{\mathbb{L}^2}^2} (l + \tilde{l}) \right)$$

where  $A, B \in T_O \text{SO}(n)$ ,  $a, b \in \mathbb{R}$ ,  $w, v \in T_l \mathcal{L}$  and  $\tilde{l} = l \cos(\|v\|_{\mathbb{L}^2}) + \frac{v}{\|v\|_{\mathbb{L}^2}} \sin(\|v\|_{\mathbb{L}^2})$ .

## V. EXPERIMENTS

The MPEG-7 dataset [Uni] is chosen in the experiments and it contains 70 clusters each of which has 20 shapes, i.e., 1400 shapes in total. Figure 2 shows an example shape from each cluster. Matlab function `BWBOUNDARIES` is used to extract the boundary curves of the shapes and 100 uniformly-space points are chosen to represent each shape. A path in  $\mathcal{L}_n^c$  is represented by 11 curves.

The tests are performed in Matlab R2014a on a 64 bit Ubuntu system with 3.6GHz CPU (Intel (R) Core (TM) i7-4790).

We compare the performance of LRBFGS, a Riemannian steepest descent (RSD) algorithm based on the framework in Section IV, and Algorithm 2 with multiple choices of step

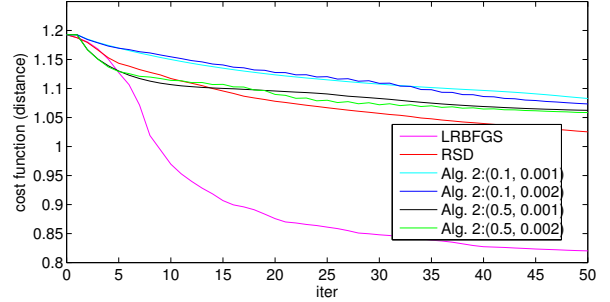
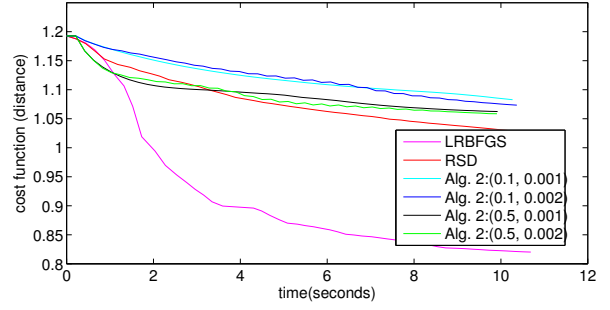


Fig. 3. Comparisons of algorithms with 50 iterations. The top figures shows the average relationship between the computational time and the cost function values. The bottom figures the average relationship between the number of iterations and the cost function values. The notation  $\text{Alg.2}:(t_1, t_2)$  denotes the step sizes in Algorithm 2.

sizes  $t_1$  and  $t_2$  over many randomly chosen pairs of curves from the data set. The initial rotation and reparameterization for all algorithms are given by the approach in [HGSA14]. The average of computational times and the average of the cost function values (distance) after each iteration is computed and the results are shown in Figure 3.

Since the step sizes,  $t_1$  and  $t_2$  are fixed, the choice of their values is important in Algorithm 2. If taken too small then the convergence is slow. If taken too large then the cost function may not decrease. It is shown in Figure 2 that the average function values with  $t_2 = 0.002$  oscillate. This is due to the fact that for that value of  $t_2$  Algorithm 2 does not converge for some of the pairs of curves – the cost function value oscillates and does not decrease. Note, however, that all algorithms that converge for a particular pair of curves converge to the same critical point. RSD and LRBFGS both use an efficient line search algorithm to determine a step size for each iteration that satisfies appropriate termination criteria to guarantee convergence [Hua13] and are therefore more robust than Algorithm 2.

RSD and Algorithm 2 have similar performance when the latter converges. LRBFGS is clearly the best algorithm is able to reach a lower cost function value within a much smaller computational time.

## VI. CONCLUSION AND FUTURE WORK

In this paper, we consider computing the geodesic in the shape space of elastic curves. A Riemannian manifold optimization approach is proposed as a replacement for the current state-of-the-art coordinate-descent/path-straightening ap-

proach in [SKJJ11]. The Riemannian quasi-Newton method, LRBFGS, is shown to be superior in both robustness and computational efficiency.

In the future, we will test the quality of the distance obtained by LRBFGS in the sense of superior clustering, classification, and Karcher mean computations. The Riemannian approach will be included in the C++ Riemannian optimization library on <http://www.math.fsu.edu/ROPTLIB>.

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