

ENDPERIODIC AUTOMORPHISMS OF SURFACES AND FOLIATIONS

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ABSTRACT. We extend the unpublished work of M. Handel and R. Miller on the classification, up to isotopy, of endperiodic automorphisms of surfaces. We give the Handel-Miller construction of the geodesic laminations, give an axiomatic theory for pseudo-geodesic laminations, show the geodesic laminations satisfy the axioms, and prove that pseudo-geodesic laminations satisfying our axioms are ambiently isotopic to the geodesic laminations. The axiomatic approach allows us to show that the given endperiodic automorphism is isotopic to a smooth endperiodic automorphism preserving smooth laminations ambiently isotopic to the original ones. Using the axioms, we also prove the “transfer theorem” for foliations of 3-manifolds, namely that, if two depth one foliations \mathcal{F} and \mathcal{F}' are transverse to a common one-dimensional foliation \mathcal{L} whose monodromy on the noncompact leaves of \mathcal{F} exhibits the nice dynamics of Handel-Miller theory, then \mathcal{L} also induces monodromy on the noncompact leaves of \mathcal{F}' exhibiting the same nice dynamics. Our theory also applies to surfaces with infinitely many ends.

1. INTRODUCTION

The Nielsen-Thurston theory of automorphisms of compact surfaces [19, 43, 29, 33] classifies the isotopy class of an automorphism f of a compact, hyperbolic surface. For endperiodic automorphisms of noncompact surfaces, M. Handel and R. Miller outlined an analogous theory (unpublished).

Both theories produce a pair of transverse geodesic laminations and a map h (endperiodic in the Handel-Miller case), isotopic to f and preserving the laminations.

In the compact case, there are h -invariant reducing circles which decompose the surface into periodic pieces and pseudo-Anosov pieces. Similarly, in the endperiodic case, there are reducing circles and reducing lines. These reduce the surface into finitely many (finite or infinite) h -orbits of compact subsurfaces, finitely many noncompact pieces on which a power of h is a translation, and finitely many noncompact “pseudo-anosov” pieces. (The lower case “a” indicates that the analogy with pseudo-Anosov automorphisms of compact surfaces is weak.) In our exposition, reduction arises late in the game when an analysis of the laminations yields the reducing curves in a very natural way.

The first nine sections of this paper treat the basics, defining endperiodicity, presenting the Handel-Miller construction of the geodesic laminations arising from an endperiodic automorphism, giving a detailed analysis of their structure, constructing the endperiodic automorphism preserving the laminations, and analyzing

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its dynamics. These sections fill in roughly a thirty year gap in the literature, the first goal of this paper.

Our second goal is to prove two important new theorems, Theorem 11.1 and Theorem 12.7 (see below), which are critical for applications to foliation theory. For this, it becomes necessary to relax the condition that the laminations be geodesic. We do this in Section 10 where we state four axioms for the “pseudo-geodesic” laminations and show that these laminations are ambiently isotopic to the geodesic laminations of the Handel-Miller theory. We note that the geodesic laminations satisfy our axioms, so we are not axiomatizing the empty set, and the isotopy theorem then shows that our axioms are complete. The entire theory developed in the geodesic case becomes immediately available in the pseudo-geodesic case.

As in the Nielsen-Thurston theory, smoothness is a problem. Thurston’s technique of “blowing down” the laminations to produce a pair of transverse foliations with finitely many p -pronged singularities made it possible to smooth h and the foliations except at the singularities. This used a pair of projectively invariant measures to produce the smooth coordinate atlas on the complement of the singular set. In our case, the blow-down does not yield foliations and the projectively invariant measures may not have full support. In our “Smoothing Theorem” (Theorem 11.1), having relaxed the geodesic condition on the laminations, we directly construct a pair of transverse smooth laminations preserved by an endperiodic diffeomorphism h and verify the axioms.

Handel-Miller theory has applications to foliations analogous to the applications of Nielsen-Thurston theory to fibrations (cf. [25]). In smooth foliations of depth one, the monodromy f of the noncompact leaves is endperiodic and there is a smooth representative h of the isotopy class of f preserving the laminations. We will prove the fundamental “Transfer Theorem” (Theorem 12.7). By this theorem, if \mathcal{F} and \mathcal{F}' are depth one foliations on M , both transverse to a 1-dimensional foliation \mathcal{L} , and if the first return map (monodromy) induced by \mathcal{L} on a depth one leaf L of \mathcal{F} preserves a pair of pseudo-geodesic laminations satisfying the axioms, then the monodromy it induces on a depth one leaf L' of \mathcal{F}' also preserves such a pair of laminations. For this, the laminations on the leaf of \mathcal{F}' cannot be assumed to be geodesic, even if those on the leaf of \mathcal{F} are, necessitating our axiomatic characterization. The proof of the theorem proceeds by showing that the truth of our axioms for the monodromy of \mathcal{F} implies their truth for the monodromy of \mathcal{F}' .

2. ENDPERIODIC AUTOMORPHISMS

We fix a temporary hypothesis,

Temporary Hypothesis. **Until Section 2.1, we assume that L is a non-compact, connected n -manifold with finitely or infinitely many ends and possibly with boundary.**

Let $f : L \rightarrow L$ be a homeomorphism. We do not assume that L is orientable nor, if it is, that f is orientation preserving.

Suppose $L \supset V_1 \supset \dots \supset V_n \supset \bar{V}_{n+1} \supset V_{n+1} \supset \dots$ with the V_n open, connected, $\bigcap_{n=1}^{\infty} V_n = \emptyset$, and $\bar{V}_n \setminus V_n$ compact. Then the nested sequence of sets $\{V_n\}$ defines an *end* of L .

If $\{V_n\}$ and $\{U_n\}$ define ends of L , then $\{V_n\}$ is said to be equivalent to $\{U_n\}$ if for every n there exists an m such that $V_n \supset U_m$ and for every m there exists an n

such that $U_m \supset V_n$. The equivalence classes, $e = [\{V_n\}]$, are called the *ends* of L and the set $\mathcal{E}(L)$ of equivalence classes is called the *endset* of L .

Often one gives an “exhaustion” $K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots \subset L$ where the K_n are compact and $\bigcup_{n=0}^{\infty} K_n = L$. Then for any $e \in \mathcal{E}(L)$, $e = [\{U_n\}]$ where U_n is an unbounded component of the complement of K_n .

Let \mathcal{T} be the topology on L , that is \mathcal{T} is the set of open sets in L . For $V \in \mathcal{T}$ let,

$$\widehat{V} = V \cup \{e = [\{V_n\}] \in \mathcal{E}(L) \mid \text{there exists an } n \text{ with } V \supset V_n\}.$$

Then it is well known that $\widehat{\mathcal{B}} = \mathcal{T} \cup \{\widehat{V} \mid V \in \mathcal{T}\}$ is a base for a compact, separable metrizable topology on $L \cup \mathcal{E}(L)$ which restricts to a totally disconnected topology on the closed set $\mathcal{E}(L)$.

If $e \in \mathcal{E}(L)$, we will say that $U \subset L$ is a neighborhood of the end e if \widehat{U} is a neighborhood of e in the space $(L \cup \mathcal{E}(L), \widehat{\mathcal{T}})$.

Notice that f induces an automorphism on the space of ends of L which, by abuse, we will also denote by f .

Definition 2.1 (p_e). An end e of L is *periodic* of *period* $p_e > 0$ if $f^{p_e}(e) = e$ and p_e is the least positive integer with this property.

Remark. If there are finitely many ends, every end is periodic.

Definition 2.2 (positive and negative ends, $\mathcal{E}_{\pm}(L)$). An end e of period p_e is a *positive end* if there is a closed, connected neighborhood U_e of e such that $L \setminus U_e$ is connected and

- (1) $f^{p_e}(U_e) \subset U_e$;
- (2) $\bigcap_{n=0}^{\infty} f^{np_e}(U_e) = \emptyset$;
- (3) $\text{Fr } U_e$ is compact.

The end e is a *negative end* if the parallel assertions hold with p_e replaced by $-p_e$. We denote the set of positive (respectively negative) ends by $\mathcal{E}_+(L)$ (respectively $\mathcal{E}_-(L)$).

Remark. We will write ∂M for the boundary of a manifold M and the symbol $\text{Fr } A$ for the topological boundary (frontier) of the subset A of a topological space.

Remark. The positive ends are the attracting ends and the negative ends are the repelling ends.

Ends may fail to be positive or negative. For instance, on a four times punctured sphere, one easily produces a homeomorphism f that cyclically permutes three punctures, fixes the fourth and satisfies $f^3 = \text{id}$. These ends are neither positive nor negative. Similar examples can be produced in which the ends are nonplanar.

Definition 2.3 (f -neighborhood of an end). A set U_e as in Definition 2.2 will be called an f -*neighborhood* of e .

Definition 2.4 (endperiodic automorphism). The homeomorphism $f : L \rightarrow L$ is called an *endperiodic automorphism* of L if all periodic ends are positive or negative.

Example 2.5. A simple example of an endperiodic automorphism is depicted in Figure 1. Here, the ends e_1 and e_2 are periodic of period 2 and negative, with $p_{e_2} = p_{e_1} = 2$. The end e is periodic of period 1, positive, with $p_e = 1$. The circle f -junctures (Definition 2.9) separate the neighborhoods of the ends into f -domains (Definition 2.24) having negative Euler characteristic. The arrows indicate

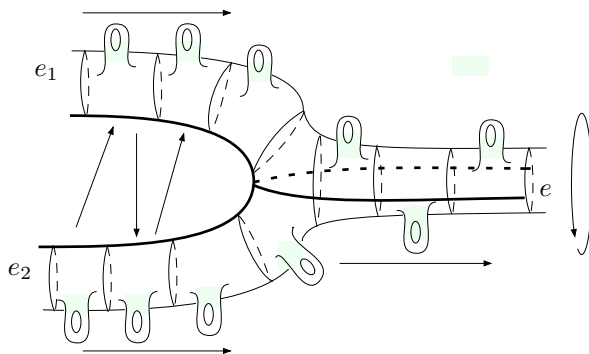


FIGURE 1. An example with two negative ends

the action of f . Notice that all of the “interesting” dynamics of f occurs in the compact region complementary to neighborhoods of the three ends. This is called the “core” and it is only there that fixed points, periodic points, other invariant sets can occur. Actually, this example can be constructed so that the only interesting dynamics is a single fixed point. The two boldface curves will be explained later.

Remark. In this paper, we study surfaces with finitely many ends but the definition of endperiodic automorphism also makes sense for surfaces with infinitely many ends and for n -manifolds with $n > 2$. In Section 13.4 we show how the theory of endperiodic automorphisms of surfaces L with infinitely many ends reduces to the case of a surface L with finitely many ends.

Remark. In [8, Section 4], we give an infinite family of examples of endperiodic automorphisms of 1-ended 3-manifolds in which the end is negative. Included are Whitehead’s example of a contractible open three manifold which is not \mathbb{R}^3 [37, 44] and 3-manifolds with nontrivial fundamental group.

Lemma 2.6. *If e and e' are distinct positive or distinct negative ends of L and U_e and $U_{e'}$ are f -neighborhoods of e and e' respectively, then $U_e \cap U_{e'} = \emptyset$.*

Proof. Suppose e and e' are both positive ends. Let V and V' be disjoint neighborhoods of e and e' respectively. If there exists $x \in U_e \cap U_{e'}$, then for k sufficiently large, $f^{kp_{e'}}(x)$ lies in both V and V' which is a contradiction. The case where e and e' are negative ends is parallel. \square

Lemma 2.7. *Let $f : L \rightarrow L$ be a homeomorphism. Then f is endperiodic if and only if f^p is endperiodic, for some integer $p > 0$.*

Proof. The “only if” direction is trivial. Assume that f^p is endperiodic and let e be an end with period p_e . Then, $f^{pp_e}(e) = e$. Since f^p is endperiodic, it follows that e is either a positive or negative end under the homeomorphism f^p . Without loss, assume that e is positive. Thus, there is a closed neighborhood U_e of e such that $L \setminus U_e$ is nonempty and connected, $f^{pp_e}(U_e) \subset U_e$, and

$$\bigcap_{n=1}^{\infty} f^{npp_e}(U_e) = \emptyset.$$

Set

$$V_e = U_e \cap f^{p_e}(U_e) \cap f^{2p_e}(U_e) \cap \dots \cap f^{(p-1)p_e}(U_e).$$

Then,

$$L \setminus V_e = (L \setminus U_e) \cup (L \setminus f^{p_e}(U_e)) \cup \dots \cup (L \setminus f^{(p-1)p_e}(U_e))$$

is connected as the union of connected sets with intersection

$$\bigcap_{i=0}^{p-1} (L \setminus f^{ip_e}(U_e))$$

which is nonempty as the finite intersection of open neighborhoods of each negative end and

$$f^{p_e}(V_e) = f^{p_e}(U_e) \cap f^{2p_e}(U_e) \cap \dots \cap f^{pp_e}(U_e) \subset V_e.$$

Since V_e is a neighborhood of e , it has exactly one noncompact component V'_e which is a neighborhood of the end e . Hence $f^{p_e}(V'_e) \subset V'_e$ and we can replace V_e with V'_e . Finally, $L \setminus V'_e$ is connected and,

$$\bigcap_{n=1}^{\infty} f^{np_e}(V'_e) \subset \bigcap_{n=1}^{\infty} f^{np_e}(U_e) \subset \bigcap_{n=1}^{\infty} f^{npp_e}(U_e) = \emptyset.$$

Thus e is a positive end. A similar argument works for negative ends, proving that f is endperiodic. \square

Lemma 2.8. *If e is a positive (respectively negative) end and U_e is an f -neighborhood of e , then there is an integer $p > 0$, divisible by p_e , such that $f^p(U_e) \subset \text{int } U_e$ (respectively $f^{-p}(U_e) \subset \text{int } U_e$).*

Proof. Recall from Definition 2.2 that U_e has compact frontier. If $f^{np_e}(U_e)$ meets $\text{Fr } U_e$ for all $n \geq 0$ then, by the compactness of $\text{Fr } U_e$, there exists

$$x \in \left(\bigcap_{n=0}^{\infty} f^{np_e}(U_e) \right) \cap \text{Fr } U_e$$

contradicting $\bigcap_{n=0}^{\infty} f^{np_e}(U_e) = \emptyset$. Thus there exists an $n > 0$ such that $f^{np_e}(U_e) \subset \text{int } U_e$. \square

Definition 2.9 (f -junctures). For each f -neighborhood U_e (Definition 2.3) of a positive or negative end e , the set $J = \text{Fr } U_e$ is called an f -*juncture* for e . The f -juncture is *positive* (respectively *negative*) if U_e is an f -neighborhood of a positive (respectively negative) end.

Remark. A given f -juncture J for a positive or negative end e gives rise to a whole bi-infinite sequence $\{J_n = f^n(J)\}_{n \in \mathbb{Z}}$ of f -junctures for the ends in the f -cycle of e .

Remark. Given an end e , there are uncountably many choices of f -neighborhood U_e and f -juncture $J = \text{Fr } U_e$. In Section 4.3.1, we will pick and fix a countable set of f -junctures.

Example 2.10. We give an example to show how the images of f -junctures behave under iteration of f . Let L be the surface depicted in Figure 2, two strips connected by an infinite sequence of tubes. The endperiodic automorphism is $f = \tau \circ g$, where g is the translation from the negative end to the positive end moving each handle to the next handle as indicated by the arrow and τ is a Dehn twist in the oval C . The

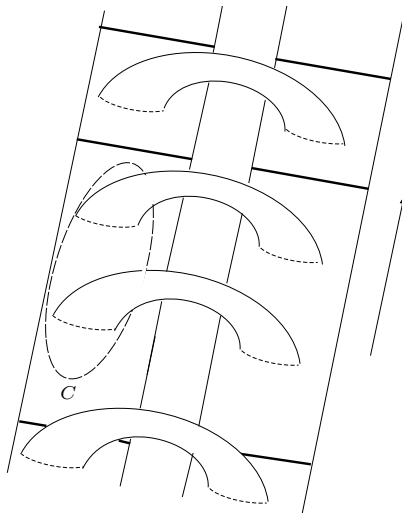


FIGURE 2. f is not a translation but some components of f -junctures escape

f -junctures are each the pair of properly embedded arcs as pictured in boldface. Under forward iteration of f , the left component of an f -juncture in the negative end gets caught by the Dehn twist and starts stretching unboundedly into the positive end. Note that the right component of the f -juncture escapes without distortion to the positive end. A similar description holds for f -junctures in the positive end and their images under backward iteration of f . This distortion of f -junctures (and of the geodesic and pseudo-geodesic junctures defined in Sections 4 and 10) is typical behavior and is what will create the stable and unstable laminations of Handel-Miller theory.

We replace our temporary hypothesis by the hypothesis,

Hypothesis 1. Hereafter, unless we explicitly state otherwise, L is a non-compact surface with finite endset.

2.1. Asymptotic construction of f -junctures. We are going to give a method of constructing f -junctures which, being closely related to the way that they arise in foliations, will be called an *asymptotic* construction. The f -junctures constructed in this way will intersect only in common components. We will see this in Example 2.29 which, in fact, is a thinly disguised example of the asymptotic construction.

The junctures in the following definition are either f -junctures or sets of geodesic or pseudo-geodesic junctures as defined in Sections 4 or 10.

Definition 2.11 (juncture intersection property). A juncture J has the *juncture intersection property* if any two junctures in the set $\{J_n = f^n(J) \mid n \in \mathbb{Z}\}$ of junctures intersect, if at all, in common components. A set of junctures has the *juncture intersection property* if every juncture in the set has the juncture intersection property.

Assume that $f : L \rightarrow L$ is endperiodic and e is a positive or negative end of period p_e .

Proposition 2.12. *The choice of f -neighborhood U_e can be made so that the f -junction $J = \text{Fr } U_e$ is a compact, properly embedded, transversely oriented 1-manifold satisfying the junction intersection property.*

Proposition 2.12 will be proven in a series of lemmas. Let e be a positive end of L and set

$$c = \{e_0 = e, e_1 = f(e), e_2 = f^2(e), \dots, e_{p_e-1} = f^{p_e-1}(e)\},$$

the complete f -cycle of ends containing e . The reader can adapt the following discussion for the case that e is a negative end.

For e' an end in the cycle c , set $\mathcal{U}_{e'} = \bigcup_{n=0}^{\infty} f^{-np_{e'}}(U_{e'})$, where $p_{e'} = p_e$ and $U_{e'}$ is any f -neighborhood of e' . Set

$$\mathcal{U}_c = \bigcup_{n=-\infty}^{\infty} f^n(U_e) = \mathcal{U}_{e_0} \cup \mathcal{U}_{e_1} \cup \dots \cup \mathcal{U}_{e_{p_e-1}}.$$

Remark that \mathcal{U}_c is an open, f -invariant set with no periodic points. The connected components \mathcal{U}_{e_i} of \mathcal{U}_c are permuted cyclically by f .

The action of f partitions \mathcal{U}_c into orbits $x = \{x_n\}_{n \in \mathbb{Z}}$, where $x_n = f^n(x_0)$. Let F be the space of orbits with the quotient topology, and remark that the quotient map $q : \mathcal{U}_c \rightarrow F$ is a regular covering map. The group of deck transformations is infinite cyclic generated by f . In particular F is a surface.

Remark. Let e be a positive (respectively negative) end of L and U_e be any f -neighborhood of e . Since L has a finite endset, the set $X = U_e \setminus \text{int } f^{p_e}(U_e)$ (respectively $X = U_e \setminus \text{int } f^{-p_e}(U_e)$) is compact. Since the compact set X and the connected set U_e both surject onto F , it follows that F is compact and connected.

Notation. Fix a basepoint $* \in F$, $* = \{x_n\}_{n \in \mathbb{Z}}$, where $x_{n+1} = f(x_n)$, $-\infty < n < \infty$, and $x_0 \in \mathcal{U}_e$.

If σ is a directed (oriented) loop in F based at $*$, σ lifts to a directed path τ in \mathcal{U}_e starting at x_n and ending at a point $x_{n+\kappa(\sigma)}$. Here, one sees that $\kappa(\sigma) \in \mathbb{Z}$ is independent of the choice of n . Simply apply powers of f to τ . Furthermore, a basepoint-preserving homotopy of σ lifts to an endpoint preserving homotopy of all lifts τ . This construction defines a group homomorphism

$$\kappa : \pi_1(F, *) \rightarrow \mathbb{Z},$$

which can be viewed as a cohomology class $\kappa \in H^1(F; \mathbb{Z})$. The period of such a class κ is the least positive value taken by κ , evidently the greatest common divisor of the the set of values of κ . The class is divisible if its period is greater than 1.

Lemma 2.13. *The class κ is divisible if and only if $p_e > 1$, in which case the period of κ is the period p_e of e .*

Indeed, for $n \in \mathbb{Z}$, there is a path in \mathcal{U}_c from x_n to x_{n+p_e} which projects by q to a loop σ in F . Thus $\kappa(\sigma) = p_e$. Evidently, p_e is the smallest positive value of κ .

Lemma 2.14. *There is a compact, transversely oriented, properly embedded 1-manifold J_κ in F such that $\kappa(\sigma) = \sigma \cdot J_\kappa$, the algebraic intersection number.*

Proof. As is well known, there is a map $f_\kappa : F \rightarrow S^1$, unique up to homotopy, such that $f_\kappa^*[S^1] = \kappa$, $[S^1] \in H^1(S^1)$ being the fundamental class. We may take f_κ to be smooth. By Sard's theorem, there is a regular value p both for f_κ and for $f_\kappa|_{\partial F}$.

Then, $J_\kappa = f_\kappa^{-1}(p)$ is a properly embedded, compact 1-manifold, transversely oriented by the orientation of S^1 . Let $\sigma : S^1 \rightarrow F$ be in general position relative to J_κ . Then $f_\kappa \circ \sigma : S^1 \rightarrow S^1$ has p as a regular value and has degree equal to the number of times (counted with sign) that it crosses p . Clearly, this degree is equal to $\sigma \cdot J_\kappa$ and is the value of κ on σ . \square

Remark. If F is oriented, the transverse orientation of J_κ induces an orientation, allowing us to view J_κ as a 1-cycle. In this case, Lemma 2.14 is really just Poincaré duality. But we do not require orientability and the transversely oriented 1-manifold J_κ can only be thought of as a 1-cocycle, evaluating on 1-cycles via the algebraic intersection product. Throughout this discussion, cocycles typically will be transversely oriented, properly embedded 1-manifolds.

Definition 2.15 (κ -junction). We call J_κ a κ -junction.

Remark. Note that a κ -junction is a compact submanifold of the compact surface F and is not a junction in the sense that we usually use the term junction in this paper.

Remark. The κ in the symbol J_κ for a κ -junction refers to the fact that the geometric object J_κ represents the specific cohomology class $\kappa \in H^1(F; \mathbb{Z})$.

One sometimes calls the cohomology class $\kappa \in H^1(F; \mathbb{Z})$ a “junction”, but we prefer to reserve this term for a geometric object representing κ .

Notice that two components of J_κ might be “parallel” circles if they cobound an annulus $A \subset \text{int } F$. They will be parallel, properly embedded arcs if, together with two arcs in ∂F , they bound a rectangle A with $\text{int } A \subset \text{int } F$. In either case, their transverse orientations are said to be coherent if one is oriented out of A and the other into A .

We want to modify a κ -junction J_κ to be *weakly groomed* in the following sense.

Definition 2.16 (weakly groomed). A compact, properly embedded, transversely oriented 1-manifold $J \subset L$ is *weakly groomed* if every pair of parallel circle (respectively arc) components have coherent transverse orientations.

Remark. The term “groomed” is already in use by 3-manifold topologists, having been introduced by D. Gabai in [28]. It places stronger conditions on J than we require, but includes our condition of “weakly groomed”. It also assumes that F is orientable, which we do not. The metaphor, of course, has to do with a nicely combed head of hair.

Lemma 2.17. *Every κ -junction J_κ is cohomologous to a weakly groomed κ -junction.*

Proof. If J_κ is not weakly groomed, let τ_1 and τ_2 be parallel components with non-coherent transverse orientation. It is clear that the algebraic intersection number of any closed, oriented curve σ with $\tau_1 \cup \tau_2$ is zero, hence these components can be removed. Repeating this procedure finitely often produces the desired weakly groomed κ -junction. \square

Thus, one can assume that the properly embedded arc components of J_κ fall into “packets” of parallel, coherently transversely oriented arcs and that no two parallel components of J_κ have opposing transverse orientation. Similarly, the components that are essential, embedded, transversely oriented circles fall into packets of parallel, coherently transversely oriented circles with no oppositely oriented parallel

packets. Each of these packets can be represented by one of its elements together with a positive integer weight (the number of components of the packet). That is, the cohomology class κ can be represented as a union of disjoint, transversely oriented arcs and circles s , no two of which are parallel, each with an attached integer weight $w_s > 0$. The union of these arcs and circles, neglecting the weights and transverse orientations, is called the *support* of the κ -juncture and will be denoted by $|J_\kappa|$.

Remark. Note that the term support and notation $|\cdot|$ for support has a different meaning for laminations in the rest of the paper.

Lemma 2.18. *If $|J_\kappa|$ separates F , there is a weakly groomed κ -juncture J_κ^* such that $|J_\kappa^*|$ does not separate F and the set of its components is a subset of the set of components of $|J_\kappa|$.*

Proof. Let s_1, s_2, \dots, s_r be oriented components of $|J_\kappa|$ that separate off a connected subsurface S of F . Let w_i be the weight associated to s_i and let w_j be the minimum, $1 \leq j \leq r$. Let s'_i be the same arc or circle as s_i , but with transverse orientation inward to S . Then the coboundary $\sum_{i=1}^r w_j s'_i$ can either be added to or subtracted from J_κ , reducing the number of components of $|J_\kappa|$. Finite repetition of this process produces the desired weakly groomed κ -juncture J_κ^* . \square

We fix a choice of J_κ satisfying the conclusions of Lemma 2.18. In particular, through each component of $|J_\kappa|$ there is a transverse loop that does not intersect any other component of $|J_\kappa|$. The following is a consequence of Lemma 2.18 and Lemma 2.13.

Corollary 2.19. *Each weight w_s is divisible by p_e .*

We can assume that our basepoint $*$ is disjoint from $|J_\kappa|$. Let F' denote the compact, connected surface with boundary (and possibly corners) obtained by cutting F apart along the components of $|J_\kappa|$. Then each component s of $|J_\kappa|$ determines two copies s_\pm of itself in $\partial F'$, s_+ being the copy along which the transverse orientation points out of F' and s_- the one along which the transverse orientation is inward.

Loops in F based at $*$ which do not properly intersect $|J_\kappa|$ remain loops in F' based at $*$. The following is an easy consequence.

Lemma 2.20. *For each $n \in \mathbb{Z}$, there is a unique copy F_n of F' embedded in \mathcal{U}_c and containing x_n .*

The projection $q : F_n \rightarrow F$ induces a homeomorphism $q' : F_n \rightarrow F'$. The curve (an arc or circle) $s_\pm^n \subset \partial F_n$ is the one carried by q' onto $s_\pm \subset \partial F'$.

Lemma 2.21. *For each $n \in \mathbb{Z}$ and each component s of $|J_\kappa|$, F_n is attached to F_{n+w_s} by an identification $s_+^n \equiv s_-^{n+w_s}$.*

Proof. By Lemma 2.18, there is a loop σ in F , based at $*$, which intersects s once and has algebraic intersection number $+1$ at that point. Thus, σ lifts to a path joining x_n to x_{n+w_s} , exiting F_n through s_+^n and entering F_{n+w_s} through $s_-^{n+w_s}$. \square

List the components of $|J_\kappa|$ as s_1, s_2, \dots, s_p . Write $w_{s_i} = w_i$. The juncture components s_{i-}^n , $1 \leq i \leq p$, are inwardly oriented components of $\text{Fr } F_n$ and the juncture components s_{i+}^n , $1 \leq i \leq p$, are outwardly oriented ones.

Let

$$V_0 = F_0 \cup F_{p_e} \cup F_{2p_e} \cup \cdots \cup F_{kp_e} \cup \cdots .$$

Lemma 2.22. *For each integer $1 \leq i \leq p$, the number of inwardly oriented components of $\text{Fr } V_0$ of the form s_{i-}^j is w_i/p_e .*

Proof. Fix s_i . Since F_{kp_e} is attached to $F_{kp_e+w_i}$ by identifying $s_{i+}^{kp_e}$ with $s_{i-}^{kp_e+w_i}$, it is clear that exactly the w_i/p_e inwardly oriented components s_{i-}^j of F_j ,

$$j = 0, p_e, 2p_e, \dots, w_i - p_e,$$

are in $\text{Fr } V_0$. □

Thus, the union of the components s_{i-}^j in $\text{Fr } V_0$ is a compact, properly embedded, transversely oriented 1-manifold. We take $U_e = V_0$ as f -neighborhood of e in Proposition 2.12 and $J = \text{Fr } V_0$.

Corollary 2.23. *The image of J under the covering projection $q : \mathcal{U}_c \rightarrow F$ is exactly $|J_\kappa|$, each component s of $|J_\kappa|$ being the image of exactly w_s/p_e components of J .*

Since f is a deck transformation, this remains true for all $f^n(J)$, all $n \in \mathbb{Z}$, and it is clear that any two of these positive f -junctures intersect, if at all, only in common components.

The proof of Proposition 2.12 is now complete.

Hypothesis 2. Hereafter, we will require that any f -juncture J is a compact 1-manifold and has the juncture intersection property.

That is, hereafter we are modifying the definition of f -juncture (Definition 2.9) to require that any f -juncture J be a compact 1-manifold and have the juncture intersection property.

Remark. Note that $q : V_0 \rightarrow F$ is a semi-covering with covering semi-group \mathbb{Z}^+ generated by f . It mirrors perfectly the way that an end e of a depth one leaf of a foliation semi-covers the compact leaf F to which it is asymptotic. (One commonly says that e “spirals” on F .)

2.2. Behavior of an endperiodic automorphism near the ends of L . An endperiodic automorphism is well behaved near the ends of L . In this subsection we give some lemmas that show the behavior can be more complex than expected. Example 2.29 shows how complicated the structure of a positive or negative end can be.

Remark. The notation and terminology defined in the next paragraph is used only in Section 2.

It is sometimes necessary to work with a positive multiple kp_e of p_e as power of f . If e is a positive (respectively negative) end, choose an f -neighborhood U_e . Let p be a positive multiple of p_e and define $U_e^i = f^{ip}(U_e)$ (respectively $U_e^i = f^{-ip}(U_e)$), $J_e^i = \text{Fr } U_e^i$, and $B_e^i = \overline{U_e^i} \setminus U_e^{i+1}$, $i \in \mathbb{Z}$.

Remark. This notation depends on the choice of positive multiple p of p_e .

Remark. The notation is such that all the sets $U_e^n, B_e^n, J_e^n \subset U_e$ and tend to e in the topology of $L \cup \mathcal{E}(L)$ as $n \rightarrow +\infty$ whether e is a positive or negative end. We will also use the notation introduced above, $J_n = f^n(J)$, for J an f -juncture. For J a positive f -juncture, J_n will approach a cycle of positive ends as $n \rightarrow +\infty$ and for J a negative f -juncture, J_n will approach a cycle of negative ends as $n \rightarrow -\infty$.

Definition 2.24 (f -domain). The set $B_e^i, i \in \mathbb{Z}$, is the f -domain for e corresponding to p .

Lemma 2.25. If f be an endperiodic automorphism, then the following are equivalent,

- (1) If e is a positive end, $f^p(U_e) = U_e^1 \subset \text{int } U_e$.
If e is a negative end, $f^{-p}(U_e) = U_e^1 \subset \text{int } U_e$.
- (2) $U_e^{i+1} \subset \text{int } U_e^i, i \in \mathbb{Z}$.
- (3) $J_e^i \cap J_e^{i+1} = \emptyset, i \in \mathbb{Z}$.
- (4) $\text{Fr } B_e^i = J_e^i \cup J_e^{i+1}, i \in \mathbb{Z}$.
- (5) B_e^i separates $L, i \in \mathbb{Z}$.

Proof. Clearly (2) \Leftrightarrow (1). In fact (1) is a special case of (2) with $i = 0$ and (2) follows from (1) since f is a homeomorphism. We will show that (2) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2).

(2) \Rightarrow (5). Since $U_e^{i+1} \subset \text{int } U_e^i$, it follows that $J_e^i = U_e^i \setminus \text{int } U_e^i \subset U_e^i \setminus U_e^{i+1} \subset B_e^i$. Since J_e^i separates L, B_e^i separates L .

(5) \Rightarrow (4). First remark that since U_e^{i+1} is a connected subset of a surface with $\text{Fr } U_e^{i+1}$ a compact 1-manifold, $\text{int } U_e^{i+1}$ is connected. Further it follows from the definition of f -neighborhood that $L \setminus U_e^i$ is connected. By (5), $L \setminus B_e^i$ is not connected so $L \setminus B_e^i$ is the union of nonempty disjoint open sets U, V . Since $\text{int } U_e^{i+1}$ and $L \setminus U_e^i$ are connected, only one of U, V can meet each of $\text{int } U_e^{i+1}$ and $L \setminus U_e^i$. It follows that $U = L \setminus U_e^i$ and $V = \text{int } U_e^{i+1}$. Therefore

$$L \setminus B_e^i = U \cup V = (L \setminus U_e^i) \cup \text{int } U_e^{i+1} = L \setminus (U_e^i \setminus \text{int } U_e^{i+1})$$

so $B_e^i = U_e^i \setminus \text{int } U_e^{i+1} \supset J_e^i \cup J_e^{i+1}$. Further, if $x \in J_e^i$ (respectively $x \in J_e^{i+1}$) then x has a neighborhood meeting $L \setminus U_e^i$ (respectively $\text{int } U_e^{i+1}$). Thus, every $x \in J_e^i \cup J_e^{i+1}$ has a neighborhood meeting both B_e^i and its complement so $J_e^i \cup J_e^{i+1} \subset \text{Fr } B_e^i$. The reverse containment is clear so (4) follows.

(4) \Rightarrow (3). If $J_e^i \cap J_e^{i+1} \neq \emptyset$, then by assumption J_e^i and J_e^{i+1} have a component σ in common. If $x \in \text{int } \sigma$, then the point x has a neighborhood V disjoint from $U_e^i \setminus U_e^{i+1}$. Thus, $x \notin B_e^i$ so $x \notin \text{Fr } B_e^i$ contradicting (4).

(3) \Rightarrow (2). If $J_e^i \cap J_e^{i+1} = \emptyset$, then $J_e^{i+1} \subset \text{int } U_e^i$ so $U_e^{i+1} \subset \text{int } U_e^i$. \square

Lemma 2.26. The integer $p > 0$ of Lemma 2.8 can be chosen so that the B_e^i are connected.

Proof. Without loss assume that e is a positive end. Suppose that

$$B_e^0 = \overline{U_e \setminus f^p(U_e)}$$

has more than one connected component. By Definition 2.3, $\text{Fr } B_e^0$ is a compact 1-manifold, hence has finitely many components. Since U_e is connected, we find finitely many paths s_1, \dots, s_r in U_e that connect the components of B_e^0 . That is $B_e^0 \cup s_1 \cup \dots \cup s_r$ is connected. For a minimal integer $m \geq 1$, these paths

all lie in $B = B_e^0 \cup B_e^1 \cup \cdots \cup B_e^m$ and we claim that B is connected. Indeed, each component of B_e^1 attaches to B_e^0 along at least one component of J_e^1 , and so $B_e^0 \cup B_e^1 \cup s_1 \cup \cdots \cup s_r \subset B$ is connected. Repeating this reasoning finitely often, we obtain that B is connected. If we replace the integer p of Lemma 2.8 with $q = p(m+1)$, then $B = \overline{U_e \setminus f^q(U_e)}$ can be taken as a new connected B_e^0 . \square

We introduce the following nonstandard term which will come up frequently throughout this paper.

Definition 2.27 (simple end). An end of L is *simple* if it has a neighborhood homeomorphic to $S^1 \times [0, \infty)$ or $[0, 1] \times [0, \infty)$.

The next lemma follows from the fact that L has no simple ends.

Lemma 2.28. *If L has no simple ends and $p > 0$ is large enough, then the B_e^i have negative Euler characteristic.*

Example 2.29. This example is meant to illustrate how complex the structure of a positive or negative end might be.

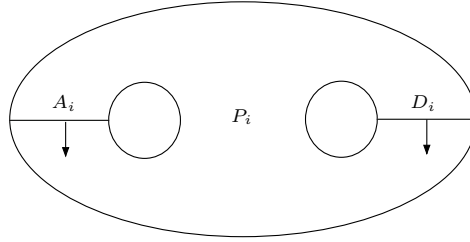


FIGURE 3. A pair of pants P_i

We describe a family of two-ended surfaces and an endperiodic automorphism on each which is a translation (this term will be carefully defined in Section 2.5). The surface $L = \bigcup_{i \in \mathbb{Z}} P'_i$ will be formed by cutting pairs of pants P_i along certain essential, properly embedded subarcs and then pasting the resulting disks P'_i to one another along these subarcs. The endperiodic automorphism $f : L \rightarrow L$ will take P'_i to P'_{i+1} .

In Figure 3, we depict the typical pair of pants and the essential arcs with transverse orientation. After cutting, P_i becomes a disk P'_i , with A_i and D_i split and indexed as indicated in Figure 4. Here i varies over the integers. Let m and n be fixed, relatively prime integers, positive and/or negative. The index i on A_i^+ , D_i^+ indicates that these arcs are identified with the original A_i and D_i , while the index on A_{i+m}^- indicates that it is to be attached to A_{i+m}^+ , forming a single arc to be labeled A_{i+m} , and the index on D_{i+n}^- indicates that it is to be attached to D_{i+n}^+ , forming a single arc to be labeled D_{i+n} . It will be convenient to represent P'_i and its boundary arcs symbolically as X_i and its vertices as in Figure 5.

The case $m = 2$ and $n = 3$ is indicated symbolically by the graph in Figure 6. The endperiodic automorphism carries X_i to X_{i+1} . An f -neighborhood of the positive end e is given by $U_e = P'_0 \cup P'_1 \cup \cdots$ and is represented symbolically as $X_0 \cup X_1 \cup \cdots$ in Figure 6. Thus, $p_e = 1$. The f -juncture $J_e^0 = \text{Fr } U_e$ consists of the five properly embedded arcs represented by the points A_0, A_1, D_0, D_1, D_2 in

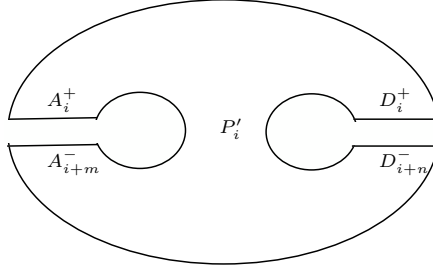


FIGURE 4. Arcs A^\pm, D^\pm with indices

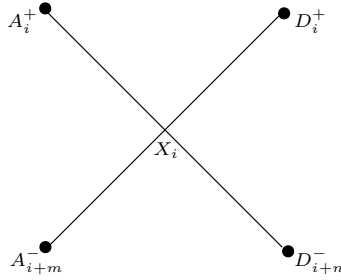


FIGURE 5. A symbolic representation of Figure 4

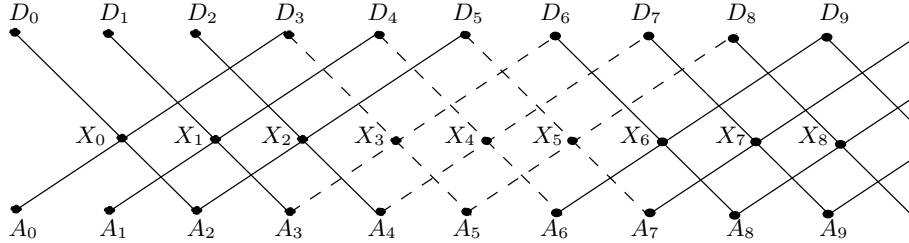


FIGURE 6. Symbolic picture of the surface with $m = 2, n = 3$

Figure 6. As usual, let $U_e^i = f^{ip}(U_e)$, $J_e^i = f^{ip}(J_e^0) = \text{Fr } U_e^i$, and $B_e^i = \overline{U_e^i \setminus U_e^{i+1}}$, $i \geq 0$ where p is a positive integer multiple of $p_e = 1$.

If $p = 1$ or 2 , the U_e^i , J_e^i , and B_e^i do not satisfy the conditions of Lemma 2.25. In particular, $J_e^0 \cap J_e^1 \neq \emptyset$ and B_e^0 does not separate L .

If $p = 3$, then B_e^0 is symbolically represented by $X_0 \cup X_1 \cup X_2$ in Figure 6 where it is drawn with solid lines, B_e^1 is represented by $X_3 \cup X_4 \cup X_5$ in Figure 6 where it is drawn with dotted lines and B_e^2 is represented by $X_6 \cup X_7 \cup X_8$ in Figure 6 where it is drawn with solid lines. In this case, B_e^0 does separate L , so the U_e^i , J_e^i , and B_e^i do satisfy the conditions of Lemma 2.25. However, B_e^0 has two components.

If $p \geq 4$, the U_e^i , J_e^i , and B_e^i do satisfy the conditions of Lemma 2.25 and B_e^0 is connected. If $p = 4$, B_e^0 is symbolically represented by $X_0 \cup X_1 \cup X_2 \cup X_3$ in Figure 6

and is a disk. If $p = 5$, B_e^0 is symbolically represented by $X_0 \cup X_1 \cup X_2 \cup X_3 \cup X_4$ in Figure 6 and is an annulus. If $p \geq 6$, B_e^0 has negative Euler characteristic, separates L , and is connected and $\text{Fr } B_e^0 = J_e^0 \cup J_e^1$ with $J_e^0 \cap J_e^1 = \emptyset$.

2.3. Properties of endperiodic automorphisms. We prove some elementary facts about endperiodic automorphisms.

Proposition 2.30. *If L has at least one nonsimple end, then the endperiodic automorphism $f : L \rightarrow L$ has both positive and negative ends.*

Proof. Assume that L has no negative ends. Let e be a positive end and let U_e be an f -neighborhood of e . Let B_e^i , $i \in \mathbb{Z}$, be the f -domains (Definition 2.24) for e corresponding to a choice of p , a large enough multiple of p_e that it can serve as a choice of p in Lemmas 2.8, 2.26 and 2.28. Thus, $B_e^{i+1} = f^p(B_e^i)$. By Lemma 2.28, the f -domains B_e^i for the end e have negative Euler characteristics. Then the sequence $\{B_e^i\}_{i=0}^{-\infty}$ is a sequence of compact surfaces with disjoint interiors, each of negative Euler characteristic, which does not accumulate at any end. This gives the contradiction that some compact subsurface $S \subset L$ has infinite Euler characteristic. \square

Remark. In particular, L must have at least two ends. The argument of this proof obviously fails if all ends are simple. For example, it is easy to produce an endperiodic automorphism on \mathbb{R}^2 with one negative end and one on $I \times \mathbb{R}$ with two negative ends.

Remark. Each end of every noncompact boundary component of a surface L limits on some end of L .

Remark. One might think that at most finitely many noncompact boundary components can have an end limiting on a given end of L . But see [4, Figure 12.5.11] for a surface with one end e and infinitely many noncompact boundary components with both ends limiting on e . Of course, this surface does not admit an endperiodic automorphism.

Lemma 2.31. *If $f : L \rightarrow L$ is endperiodic, then, only finitely many noncompact boundary components can limit on a given end e of L .*

Proof. Since L has finitely many ends, e is a periodic end. Let U_e be an f -neighborhood of e . Consider the f -junctures $\{J_e^i\}_{i=1}^{\infty}$, a sequence of mutually homeomorphic 1-manifolds. If J_e^i is a collection of simple closed curves, no noncompact boundary component limits on e . In general, J_e^i contains finitely many arc components and the finite number of endpoints of these components is independent of i . Each noncompact boundary component σ of L with an end limiting on e contains an endpoint of an arc component of J_e^i for some i and if σ contains an endpoint of an arc component of J_e^i , it contains an endpoint of an arc component of J_e^{i+r} , $r \geq 0$. Thus, if infinitely many boundary components of L have an end limiting on e , the number of endpoints of arc components of J_e^i is unbounded which is a contradiction. \square

Remark. It should be noted that it is possible that a component of ∂L might be a line ℓ joining a positive end to a positive end (possibly the same end) with the parallel possibility for negative ends. In that event, f will have at least one periodic point on ℓ . It will turn out that ℓ will then be a leaf of one of the two

transverse laminations to be constructed in Section 4, while a ray issuing from the periodic point will be a leaf of the other. This causes some difficulties for us, but the problem can be eliminated without loss of generality by doubling along all such components ℓ (and only along such components). Denote the partially doubled surface by L' and the corresponding partial double of f by f' . The original surface remains as an f' -invariant subsurface in L' . The remaining noncompact boundary components which issue from periodic ends will join negative ends to positive ones and it is easy to modify the endperiodic automorphism near these boundary lines so that there are no periodic points on them. Accordingly, we make the following assumption.

Hypothesis 3. Each noncompact component of ∂L , joins a negative end to a positive end and contains no periodic point.

2.4. Tunneling. A certain special class of geometric modifications of junctures that leave them unchanged homologically is a process which we call “tunneling”. In Figure 7 we represent tunneling between components τ_1 and τ_2 of J_κ along an arc α issuing from τ_1 and ending at τ_2 and not otherwise meeting J_κ so that, whatever orientation is given to α , it agrees with the transverse orientation of $\tau_1 \cup \tau_2$ at one end and opposes it at the other. In this paper we only need this in the proof of the transfer theorem (Theorem 12.7).

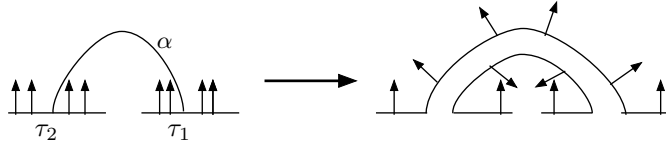


FIGURE 7. A tunneling cohomology

Let $\sigma \subset \text{int } F$ be an embedded, oriented circle. Suppose that σ intersects J_κ transversely and in finitely many points. Let $\sigma \cdot J_\kappa$ denote the algebraic intersection number. Let the transversely oriented components of $|J_\kappa|$ be τ_j with respective weights w_j , $1 \leq j \leq r$.

By tunneling we will prove the following.

Proposition 2.32. For any $\sigma \subset \text{int } F$, an embedded, oriented circle that intersects a κ -junction J_κ transversely and in finitely many points, there is a new choice of κ -junction J_κ , also intersecting σ transversely and in finitely many points, such that

- (1) $|\sigma \cdot J_\kappa| = \sum_{j=1}^r w_j \text{card}(\sigma \cap \tau_j)$,
- (2) J_κ is weakly groomed,
- (3) $|J_\kappa|$ does not separate F .

Proof. Realize J_κ not as a union of weighted, transversely oriented arcs and circles, but of packets of parallel, coherently transversely oriented arcs and circles. If the asserted equality fails, there is a subarc α of σ with endpoints x_1, x_2 on respective component(s) τ_1, τ_2 of J_κ , not intersecting J_κ in any other points, such that the orientation of α at, say, x_1 disagrees with the transverse orientation of τ_1 at that point and, at x_2 , the orientation of α and the transverse orientation of τ_2 agree.

Let A be a rectangle with a pair of opposite sides $\beta_i \subset \tau_i$ containing the points x_i in their interiors, $i = 1, 2$, and a pair of opposite sides α_i parallel to α with $\text{int } \alpha \subset \text{int } A$. Clearly A can be chosen to be disjoint from $\sigma \setminus \alpha$. Transversely orient the circle ∂A so that the orientation points outward from A if α meets τ_1 and τ_2 on the positive sides and, in the alternative case, let ∂A be transversely oriented into A . Let τ denote the cocycle which is the transversely oriented ∂A . This cocycle is clearly a coboundary and so the cocycle $J_\kappa + \tau$ is cohomologous to J_κ . The overlap of τ and τ_i is two copies of β_i with opposite transverse orientation, hence can be “erased”, $i = 1, 2$. The resulting transversely oriented, properly embedded 1-manifold J_κ^* is as in Figure 7 and the intersection points x_1, x_2 which introduced cancelling intersection numbers ± 1 have been eliminated without changing the other intersections of σ_k with J_κ^* , $1 \leq k \leq n$. Finite repetition produces J_κ^* such that the algebraic intersection numbers at each point of $\sigma_i \cap J_\kappa^*$ all have the same sign.

Finally, the modifications in Lemma 2.17 and Lemma 2.18 only involve throwing away components of κ -juncture, hence do not affect the intersection properties already established. \square

As an example of tunneling, the reader might try proving the following result which is not consequential for this paper but might be useful in studying examples.

Lemma 2.33. *The κ -juncture, chosen with all the above properties, can be assumed to be represented entirely by weighted, transversely oriented, properly embedded arcs, or one weighted, transversely oriented circle.*

2.5. Translations. The simplest endperiodic automorphisms are the translations, although even these can be surprisingly complicated (check out Example 2.29).

Definition 2.34 (translation). The endperiodic automorphism is a *translation* if (in the notation of Section 2.1) L has an end e such that $\mathcal{U}_e = L$.

Remark. In the case of a translation, L has one positive end and one negative end.

Suppose that f is a translation. It is evident that L has two ends, an attracting end e and a repelling end e' and $p_e = 1$. Lemma 2.20 gives a sequence $\{F_n\}$ of copies of F' embedded in $\mathcal{U}_e = L$ with $f(F_n) = F_{n+1}$ for $n \in \mathbb{Z}$ and $L = \bigcup_{n=-\infty}^{\infty} F_n$. It is natural to think of the F_n 's as “fundamental domains” for the homeomorphism f . Example 2.29 shows that, in general, F_n does not separate L and can be attached to more than two of the F_j 's along common boundary components.

We make the following definition only for use in the next two lemmas

Definition 2.35. If the F_n 's can be chosen to separate L and to be attached to exactly two of the F_j 's along common boundary components, we say that the translation f is a *simple translation*.

An immediate consequence of Lemmas 2.25 and 2.8 is the following.

Lemma 2.36. *If f is a translation and k is sufficiently large, then f^k is a simple translation.*

In case $\partial L = \emptyset$, hence $\partial F = \emptyset$, and f is a translation, the κ -juncture can always be chosen to be a simple closed curve s (Lemma 2.33). As a consequence, we note the following.

Lemma 2.37. *If $\partial L = \emptyset$ and $f : L \rightarrow L$ is a translation, then f is a simple translation.*

2.6. Avoiding the juncture intersection property. It was pointed out to us by the anonymous referee of [16] that, by a small isotopy of the f -junctures in \mathcal{U}_c , it can be assumed that every two f -junctures are disjoint. This has the effect of allowing part (1) of Definition 2.2 to be strengthened to read:

$$f^{p_e}(U_e) \subset \text{int } U_e.$$

While this may seem desirable, Example 2.29 suggests that it is a bit contrived and in the context of depth one foliations it is decidedly unnatural. Our asymptotic construction of f -junctures is entirely motivated by depth one foliations. We record the fact for its possible usefulness, but continue to stick with Definition 2.2 as stated and the juncture intersection property.

3. PRELIMINARIES TO THE HANDEL-MILLER THEORY

We present here some material that will be appealed to repeatedly.

Temporary Hypothesis. In Section 3, we assume that L is a noncompact, connected surface with finitely or infinitely many ends and possibly with boundary.

3.1. Some remarks on isotopies. In constructing isotopies in this paper, we will frequently use the Epstein-Baer theorems [20, Theorem 2.1 and 3.1] about homotopies and isotopies of curves. For convenient reference we state them here and refer the reader to [20] for the proofs. In what follows, L is a connected surface, compact or noncompact, with or without boundary and orientable or nonorientable. Recall that an ambient isotopy on M is a continuous map $\Phi : M \times I \rightarrow M$, written $\Phi(x, t) = \Phi^t(x)$, where $\Phi^0 = \text{id}$, with $\Phi^t : M \rightarrow M$ a homeomorphism, $0 \leq t \leq 1$.

Remark. Whether we say so or not, all of our isotopies will be ambient.

Theorem 3.1 (Epstein-Baer). *Let $\alpha, \beta : S^1 \rightarrow \text{int } L$ be freely homotopic, embedded, 2-sided, essential circles. Then there is an ambient isotopy $\Phi : L \times I \rightarrow L$, compactly supported in $(\text{int } L) \times I$, such that $\Phi^1 \circ \beta = \alpha$.*

Theorem 3.2 (Epstein). *Let $\alpha, \beta : [0, 1] \rightarrow L$ be properly embedded arcs with the same endpoints which are homotopic modulo the endpoints. Then there is an ambient isotopy $\Phi : L \times I \rightarrow L$, compactly supported in $L \times I$ with $\Phi(x, t) = x$ for $(x, t) \in \partial L \times I$, such that $\Phi^1 \circ \beta = \alpha$.*

Epstein proves these first in the PL category, then extends them to the category TOP by an approximation argument. The PL argument adapts very well to the smooth category DIFF, giving the following.

Theorem 3.3. *In the above theorems, if the curves in question are smooth embeddings, then the isotopies can be chosen to be smooth.*

Remark. One will frequently want to perform these isotopies sequentially on a possibly infinite sequence of disjoint curves. In order to make sure that already completed isotopies are not undone by a subsequent one, one resorts to the following inductive trick. If $\sigma_1, \sigma_2, \dots, \sigma_n$ are the resulting curves from isotopies already performed and τ is a curve to be isotoped to σ_{n+1} , both of these curves being disjoint from $S = \sigma_1 \cup \sigma_2 \cup \dots \cup \sigma_n$, then cut L apart temporarily along S and perform the Epstein-Baer isotopies in the resulting surface L' . Of course, we are assuming that τ and σ_{n+1} lie in a common component of L' .

3.2. Standard hyperbolic metrics. In order to construct and analyze the Handel-Miller laminations, it is necessary to introduce a hyperbolic metric on L . Here we give necessary definitions and prove a key result (Theorem 3.11). In [15] we generalized to arbitrary noncompact surfaces, with suitable hyperbolic metric, theorems we need that are well-known for hyperbolic surfaces of finite area.

Let L be a connected surface, compact or not, with or without boundary and orientable or not. If L has boundary and a complete hyperbolic metric making all components of ∂L geodesics, it is well known that the double $2L$ has a canonical hyperbolic metric which is complete and agrees with the given one on $L \subset 2L$.

If L has empty boundary, the open unit disk Δ , with its canonical hyperbolic metric, is the universal cover \tilde{L} . If $\partial L \neq \emptyset$, view $L \subset 2L$, identify $\tilde{2L} = \Delta$ and choose a lift $\tilde{L} \subset \Delta$ of L . The projection $\pi : \Delta \rightarrow 2L$ restricts to give the universal cover $\pi : \tilde{L} \rightarrow L$. The lifts \tilde{L} are permuted transitively by the group of deck transformations of $2L$, giving all the choices of embeddings of the universal cover of L in Δ . Those deck transformations leaving a given \tilde{L} invariant restrict to define the group of deck transformations for $\pi : \tilde{L} \rightarrow L$.

Let $\mathbb{D}^2 = \Delta \cup S_\infty^1$, the closed unit disk with boundary S_∞^1 the unit circle, known as the ‘‘circle at infinity’’. If $A \subset \mathbb{D}^2$, let \hat{A} denote the closure of A in \mathbb{D}^2 . In particular, if $\sigma \subset \Delta$ is a geodesic, $\hat{\sigma}$ will be the compact, properly embedded arc in \mathbb{D}^2 obtained by adjoining to σ its endpoints in S_∞^1 . This is called the completion of σ . We also let $\hat{\tilde{L}}$ denote the closure of \tilde{L} in \mathbb{D}^2 . This is called the completion of \tilde{L} .

Definition 3.4 (ideal boundary E). The set $E = \hat{L} \cap S_\infty^1$ is called the *ideal boundary* of L or of \tilde{L} .

Either $\partial L = \emptyset$ and $E = S_\infty^1$, or $\partial L \neq \emptyset$ and E is a compact subset of S_∞^1 . Note that the completions $\hat{\gamma}$ of the components γ of $\partial \tilde{L}$ have endpoints in E .

Definition 3.5 (standard metric, standard surface). A Riemannian metric μ on L is *standard* if it is a complete hyperbolic metric making all components of ∂L geodesics and admitting no isometrically embedded, hyperbolic half-planes (for short, no half-planes). The pair (L, μ) is called a *standard hyperbolic surface*. If L is a surface admitting a standard metric μ , we say that L is a *standard surface*.

This is nonstandard terminology.

By a hyperbolic half-plane, we mean, of course, the union in the Poincaré disk Δ of a geodesic γ and one of the components of $\Delta \setminus \gamma$. In [15, Theorem 8], we prove that, up to homeomorphism, there are exactly 13 nonstandard surfaces. None of these are interesting from the point of view of endperiodic theory.

In [15, Lemma 1] we show that the hyperbolic metric on $2L$ is standard if and only if the hyperbolic metric on L is standard.

Hypothesis 4. In this paper, all hyperbolic metrics are standard, save mention to the contrary.

3.3. Escaping curves. It is well known that if $\sigma \subset L$ is an essential closed curve which does not bound a cusp, then it is freely homotopic to a unique closed geodesic σ^g . Similarly, if σ is a boundary incompressible, properly embedded arc, it is homotopic to a unique, properly embedded geodesic arc σ^g , where the homotopy keeps the endpoints fixed.

Definition 3.6 (geodesic tightening). In either of the above cases, we say that σ^g is the geodesic tightening of σ .

Definition 3.7 (escapes). Let $\{A_k\}$, be a sequence of subsets of L , indexed either by the nonnegative integers $k \geq 0$, the nonpositive integers $k \leq 0$, or all integers $k \in \mathbb{Z}$. We say that $\{A_k\}$ *escapes* if, for every compact subset $K \subset L$, $A_k \cap K = \emptyset$ for all but finitely many values of k .

Example 3.8. If e is a positive end and J is an f -juncture corresponding to e , then $\{f^k(J)\}_{k \geq 0}$ escapes, but, except in trivial cases, $\{f^k(J)\}_{k \leq 0}$ does not escape. A similar remark, with opposite signs, applies to negative ends.

Note that, by Hypothesis 3, if an f -juncture J is an arc, then the countable set of endpoints of arc components of the set of f -junctures $\{f^k(J)\}_{k \in \mathbb{Z}}$ escapes.

Definition 3.9 (virtually escapes). Let $\{\sigma_k\}$ be a sequence of essential simple loops not bounding a cusp or a sequence of properly embedded, boundary incompressible arcs, indexed either by the nonnegative integers $k \geq 0$, the nonpositive integers $k \leq 0$, or all integers $k \in \mathbb{Z}$. We say that $\{\sigma_k\}$ *virtually escapes* if the sequence of geodesic tightenings $\{\sigma_k^g\}$ escapes.

Example 3.10. A hyperbolic half-plane causes behavior we need to exclude. In Figure 8, suppose f is a translation as indicated. The sequence of positive iterates under f of an f -juncture escapes but does not virtually escape. Their geodesic tightenings (see Definition 3.6), which cannot intersect the hyperbolic half-plane H , accumulate on ∂H . The f -junctures are depicted by dashed circles and the geodesic tightening of one of them is depicted by the boldfaced dashed circle.

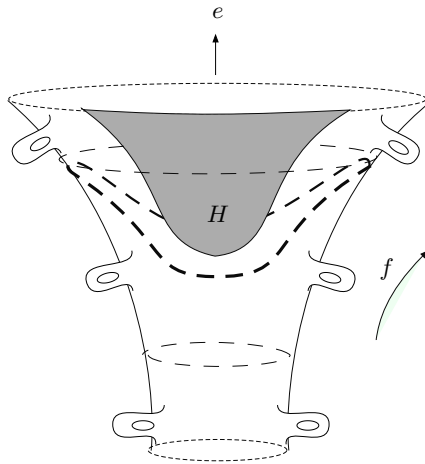


FIGURE 8. A half-plane H prevents the escaping sequence $\{f^k(J)\}_{k \geq 0}$ of f -junctures from virtually escaping

Theorem 3.11. In a standard hyperbolic surface L , if $\{\sigma_k\}$ is a sequence of essential simple loops not bounding a cusp or a sequence of properly embedded, boundary incompressible arcs and $\{\sigma_k\}$ escapes, then $\{\sigma_k\}$ *virtually escapes*.

Remark. The converse is obviously false.

Evidently, it will be enough to carry out the proof for the case of a sequence $\{\sigma_k\}_{k \geq 0}$.

Theorem 3.11 will be proven in a series of lemmas. We will suppose that the assertion fails and show that L then contains an isometrically embedded hyperbolic half-plane. This will contradict our assumption that (L, μ) is a standard hyperbolic surface.

Fix a realization of the universal cover $\tilde{L} \subseteq \Delta$.

Suppose that the sequence $\{\sigma_n\}$ escapes but does not virtually escape. Let σ_n^g be the geodesic tightening of σ_n . Then there is a compact set K such that infinitely many of the σ_n^g 's meet K . Passing to a subsequence, assume that they all do.

If only finitely many of the σ_n^g are distinct, then there exists a subsequence of the sequence $\{\sigma_n\}$ consisting of freely homotopic loops. It follows that \tilde{L} has an end e with a neighborhood homeomorphic to an open annulus. Then e is either a cusp bounded by all the σ_n in the subsequence contrary to the hypotheses of the theorem or a flaring end contrary to Hypothesis 4. Thus, infinitely many of the σ_n^g are distinct and, by passing to a subsequence, we assume they all are. Since the sequence $\{\sigma_n\}$ escapes and each σ_n is compact, by passing to a subsequence, we can further assume that the σ_n are pairwise disjoint.

Pick $y_n \in \sigma_n^g \cap K$. Then $\{y_n\}$ has a subsequence that converges to $x \in K$ and we reindex so that this subsequence is indexed by $n \in \mathbb{Z}^+$. Let D be a disk neighborhood of x .

Let s be a geodesic arc through x in D which meets every σ_n^g transversely. On at least one side of x in s , infinitely many σ_n^g intersect s in a closest point x_n to x and, again passing to a subsequence if necessary, we can assume that $x_n \rightarrow x$ monotonically as $n \rightarrow \infty$. Fix a lift $\tilde{x} \in \tilde{L}$ of x . This fixes lifts \tilde{D} , \tilde{s} , \tilde{x}_n and $\tilde{\sigma}_n^g$. If $\partial L = \emptyset$, then $\partial \tilde{L} = S_\infty^1$, and if $\partial L \neq \emptyset$, then $\partial \tilde{L}$ is a fractal curve homeomorphic to a circle.

Definition 3.12 (S_∞). Whether or not $\partial L = \emptyset$, the circle $\partial \tilde{L}$ will be denoted by S_∞ .

Each $\tilde{\sigma}_n^g$ cuts off two regions in \tilde{L} , one of which contains \tilde{x} and will be denoted by \tilde{H}_n . The σ_n were chosen above to be disjoint. It follows that the $\tilde{\sigma}_n^g$ are disjoint and have no common endpoints on S_∞ . If the σ_n^g are arcs this is clear and if the σ_n^g are closed loops this follows since distinct closed geodesics in hyperbolic geometry can not share ideal points at infinity. Therefore, $\tilde{H}_{n+1} \subset \tilde{H}_n$, a proper inclusion for all $n \geq 0$. The following, then, is evident.

Lemma 3.13. *The endpoints a_n, b_n of $\tilde{\sigma}_n^g$ lie in $\partial \tilde{L}$ and are the endpoints of a nested sequence of arcs,*

$$S_\infty \supset [a_0, b_0] \supset [a_1, b_1] \supset \cdots \supset [a_n, b_n] \supset \cdots,$$

where each $[a_n, b_n] \subset (a_{n-1}, b_{n-1})$. Consequently, the sequences $\{a_n\}, \{b_n\}$ converge strictly monotonically to points $a, b \in \partial \tilde{L}$.

Lemma 3.14. *The points a, b are in E and distinct.*

Proof. The $\tilde{\sigma}_n^g$ are geodesics in \tilde{L} passing through an arbitrarily small neighborhood of \tilde{x} . By elementary properties of the Poincaré model, the endpoints cannot be

converging to the same point on S_∞ . If all σ_n^g are closed, all $a_n, b_n \in E$, hence $a, b \in E$. If they are properly embedded arcs, the sequences of iterates of their endpoints escape, and so neither a_n nor b_n can be approaching a point on $\partial\tilde{L}$. \square

The points $a, b \in E$ are endpoints of a geodesic $\tilde{\gamma} \subset \tilde{L}$ which projects to a geodesic $\gamma \subset L$ containing the point x .

Lemma 3.15. *The geodesic γ is simple.*

Proof. If γ is not simple, another lift $\bar{\gamma}$ properly intersects $\tilde{\gamma}$. Then the sequence of lifts $\{\tilde{\sigma}_n\}$ converging on $\tilde{\gamma}$ and corresponding sequence of lifts $\{\bar{\sigma}_n\}$ converging on $\bar{\gamma}$ will force proper intersections $\tilde{\sigma}_n \cap \bar{\sigma}_n$. This forces σ_n to self intersect for n large enough. \square

Lemma 3.16. *The geodesic γ is not closed.*

Proof. If γ is closed, then $\tilde{\gamma}$ is the axis of a deck transformation T . Passing to T^2 , if necessary, assume that T is orientation preserving. Recall that the endpoints of $\tilde{\gamma}$ are $a, b \in S_\infty^1$. Assume that b attracts and a repels. Let $\tilde{\sigma}_n^g$ be the lift of the geodesic tightening of σ_n with endpoints $a_n, b_n \in S_\infty$. The sequence $\{\tilde{\sigma}_n^g\}$ converges uniformly to $\tilde{\gamma}$ in the Euclidean metric on Δ . The point $T(a_n)$ is further from a on the circle S_∞ and the point $T(b_n)$ is closer to b on the circle S_∞ . Thus, for n sufficiently large, $\tilde{\sigma}_n^g \cap T(\tilde{\sigma}_n^g) \neq \emptyset$. This implies that σ_n^g is not a simple curve contrary to assumption. \square

Let $H \subset \Delta$ be the half-plane with boundary $\tilde{\gamma}$ that is disjoint from every $\tilde{\sigma}_k^g$. The intersection $\hat{H} \cap S_\infty^1$ is a compact, nondegenerate subarc $A \subset S_\infty^1$. We will denote this subarc A by $[a, b]_\infty$ to distinguish it from the arc $[a, b] \subset \partial\tilde{L}$ which are different if $\partial L \neq \emptyset$.

Lemma 3.17. *$H \subset \tilde{L}$*

Proof. The lemma is trivial if $\partial L = \emptyset$ so we assume that $\partial L \neq \emptyset$. Let τ be the geodesic in Δ which contains the geodesic $\tilde{\sigma}_0^g \subset \hat{L}$. Let H^* be the half-plane in Δ with boundary τ containing the half-plane H . We will work entirely in H^* . Since the sequence $\{\sigma_n\}$ escapes and σ_0^g is compact, for n sufficiently large, $\sigma_n \cap \sigma_0^g = \emptyset$ so $\tilde{\sigma}_n \cap \tilde{\sigma}_0^g = \emptyset$. Since the endpoints a_n, b_n of $\tilde{\sigma}_n$ lie in H^* it follows that $\tilde{\sigma}_n$ lies in H^* for n sufficiently large. We restrict ourselves to such sufficiently large n .

Let $S_\varepsilon \subset \Delta$ be the Euclidean circle, concentric with S_∞^1 , of Euclidean radius $1-\varepsilon$, $0 < \varepsilon < 1$. This circle and S_∞^1 cobound an annulus V_ε in \mathbb{D}^2 . Let $D_\varepsilon \subset \Delta$ denote the closed disk of radius $1-\varepsilon$ so $\partial D_\varepsilon = S_\varepsilon$. For a fixed $\varepsilon > 0$, if infinitely many $\tilde{\sigma}_n$'s intersect the disk D_ε , then projecting down into L produces infinitely many σ_n 's meeting the compact set $p(D_\varepsilon) \cap L$, contradicting the fact that the sequence $\{\sigma_n\}$ escapes. Thus, for n sufficiently large, $\tilde{\sigma}_n \cap D_\varepsilon = \emptyset$ so $\tilde{\sigma}_n \subset V_\varepsilon$.

Recall that $[a, b]_\infty$ denotes the interval $\bar{H} \cap S_\infty^1$ in S_∞^1 with endpoints a, b . We are required to show that no boundary component of L can lift to an arc α issuing transversely from $(a, b)_\infty$. If such an arc α exists then there is an $\varepsilon > 0$ such that $S_\varepsilon \cap H \cap \alpha \neq \emptyset$. By the previous paragraph there is a $\tilde{\sigma}_n$ (in fact infinitely many) meeting α which is a contradiction. Since no lift of a boundary component of L can have endpoint in $(a, b)_\infty$ it follows that $H \subset \hat{L}$. \square

Lemma 3.18. *The projection $\pi : \tilde{L} \rightarrow L$ embeds int H isometrically in L . Thus, L contains an embedded half-plane.*

Proof. If an endpoint of an axis of a deck transformation lies in $[a, b]_\infty$, then, as in the proof of Lemma 3.17, that axis meets infinitely many $\tilde{\sigma}_n$. This is a contradiction since the sequence $\{\sigma_n\}$ escapes and the axis projects to a compact subset of L . Thus the axis of every deck transformation ψ has both endpoints outside of $[a, b]_\infty$. If $(a, b)_\infty$ properly overlaps its ψ -image for some deck transformation ψ , then γ has two distinct lifts that intersect. Thus γ intersect itself contradicting Lemma 3.15. If there is a deck transformation that takes $(a, b)_\infty$ to itself then $\hat{\gamma}$ is that axis. Lemma 3.16 rules out that possibility. Thus, the images of $(a, b)_\infty$ under the deck transformations are disjoint and the projection $\pi : \tilde{L} \rightarrow L$ embeds into H isometrically in L so L contains a half-planes. \square

Theorem 3.11 is proven.

Our definition of virtually escaping is metric dependent. We show that, for standard hyperbolic metrics, the notion of virtually escaping is independent of choice of metric.

Corollary 3.19. *If μ_i are standard hyperbolic metrics on L , $i = 1, 2$, then the sequence $\{\sigma_k\}$ virtually escapes relative to μ_1 if and only if it virtually escapes relative to μ_2 .*

Proof. Let σ_k^g be the tightening of σ_k to a μ_1 -geodesic, σ_k^{g*} its μ_2 -tightening. Then σ_k^{g*} is the μ_2 -tightening of σ_k^g which, in turn, is the μ_1 -tightening of σ_k^{g*} . By Theorem 3.11, the sequence $\{\sigma_k^g\}$ escapes if and only if the sequence $\{\sigma_k^{g*}\}$ escapes. \square

Remark. Theorem 3.11 is used in the proof of Theorem 4.24 whose most important application is to ensure there are no spurious leaves in the geodesic laminations, constructed in Section 4.3, that are associated to the endperiodic automorphism f .

4. THE LAMINATIONS AND ENDPERIODIC AUTOMORPHISM PRESERVING THEM

In the Handel-Miller theory, given an endperiodic automorphism $f : L \rightarrow L$, one tightens the f -junctures to geodesic junctures, and uses these to construct a pair of transverse geodesic laminations and an endperiodic automorphism $h : L \rightarrow L$, preserving the geodesic laminations and the geodesic junctures. This is reminiscent of the Nielsen-Thurston theory for automorphisms of compact surfaces.

4.1. Laminations. Roughly speaking, a p -dimensional lamination of an n -manifold M is a foliated subset of leaf dimension p . The codimension is $q = n - p$. The leaves are one-one immersed p -dimensional submanifolds, but transverse q -disks intersect the lamination in relatively closed subsets which may be quite messy and typically are totally disconnected.

In this paper we are interested in laminations of surfaces by curves. We give careful definitions here in that setting, but everything works for laminations of arbitrary dimension p in manifolds of arbitrary dimension $n > p$. (cf. [34, pp. 404-405] for codimension 1 laminations of n -manifolds. This easily adapts to codimension $n - p$.) We are interested in surfaces with boundary and in laminations with some leaves transverse to the boundary. We will use “laminated charts” in analogy with “foliated charts” in foliation theory. Since we allow boundary, these charts will either be homeomorphic to $(a, b) \times (c, d) \subset \mathbb{R}^2$ or to $(a, 0] \times (c, d) \subset \mathbb{R}_-^2$, where $\mathbb{R}_-^2 = (-\infty, 0] \times \mathbb{R}$ is the Euclidean half plane. We let \mathbb{F}^2 denote either \mathbb{R}^2 or \mathbb{R}_-^2 and we denote by I either (a, b) or $(a, 0]$ and $J = (c, d)$. There is no reason to

require I and J to be finite intervals but there is also no good reason not to, so we assume $a, b, c, d \in \mathbb{R}$.

Definition 4.1 (laminated chart). A *laminated chart* in a surface L (possibly with boundary) is a triple (U, Y, φ) , where $U \subset L$ is open, φ is a homeomorphism of U onto $I \times J \subset \mathbb{F}^2$ and $Y \subseteq J$ is a relatively closed subset.

Let $N \subseteq L$ be a subset which is the union of a disjoint set $\Lambda = \{\lambda_\beta\}_{\beta \in \mathfrak{B}}$ of one-one immersed, connected 1-manifolds in L .

Definition 4.2 (lamination). We say that Λ is a *lamination* of L by curves if there is a set $\{(U_\alpha, Y_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$ of laminated charts such that $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ covers N and, for each $\alpha \in \mathfrak{A}$, the path connected components of $\lambda \cap U_\alpha$, where λ ranges over Λ , called *plaques* of the laminated chart $(U_\alpha, Y_\alpha, \varphi_\alpha)$, are carried by φ_α exactly onto the sets $I \times \{y\}$, $y \in Y_\alpha$. The *support* of the lamination is $|\Lambda| = N$ and each $\lambda \in \Lambda$ is called a *leaf* of the lamination. If $|\Lambda|$ is closed in L we will say that the lamination is *closed*.

Definition 4.3 (laminated (partial) atlas). The set $\mathcal{A}_\Lambda = \{(U_\alpha, Y_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$ is called a *laminated partial atlas*. If $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ covers L , we say that $\mathcal{A}_\Lambda = \{(U_\alpha, Y_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$ is a *laminated atlas*.

Remark. \mathcal{A}_Λ is a *partial atlas* because it may not cover L . If, Λ is a *closed lamination*, one can produce a laminated atlas. For each point $x \notin |\Lambda|$, choose a trivially laminated chart (U, \emptyset, φ) about x such that $U \cap |\Lambda| = \emptyset$. In this paper, we work mainly with closed laminations and associated laminated atlases.

Remark. If a plaque P of U_α meets a plaque Q of $U_{\alpha'}$, then it is clear that $P \cap Q$ is an open subset of P and of Q . Without being given N and Λ , one can define an abstract “laminated atlas”, by this property. Then, as in the case of foliations, one can recover the leaves via chains of overlapping plaques. A lamination can then be defined to be an (abstract) laminated atlas. Note that, in our definition, two laminated atlases for the same lamination have union a laminated atlas for that lamination.

Two (abstract) laminated atlases are said to be *coherent* if their union is a laminated atlas. This is an equivalence relation, hence a lamination can be defined as a coherence class of laminated atlases. The union of all atlases in the coherence class is the maximal element of the class and can also be identified as the lamination.

All of this is closely analogous to foliations and foliated atlases, a very detailed treatment of which will be found in [4, pp. 19-31].

Remark. Note that, for a laminated chart (U, Y, φ) , φ restricts to a homeomorphism $\psi : N \cap U \rightarrow I \times Y$. Thus N , equipped with an “atlas” $\{N \cap U_\alpha, \psi_\alpha\}_{\alpha \in \mathfrak{A}}$, satisfies the standard definition of a “foliated space” [4, Chapter 11]. A lamination, therefore, is a topological embedding of a foliated space, but a very special embedding admitting continuous sets of local transversals. The lamination community is typically vague on this point, commonly defining a lamination as a foliated space embedded in a manifold as a closed subset, each leaf of which is complete in its path metric (cf. [2, Definition 6.9]), but for some delicate arguments in this paper that definition seems to be inadequate. Note also that laminations in our sense locally extend to foliations.

In this paper we study a special class of 1-dimensional laminations of surfaces L . The transverse sets $Y_\alpha \subset \mathbb{R}^1$ will be totally disconnected. As a result, the support $|\Lambda|$ determines the lamination and it is fairly customary to make no distinction between the lamination and its support. We prefer to maintain the distinction throughout this paper.

Definition 4.4 (transversely totally disconnected). A lamination Λ is *transversely totally disconnected* if, for each of its laminated charts $(U_\alpha, Y_\alpha, \varphi_\alpha)$, the space Y_α is totally disconnected.

All the laminations we study will be transversely totally disconnected.

Remark. By abuse of notation, we may denote a laminated chart containing a plaque P by $P \times (-1, 1)$. We can then denote by $P \times (-\delta, \varepsilon)$ subcharts with $-\delta < 0$ and $0 < \varepsilon$, identifying $P \times \{0\}$ with P . The idea is that by making δ and ε sufficiently “small”, we get arbitrarily thin normal neighborhoods of P . By passing to the maximal laminated partial atlas, the plaque P may be as “long” as desired, generally requiring that δ, ε be sufficiently small.

If $x \in \lambda \in \Lambda$ and if λ is not isolated on at least one side, then points of $|\Lambda|$ accumulate on x from a non-isolated side of λ . The leaves containing these points accumulate *locally uniformly* on λ in the following sense.

Definition 4.5 (locally uniform accumulation). Let $\lambda \in \Lambda$. We say that $\{\lambda_\alpha\}_{\alpha \in \mathfrak{A}} \subset \Lambda$ *accumulates locally uniformly* on λ if, for every bounded subarc $P \subset \lambda$, there is a lamination chart $V_P \cong P \times (-\delta, \varepsilon)$, having $P = P \times \{0\}$, such that the set of plaques of $V_P \cap \bigcup_{\alpha \in \mathfrak{A}} \lambda_\alpha = \{P \times \{t_\beta\}\}$ is a set of plaques accumulating uniformly on P .

Remark that \mathfrak{A} might be a singleton, but the single leaf λ_α might accumulate locally uniformly on λ . Indeed, λ might accumulate locally uniformly on itself. The laminations that we study in this paper typically exhibit such behavior.

4.1.1. *Bilaminations.* We will be studying a pair of mutually transverse laminations. In the theory of foliations of n -manifolds, a pair of mutually transverse foliations \mathcal{F} and \mathcal{F}' , of respective leaf dimensions p and q , $p + q = n$, gives rise to a “bifoliated atlas”, but for C^0 laminations, a proof of this eludes the authors. We will need such a “bilaminated (partial) atlas”, so we will define the term “bilamination” accordingly. This will not get us into trouble since we are generalizing the Handel-Miller theory where the pair of mutually transverse geodesic laminations will be shown to define a bilamination.

In what follows, I may be of the form (a, b) or $(a, 0]$ and J may be of the form (c, d) or $[0, d)$. Thus L might have boundary and corners.

Definition 4.6 (bilaminated chart). A *bilaminated chart* (U, X, Y, φ) is an open subset U of L together with a homeomorphism $\varphi : U \rightarrow I \times J$, where $X \subseteq I$ and $Y \subseteq J$ are relatively closed subsets.

Let Λ, Λ' be a pair of mutually transverse sets of disjoint, connected, one-one immersed 1-manifolds. Denote by $|\Lambda, \Lambda'|$ the union $|\Lambda| \cup |\Lambda'|$.

Definition 4.7 (bilamination). The pair (Λ, Λ') is a *bilamination* if there is given a set $\mathcal{A}_{(\Lambda, \Lambda')} = \{(U_\alpha, X_\alpha, Y_\alpha, \varphi_\alpha)\}_{\alpha \in \mathfrak{A}}$ of bilaminated charts such that $\{U_\alpha\}_{\alpha \in \mathfrak{A}}$ covers $|\Lambda, \Lambda'|$ and, for each $\alpha \in \mathfrak{A}$, $(U_\alpha, X_\alpha, \varphi_\alpha)$ is a laminated chart for Λ' and

$(U_\alpha, Y_\alpha, \varphi_\alpha)$ is a laminated chart for Λ . We call \mathcal{A} a *bilaminated partial atlas*. If it covers L , we call it a *bilaminated atlas*. The *support* of the bilamination is $|\Lambda, \Lambda'|$.

Remark. In particular, the bilaminated partial atlas can serve as a laminated partial atlas for each of Λ and Λ' , hence these are laminations. The laminations that make up a bilamination are mutually transverse and extend locally to a pair of transverse foliations. If Λ and Λ' are each closed we say that the bilamination is closed. In this case, the support $|\Lambda, \Lambda'|$ is closed and the bilaminated partial atlas can be extended to a bilaminated atlas by assigning bilaminated charts of the form $(U, \emptyset, \emptyset, \varphi)$ as neighborhoods of $x \in \mathcal{C}|\Lambda, \Lambda'|$ (here \mathcal{C} denotes “complement”) where $U \cap |\Lambda, \Lambda'| = \emptyset$.

It is sometimes useful to use closed bilaminated charts. For these, one takes $I = [a, b]$ and $J = [c, d]$. It is obvious that, for every $x \in |\Lambda, \Lambda'|$ and every bilaminated chart containing x , there is a closed bilaminated subchart containing x . The proof of the following is elementary.

Lemma 4.8. *Let (Λ, Λ') be a bilamination and $R \subset L$ a compact, simply connected “rectangle” with top and bottom edges subarcs of leaves of Λ and right and left edges subarcs of leaves of Λ' . Then R has a natural structure of a closed, bilaminated chart for (Λ, Λ') .*

Example 4.9. Important examples of bilaminations are given by a pair of mutually transverse, closed geodesic laminations on a complete hyperbolic surface L . Such an example is given by a pair (Λ^s, Λ^u) of stable and unstable geodesic laminations in the Nielsen-Thurston theory (see Casson and Bleiler [19]). Another example is given by the geodesic bilamination (Λ_+, Λ_-) associated to an endperiodic automorphism f constructed in Section 4.3.

4.1.2. *Smoothness.* The material in this subsection is technical and not immediately important to our development and can be skipped on first reading. The authors are not aware of a generally agreed upon definition of “smoothness” for laminations. We propose a definition which generalizes the standard definition for foliations.

The surface L has a C^r structure, $1 \leq r \leq \infty$ or $r = \omega$ (a real analytic structure), namely a maximal C^r atlas \mathcal{A}_r .

Definition 4.10 (smooth lamination, smooth bilamination). The lamination Λ (respectively bilamination (Λ, Λ')) is *smooth of class C^r* if it admits a laminated partial atlas $\mathcal{A}_\Lambda \subset \mathcal{A}_r$ (respectively a bilaminated partial atlas $\mathcal{A}_{\Lambda, \Lambda'} \subset \mathcal{A}_r$). If $r = \infty$, the lamination is just said to be *smooth*.

As usual, if the lamination or bilamination is closed, a partial laminated atlas of class C^r extends to a laminated atlas on L of class C^r .

Remark. It would not be equivalent to require only that the leaves be C^r immersed curves. This is well known for foliations, but is also true for transversely totally disconnected laminations. The problem is that gaps in a lamination may accumulate on a leaf, but because the lengths of the gaps at one end of a laminated chart may not compare uniformly to the lengths at the other, the mean value theorem can make it impossible to extend the lamination over the gaps to a foliation smooth on the whole chart. It is not obvious whether the geodesic bilaminations (Λ_+, Λ_-) associated to an endperiodic automorphism f constructed in Section 4.3 are smooth, but in general we think not. As the following example shows, a geodesic lamination of a hyperbolic surface need not be smooth of class C^2 .

Example 4.11. Consider the Denjoy exceptional minimal set Λ in T^2 , a transversely totally disconnected lamination. It is constructed by suspending a C^1 diffeomorphism of the circle which has a minimal invariant Cantor set [40, Appendix]. While Λ is C^1 , it is not even homeomorphic to a C^2 lamination. This is classically known. To get this lamination on a hyperbolic surface, proceed as follows. In the open subset of T^2 complementary to $|\Lambda|$, attach a handle, making the surface a 2-holed torus Σ_2 and leaving Λ intact. One can then put a hyperbolic metric g on Σ_2 . Relative to this metric, each leaf $\lambda \in \Lambda$ is a pseudo-geodesic (Definition 4.18). This is seen by noting that an essential circle σ in T^2 which the oriented leaves of the Denjoy lamination periodically cross, always in the same direction, survives as such a circle $\sigma \subset \Sigma_2$. We can even take σ to be a closed geodesic. For each lift $\tilde{\lambda}$ of a leaf $\lambda \in \Lambda$, suitable lifts of σ crossed by $\tilde{\lambda}$ form a nested family of geodesic arcs defining a unique point on S^1_∞ . This gives one ideal endpoint of $\tilde{\lambda}$ and the other endpoint is found similarly by reversing the orientation of λ .

Thus $\lambda \in \Lambda$ is homotopic to a unique geodesic λ^g . The collection of these geodesics is readily shown to be a lamination Λ^g of Σ_2 . It is highly doubtful that Λ^g is still C^1 , but we claim that it certainly cannot be C^2 . Indeed, if it were C^2 , the positively directed unit velocity field tangent to Λ^g extends locally to a C^2 vector field of unit length and these local extensions can be assembled by a C^2 partition of unity to get a nowhere zero vector field v defined on an open set U containing $|\Lambda^g|$ and agreeing with the unit velocity field along the leaves of Λ^g . This defines a C^2 foliation \mathcal{F} on U which incorporates $|\Lambda^g|$ as a compact \mathcal{F} -saturated subset. The holonomy pseudogroup of \mathcal{F} is a (possibly infinitely generated) C^2 pseudogroup on \mathbb{R} defined by a foliated atlas $\mathcal{A}_\mathcal{F}$ for \mathcal{F} (e.g., see [4, Section 2.2] for details). A finitely generated sub-pseudogroup Γ is obtained from a finite subcover of $|\Lambda^g|$ by charts of $\mathcal{A}_\mathcal{F}$ and $|\Lambda^g|$ defines a Γ -invariant Cantor set $X \subset \mathbb{R}$. A deep result of R. Sacksteder [38, Theorem 1] provides a point $x \in X$ and an element $\gamma \in \Gamma$ such that $\gamma(x) = x$ and $0 < \gamma'(x) < 1$. But the leaves of Λ^g are simply connected, implying the germinal holonomy group at x (cf. [4, Section 2.3]) is trivial. Thus $\gamma'(x) = 1$ and this contradiction proves the claim that Λ^g is not C^2 .

Without smoothness, it is not clear that a pair of transverse laminations is a bilamination. However, we have the following.

Lemma 4.12. *If Λ and Λ' are closed, mutually transverse, C^1 laminations, then (Λ, Λ') is a bilamination.*

Proof. If $x \in L \setminus (|\Lambda| \cup |\Lambda'|)$, then there is a C^1 coordinate chart (U, φ) about x such that $U \cap (|\Lambda| \cup |\Lambda'|) = \emptyset$. We can view $(U, \emptyset, \emptyset, \varphi)$ as a bilaminated chart for (Λ, Λ') . If $x \in |\Lambda| \setminus |\Lambda'|$, there is a C^1 laminated chart (U, Y, φ) for Λ about x such that $U \cap |\Lambda'| = \emptyset$, hence we can view $(U, Y, \emptyset, \varphi)$ as a bilaminated chart for (Λ, Λ') . We argue similarly for $x \in |\Lambda'| \setminus |\Lambda|$. For the case that $x \in |\Lambda| \cap |\Lambda'|$, the C^1 hypothesis becomes important. There are two C^1 foliations defined in a rectangular neighborhood R of x , \mathcal{F} having plaques of Λ among its leaves, and \mathcal{F}' incorporating the plaques of Λ' . They may not be transverse throughout R . Let v be a continuous, nowhere vanishing vector field tangent to \mathcal{F} , v' such a field tangent to \mathcal{F}' . At x , the vectors v_x and v'_x are linearly independent, hence the fields, being continuous, are linearly independent at every point of a small enough neighborhood of x . Thus, the foliations are transverse in a smaller rectangular neighborhood of x . Thus, the foliations are transverse in a smaller rectangular neighborhood of

x . These foliations can be taken as a smooth coordinate grid about x defining a bilaminated chart. \square

4.2. A heuristic example. Before presenting the formal construction of the geodesic laminations associated to an endperiodic automorphism, we present an example at an intuitive level.

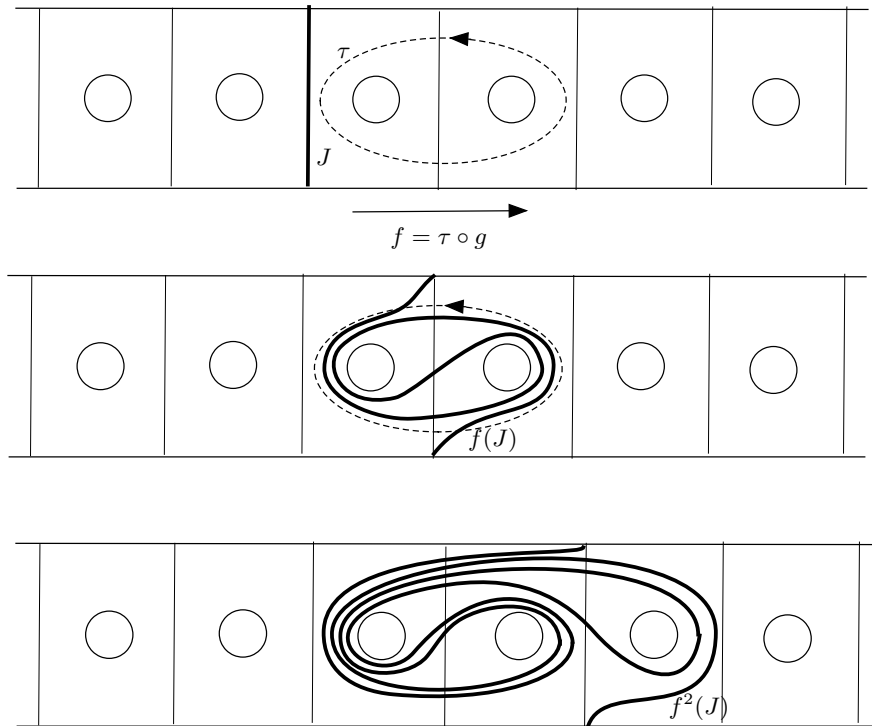


FIGURE 9. Juncture J and geodesic tightenings of $f(J)$, $f^2(J)$

Example 4.13. Let L be a two-ended strip with disks removed approaching both ends, g a translation of the strip from left to right which is an isometry relative to a standard hyperbolic metric quasi-isometric to the Euclidean metric implicit in Figure 9. Let τ be a Dehn twist in the dashed curve as indicated in Figure 9 (top). Let the endperiodic automorphism $f = \tau \circ g$. A geodesic juncture J is drawn on the left in Figure 9 (top). The Handel-Miller theory studies endperiodic automorphisms by studying the limit laminations of the geodesic tightenings of the distorted f -junctures $f(J), f^2(J), \dots$. We call them “distorted” because under iterated applications of f they become longer and longer without a finite upper bound, “looking” less and less like junctures and more and more (to a myopic observer) like noncompact leaves of a lamination. The first two distorted geodesic junctures

are drawn in Figure 9 (middle and bottom). The sequence of geodesic tightenings of distorted f -junctures accumulates on the positive geodesic limit lamination Λ_+ carried by the traintrack in Figure 10 (top) and disjoint from every f -neighborhood of the negative end. By using f^{-1} instead of f , an analogous negative geodesic lamination Λ_- , transverse to Λ_+ , is constructed, disjoint from every f -neighborhood of the positive end.

A rigorous presentation of the Handel-Miller construction is given in Section 4.3.

We will replace the endperiodic automorphism f with an isotopic one h which preserves the laminations. Figure 10 (top) represents a traintrack T which carries the positive geodesic lamination Λ_+ for this endperiodic automorphism. Figure 10 (bottom) represents $h(T) \subset T$. We leave it as an exercise for the reader to verify that these figures are correct. Note that $h(T)$ is obtained from T by blowing air from A to A . Similarly $h^2(T) \subset h(T) \subset T$ is obtained by blowing air from B to B . Once one sees $h(T)$, it is much easier to see $h^2(T), h^3(T), \dots$. The positive lamination itself is $\Lambda_+ = \bigcap_{n=1}^{\infty} h^n(T)$. This example exhibits behavior that is typical and will be visited again in Example 4.35.

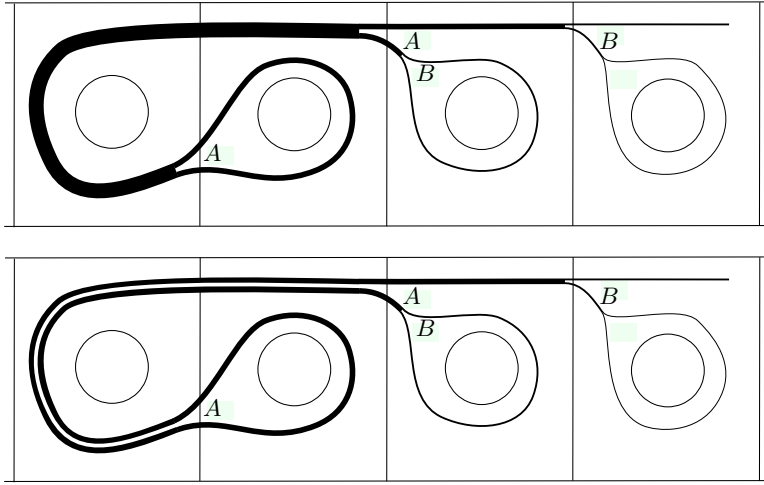


FIGURE 10. The traintracks T , $h(T) \subset T$, etc.

4.3. The Handel-Miller construction. Here we give the simple, classical and unpublished construction of Handel and Miller of the geodesic laminations associated to a given endperiodic automorphism $f : L \rightarrow L$.

Definition 4.14 (admissible). An *admissible surface* is a standard hyperbolic surface with finitely many ends, none of which are simple (Definition 2.27).

For an admissible surface L , relative to an endperiodic automorphism, $\mathcal{E}(L) = \mathcal{E}_-(L) \cup \mathcal{E}_+(L)$.

Lemma 4.15. *If L is an admissible surface and $e \in E$, then the orbit of e under the group of covering transformations is dense in E .*

Indeed, it is standard that a covering transformation ψ induces a homeomorphism $\hat{\psi} : E \rightarrow E$. Admissible surfaces satisfy the hypotheses of [15, Corollary 1], proving the lemma.

Hypothesis 5. Hereafter, L is an admissible surface.

Remark that an admissible surface has no cusps, hence there are no parabolic deck transformations of the universal cover. All deck transformations are either hyperbolic or are orientation reversing and have square a hyperbolic transformation. In any case, they have a unique geodesic axis in \tilde{L} .

Definition 4.16 (geodesic lamination). A lamination of a hyperbolic surface in which all the leaves are geodesics is a *geodesic lamination*.

4.3.1. *Choosing and fixing the f -junctures.* We first choose and fix a countable set of f -junctures in the uncountable set of all f -junctures as follows. Let e be an end of L and set

$$c = \{e_0 = e, e_1 = f(e), e_2 = f^2(e), \dots, e_{p_e-1} = f^{p_e-1}(e)\},$$

the f -cycle of ends containing e . Let $N_c = J = \text{Fr } U_e$ be the f -juncture associated to the end e defined in Proposition 2.12 (where $U_e = V_0$ is defined in the proof of Proposition 2.12 in the paragraph before Corollary 2.23) and consider the set of f -junctures $\{f^n(N_c) \mid n \in \mathbb{Z}\}$. Choose and fix such an N_c for each of the finitely many f -cycles c of ends and fix as our countable set of f -junctures the union of the sets $\{f^n(N_c) \mid n \in \mathbb{Z}\}$ as c ranges over the set of f -cycles of ends.

Remark. From now on we will use the symbol N to denote one of these countably many f -junctures and use the symbol J to denote a juncture which we define below.

Definition 4.17 (fixed set of f -junctures, $\mathcal{N}, \mathcal{N}_+, \mathcal{N}_-$). Fix the set of f -junctures constructed above. The set of all components of the fixed set of f -junctures will be denoted by \mathcal{N} . The subset of \mathcal{N} consisting of components of positive (respectively negative) f -junctures will be denoted by \mathcal{N}_+ (respectively \mathcal{N}_-).

We extend the definition of pseudo-geodesic given in [15, Definition 5] to include properly embedded, boundary incompressible compact arc.

Definition 4.18 (pseudo-geodesic). A curve $\gamma \subset L$ is a *pseudo-geodesic* if either some (hence every) lift $\tilde{\gamma}$ has two distinct, well defined endpoints on S_∞^1 or γ is a properly embedded, boundary incompressible compact arc.

Remark that essential embedded circles in L that do not bound cusps and geodesics are pseudo-geodesics. Note that since the surface is admissible, every essential closed curve is a pseudo-geodesic.

In Definition 3.6, we defined the geodesic tightening of an essential closed curve or a boundary incompressible, properly embedded arc. Here we extend the definition to an arbitrary pseudo-geodesic.

Definition 4.19 (geodesic tightening, γ^g). If γ is a pseudo-geodesic, then the *geodesic tightening* of γ is the unique geodesic γ^g whose lifts have the same endpoints on S_∞ as the lifts of γ .

Definition 4.20 (geodesic tightening map, ι). We will call the map that sends a pseudo-geodesic to its geodesic tightening the *geodesic tightening map* and denote it by ι . Thus, if γ is a pseudo-geodesic, then $\iota(\gamma)$ is the geodesic whose lifts have the same endpoints on S_∞ as γ .

Definition 4.21 (juncture, \mathcal{J} , \mathcal{J}_+ , \mathcal{J}_-). The set \mathcal{J} of *juncture components* consists of the geodesic tightenings of the f -juncture components in the set \mathcal{N} . Let $\iota : \mathcal{N} \rightarrow \mathcal{J}$ be the geodesic tightening map. Then $\mathcal{J}_+ = \iota(\mathcal{N}_+)$ is the set of *positive juncture components* and $\mathcal{J}_- = \iota(\mathcal{N}_-)$ is the set of *negative juncture components*. Further the map ι extends in a natural way to the fixed set of f -junctures to define a fixed set of *junctures* of the form $J = \iota(N)$ where N is one of the fixed f -junctures.

Remark. It is important to keep clear the distinction between f -junctures and junctures. The set of f -junctures is f -invariant, but generally not geodesic. The set of junctures is geodesic but generally not f -invariant.

We assume the set of f -junctures, the set of junctures, the set \mathcal{N} of f -juncture components, and the set \mathcal{J} of juncture components has been constructed as above and fixed.

Definition 4.22 (J_n). Given a juncture $J = \iota(N)$, let J_n denote the geodesic tightening of $f^n(J)$ which is the same as the geodesic tightening of $f^n(N)$, $n \in \mathbb{Z}$.

Since the positive (respectively negative) f -junctures are constructed only to intersect in common components, the same is true after tightening. We state this formally for future reference.

Proposition 4.23. *The set of junctures has the juncture intersection property.*

By Theorem 3.11, the junctures have the following critically important property.

Theorem 4.24. *If $J = \iota(N)$ where the f -juncture N cuts off a neighborhood of the positive end e of period p , then $\{J_n\}_{n \geq 0}$ escapes. If the end e is negative, then $\{J_{-n}\}_{n \geq 0}$ escapes.*

Definition 4.25 (juncture escapes). A component σ of a juncture J *escapes* if $\{i(f^n(\sigma))\}_{n \in \mathbb{Z}}$ escapes. The juncture J *escapes* if each of its components escapes or, equivalently, if $\{J_n\}_{n \in \mathbb{Z}}$ escapes.

Remark. We will see that a juncture escapes if and only if f is isotopic to a translation (Proposition 4.76). But escaping components of junctures can easily arise, as illustrated in Example 2.10. They do not accumulate anywhere. We will mainly be interested in the nonescaping components.

Definition 4.26 (\mathcal{X}_\pm). We denote by \mathcal{X}_+ the set of nonescaping components of positive junctures and by \mathcal{X}_- the corresponding set of nonescaping components of negative junctures.

Remark. $\mathcal{X}_+ \subset \mathcal{J}_+$ and $\mathcal{X}_- \subset \mathcal{J}_-$.

4.3.2. *The Handel-Miller bilaminations.* Define $\mathfrak{G}_\pm = \overline{|\mathcal{X}_\mp|}$ and set $\mathfrak{L}_\pm = \mathfrak{G}_\pm \setminus |\mathcal{X}_\mp|$. The goal of this section is to show that \mathfrak{G}_\pm and \mathfrak{L}_\pm are the supports of closed geodesic laminations Γ_\pm and Λ_\pm , respectively.

Lemma 4.27. *The space \mathfrak{G}_\pm consists of the disjoint union of one-one immersed, complete geodesics lines or compact geodesic arcs or circles which are the path components of \mathfrak{G}_\pm .*

Proof. We consider \mathfrak{G}_+ . The proof for \mathfrak{G}_- is analogous. We have already seen that the union of the positive junctures is a disjoint union of isolated geodesic circles and/or properly embedded geodesic arcs. Therefore it remains to show that \mathfrak{L}_+

is a disjoint union of geodesics which are the path components of $|\mathfrak{L}_+|$. It will be enough to prove this for the lift $\tilde{\mathfrak{L}}_+ \subset \tilde{L}$.

Choose $x \in \tilde{\mathfrak{L}}_+$ and $x_n \in |\tilde{\mathfrak{X}}_-|$ such that $x_n \rightarrow x$ as $n \rightarrow \infty$. Let σ_n be the unique element of $\tilde{\mathfrak{X}}_-$ containing x_n . Let v_n be a vector tangent to σ_n at x_n of hyperbolic length 1. Passing to a subsequence, if necessary, we can assume that $v_n \rightarrow v$ as $n \rightarrow \infty$, where v is a vector at x of hyperbolic length 1. Then, there is a unique geodesic σ through x tangent to v such that the sequence $\{\hat{\sigma}_n\}$ converges uniformly to $\hat{\sigma}$ in the Euclidean metric on \mathbb{D}^2 . In particular, the endpoints (finite or ideal) on S_∞^1 of the σ_n converge to the endpoints of σ on S_∞^1 .

Since $x \in \tilde{\mathfrak{L}}_+$ is arbitrary, we see that $\tilde{\mathfrak{L}}_+$ is a union of such geodesics and $\tilde{\mathfrak{X}}_-$ accumulates on each point of $\tilde{\mathfrak{L}}_+$. Thus, $\tilde{\mathfrak{L}}_+$ has empty interior. In fact, if $x \in \tilde{\mathfrak{L}}_+$ and U is a neighborhood of x , \mathfrak{X}_- meets U and each point of \mathfrak{X}_- has a neighborhood disjoint from $\tilde{\mathfrak{L}}_+$. If any two of these geodesics in $\tilde{\mathfrak{L}}_+$ intersect and are not identical, they intersect transversely, implying that two geodesics in $\tilde{\mathfrak{X}}_-$ also intersect transversely. We know this is false, so it follows that $\tilde{\mathfrak{L}}_+$ is a disjoint union of geodesics with endpoints on S_∞^1 and has empty interior. These geodesics, then, are path components of $\tilde{\mathfrak{L}}_+$.

Projecting to L , we see that \mathfrak{L} has no interior and is also a union of nonintersecting geodesics. None of these geodesics can self intersect since this would imply that two geodesics in \mathfrak{X}_- intersect transversely. Thus, the path components of \mathfrak{L}_+ are one-one immersed geodesics which are complete since the endpoints of their lifts are on S_∞^1 . \square

From now on we will write the set of geodesics forming the path components of \mathfrak{L}_\pm as Λ_\pm and the set of geodesics forming the path components of \mathfrak{G}_\pm as Γ_\pm . Thus, $\mathfrak{L}_\pm = |\Lambda_\pm|$ and $\mathfrak{G}_\pm = |\Gamma_\pm|$. These path components will be called ‘‘leaves’’ of Λ_\pm and Γ_\pm , respectively, even before the proof of the following proposition is completed.

Proposition 4.28. *The pairs (Γ_+, Γ_-) and (Λ_+, Λ_-) are bilaminations.*

Proof. For p an arbitrary point of L we construct a bilaminated chart (U, X, Y, φ) for (Γ_+, Γ_-) with $p \in U$. Since p is arbitrary, we obtain a bilaminated atlas for (Γ_+, Γ_-) . By ignoring \mathfrak{X}_\pm , we see that this is also a bilaminated atlas for (Λ_+, Λ_-) .

Since $|\Gamma_-| \cup |\Gamma_+|$ is a closed set, points in its complement have coordinate charts that do not meet it. These are trivially bifoliated charts for (Γ_+, Γ_-) .

The case in which $p \in |\mathfrak{X}_\pm| \setminus |\Lambda_\mp|$ is also easy. The point p will only lie on the intersection of a positive and negative juncture or on one juncture. Thus, the bifoliated chart (U, X, Y, φ) will have X or Y both singletons, or one a singleton and the other empty.

Suppose $p \in |\Lambda_+| \setminus |\Lambda_-|$. Then $p \in \lambda \in \Lambda_+$ with λ approached on one or both sides by geodesics in Γ_+ . Thus, there is a convex geodesic quadrilateral Q containing p in its interior and having as one pair of opposite sides geodesic arcs A and B chosen as follows. If both sides of λ are approached by geodesics in Γ_+ , choose A and B to be a subarc of Γ_+ . Otherwise, choose A on the side of λ not approached by elements of Γ_+ in a small neighborhood of p so that no points of $|\Gamma_+|$ lie between A and λ and choose B to be a subarc of Γ_+ . Take the other pair of opposite sides of Q to be geodesic arcs σ and τ joining the endpoints of A to the corresponding endpoints of B . Choosing the arcs A and B suitably guarantees

that $Q \cap |\Gamma_-| = \emptyset$ if $p \notin |\mathcal{X}_+|$ or $Q \cap |\Gamma_-|$ is one geodesic arc in a positive juncture if $p \in |\mathcal{X}_+|$. Let $Y' = \tau \cap (|\Gamma_+| \cup A)$ (note that the A in this definition of Y' is redundant in the case that Γ_+ approaches λ on both sides). Remark that the compact set Y' projects continuously and one-one onto $\sigma \cap (|\Gamma_+| \cup A)$ along arcs in leaves of Γ_+ (and the arc A if necessary). The continuity is a consequence of the fact that the geodesic arcs which are path components of $Q \cap |\Gamma_+|$ depend continuously on their endpoints. This map extends linearly over the gaps in Y' to produce a homeomorphism $f : \tau \rightarrow \sigma$. The geodesics with endpoints $y, f(y)$, $y \in \tau$, foliate Q and the foliation contains among its leaves the path components of $Q \cap |\Gamma_+|$. (These leaves cannot intersect since this would produce geodesic digons in Q .) It is then easy to use a linear map (or in the case that p is in a positive juncture a piecewise linear map) from A to B to produce another geodesic foliation of Q transverse to the first. Coordinatizing A with a coordinate x and τ with a coordinate y and using these transverse geodesic foliations as a coordinate grid, we obtain a closed coordinate chart (Q, x, y) with p in its interior which, together with $Y = \tau \cap |\Gamma_+|$, defines a closed laminated chart for Γ_+ . The chart is trivially bilaminated by taking $X = \emptyset$ if $p \notin |\mathcal{X}_+|$ or one point if $p \in |\mathcal{X}_+|$. Similarly, if $p \in |\Lambda_-| \setminus |\Lambda_+|$, we obtain a bilaminated chart with p in its interior.

Finally, if $p \in |\Lambda_+| \cap |\Lambda_-|$, the reader can adapt the above construction to obtain a pair of transverse geodesic foliations of Q where, as above, A, B will be arcs in leaves of Γ_+ or disjoint from $|\Gamma_+|$ and similarly σ, τ will be arcs in leaves of Γ_- or disjoint from $|\Gamma_-|$. One of these foliations incorporates the path components of $|\Gamma_+| \cap Q$ among its leaves, the other the path components of $|\Gamma_-| \cap Q$. This gives a closed bilaminated chart about p , completing our construction of a bilaminated atlas for (Γ_+, Γ_-) . \square

Corollary 4.29. *The laminations Λ_{\pm} are transversely totally disconnected (Definition 4.4).*

Proof. This follows since the leaves of Γ_{\pm} are the path components of $|\Gamma_{\pm}|$. \square

Definition 4.30 (Handel-Miller bilamination). The bilamination (Λ_+, Λ_-) is called the *Handel-Miller (geodesic) bilamination* associated to the endperiodic automorphism f . The individual laminations Λ_{\pm} will be called the *Handel-Miller (geodesic) laminations* associated to f .

The bilamination (Γ_+, Γ_-) is an *extended Handel-Miller bilamination* associated to f .

Remark. The use of the definite article “the” for the Handel-Miller bilamination and the indefinite article “an” for an extended one is important. In Corollary 4.72 we will show that the bilamination (Λ_+, Λ_-) depends only on f (and, of course, on the choice of standard hyperbolic metric). On the other hand (Γ_+, Γ_-) also depends on the choice of f -junctures, hence of their geodesic tightenings.

Remark. The analogy of the construction of the geodesic laminations in the Handel-Miller theory and in the Nielsen-Thurston theory is evident, but there are some very significant differences. The sequence of closed geodesics limiting on a geodesic lamination Λ in the Nielsen-Thurston case is not pairwise disjoint nor disjoint from Λ . The endperiodic case allows us to choose the connected geodesic 1-manifolds in \mathcal{X}_{\pm} to be pairwise disjoint and disjoint from the lamination Λ_{\mp} on

which they limit. This is a major simplification and also introduces new laminations Γ_{\pm} which will be very useful in developing the theory and for which there are no analogues in the Nielsen-Thurston theory.

Remark. The pair $(\Lambda_+ \cup \mathcal{J}_-, \Lambda_- \cup \mathcal{J}_+)$ may not be a bilamination. In fact, an escaping component of a positive juncture may coincide with an escaping component of a negative juncture. However, it is obvious that $\Lambda_+ \cup \mathcal{J}_-$ and $\Lambda_- \cup \mathcal{J}_+$ are each laminations.

4.3.3. *The strongly closed property.*

Definition 4.31 (pseudo-geodesic lamination). A lamination of the surface L in which all the leaves are pseudo-geodesics is a *pseudo-geodesic lamination*.

Definition 4.32 (converges strongly). If Λ is a pseudo-geodesic lamination and $\{\tilde{\lambda}_n\}$ is a sequence of leaves of $\tilde{\Lambda}$, then $\{\tilde{\lambda}_n\}$ *converges strongly* to $\tilde{\lambda} \in \tilde{\Lambda}$ if,

- (1) For every bounded subarc $P \subset \tilde{\lambda}$ there is a laminated chart $V_P \cong P \times (-\varepsilon, \varepsilon)$, with $P = P \times \{0\}$, such that the sequence of plaques $V_P \cap \tilde{\lambda}_n = P \times \{t_n\}$ converge to P as $n \rightarrow \infty$;
- (2) The endpoints of the $\tilde{\lambda}_n$ in S^1 converge to the endpoints of $\tilde{\lambda}$ in S^1_{∞} in the Euclidean metric of \mathbb{D}^2 as $n \rightarrow \infty$.

Definition 4.33 (strongly closed property). A lamination Λ has the *strongly closed property* if,

- (1) Whenever a sequence of points $x_n \in \ell_n \subset \tilde{\Lambda}$ converges to $x \in \tilde{\Lambda}$ and ℓ is the leaf, of $\tilde{\Lambda}$ through x , the sequence $\{\ell_n\}$ converges strongly to ℓ ;
- (2) Whenever the endpoints in S^1_{∞} of a sequence $\{\ell_n\} \subset \tilde{\Lambda}$ converge in the Euclidean metric to a pair of distinct points $a, a' \in S^1_{\infty}$, then these are the endpoints of a leaf ℓ of $\tilde{\Lambda}_{\pm}$ and the sequence $\{\ell_n\}$ converges strongly to ℓ .

The next lemma is obvious from the construction of the laminations $\tilde{\Gamma}_{\pm}$ and $\tilde{\Lambda}_{\pm}$.

Lemma 4.34. *The laminations Γ_{\pm} and Λ_{\pm} are strongly closed.*

Remark. In Section 10 we give an axiomatic treatment of the Handel-Miller theory. However, the corresponding *pseudo-geodesic* laminations satisfying the axioms are not obviously strongly closed, but, by the isotopy theorem (Theorem 10.15), the strongly closed property will follow from the geodesic case. In turn, this property will be needed at a key point of the proof of the transfer theorem (Theorem 12.7).

4.3.4. *Examples.*

Example 4.35. We continue Example 4.13. The vertical lines in Figure 11 are junctures, the ones to the left of the core K being negative junctures and those to the right of the core being positive junctures. The juncture immediately to the left of the core is the juncture J of Figure 9. The traintracks representing both Λ_+ and Λ_- (dashed) have been drawn. Along the segments of the traintrack joining two consecutive switches, the lamination looks like an uncountable, but totally disconnected packet of parallel arcs. At the switches, the packet splits along a gap, “half” of the curves veering to the left and half to the right. The intersection $\mathcal{K} = |\Lambda_+| \cap |\Lambda_-| \subset K$ is a totally disconnected set, homeomorphic to the Cantor set and living in the regions indicated by the two small circles in the figure. After f is isotoped to a homeomorphism h preserving the laminations, the set \mathcal{K} will be

invariant and the dynamics of $h|_{\mathcal{K}}$ will be isomorphic to that of a 2-ended Markov chain (cf. Section 9).

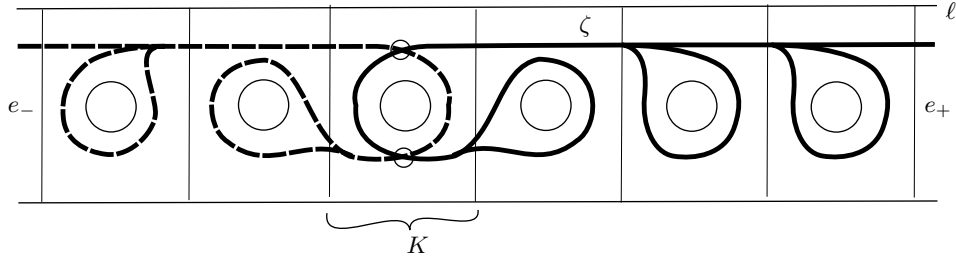


FIGURE 11. A simple example

One easily proves that Λ_+ has a single leaf λ_+ that is isolated on one side. A ray ζ of λ_+ (the “top” ray on the right hand side of the core) makes a “beeline” for e_+ , never veering back, but the complementary ray repeatedly delves arbitrarily deeply into the neighborhood of e_+ and then turns around to revisit the core before going even more deeply into the neighborhood of e_+ . Every other leaf of Λ_+ behaves like this in both directions, always returning to the core. The lamination Λ_- behaves similarly.

This illustrates typical behavior that will be proven as theorems in this paper. However not everything in this example is typical. Generally, the laminations may not be transversely Cantor as they are here. The laminations may also have isolated leaves as well as limit leaves, be transversely countable, or even finite. In Example 2.5, each lamination has a single leaf indicated in Figure 1 in boldface.

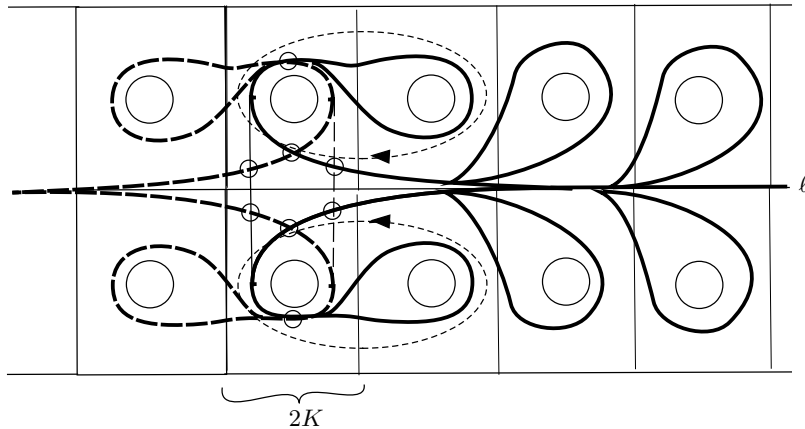


FIGURE 12. A doubled example

Example 4.36. One might think that, if $\partial L \neq \emptyset$, the Handel-Miller laminations for the double $2f : 2L \rightarrow 2L$ would just be two copies of the Handel-Miller laminations

on L . But doubling Example 4.35 shows this to be false. In Figure 12 we draw the train tracks for the laminations Λ_{\pm} for the double, where the dashed tracks carry Λ_- and the solid carry Λ_+ . In this figure, the top and bottom boundary lines are to be identified and each boundary circle in the top row should be identified with the corresponding boundary circle in the bottom row. In each lamination, there is an isolated leaf λ_{\pm} (not labeled, but easy to spot) that crosses the line ℓ which was the top boundary line in Figure 11. This leaf is dense in Λ_{\pm} . The bold part of the traintracks carries uncountably many leaves, the lighter segments each only carry a segment of λ_{\pm} . The reason that this new isolated leaf appears is that a component σ of (distorted) juncture which is a properly embedded geodesic arc has double 2σ which is not generally a geodesic circle. Tightening it to one distorts 2σ enormously, pulling it back to a closed geodesic which crosses ℓ in a point in the compact core $2K$. This sequence of crossings accumulates monotonically on a point x_{\pm} , the junctures accumulating locally uniformly near x_{\pm} on the the segment of the isolated leaf λ_{\pm} through x_{\pm} . In the figure we have drawn circles where the two laminations intersect. Remark that Λ_{\pm} is equal to the double of its counterpart in Figure 11 together with the isolated leaf λ_{\pm} . Here we have narrowed the gap between ℓ and the two copies of ζ to the point of invisibility.

4.4. Distinguished neighborhoods. Suppose $e \in \mathcal{E}(L)$ and the juncture $J = \iota(N)$ where N is the frontier of a closed f -neighborhood V of e . Applications of Theorem 3.1 and/or 3.2 to the components of N , provide an isotopy Φ such that $\Phi^1(N) = J$. Thus, $\Phi^1(V)$ is a closed neighborhood U_e of e and $J = \text{Fr } U_e$.

Definition 4.37 (distinguished neighborhood, U_e). The set U_e as above is called a *distinguished neighborhood of the end e* . The set of *distinguished neighborhoods* is the set of all such sets U_e with $\text{Fr } U_e$ a juncture.

Remark. The fixed countable set of f -junctures (Definition 4.17) is in one-one correspondence with the set of junctures (Definition 4.21) which is in one-one correspondence with the set of distinguished neighborhoods.

Lemma 4.38. *If $e \neq e'$ are any two positive (respectively negative) ends and U_e and $U_{e'}$ are distinguished neighborhoods of e and e' , then $U_e \cap U_{e'} = \emptyset$.*

Proof. Consider the junctures $J = \text{Fr } U_e$ and $J' = \text{Fr } U_{e'}$. Let N and N' be the f -junctures such that $\iota(N) = J$ and $\iota(N') = J'$. Let V_e and $V_{e'}$ be f -neighborhoods of e and e' respectively such that $N = \text{Fr } V_e$ and $N' = \text{Fr } V_{e'}$. By Lemma 2.6, $V_e \cap V_{e'} = \emptyset$. It follows that $U_e \cap U_{e'} = \emptyset$. \square

Recall the notation, J_n equals the geodesic tightening of $f^n(J)$ if J is a juncture, $n \in \mathbb{Z}$.

Definition 4.39 (U_e^n). Suppose $e \in \mathcal{E}_+(L)$ (respectively $e \in \mathcal{E}_-(L)$), U_e is a distinguished neighborhood of e , and $J = \text{Fr } U_e$. For $n \in \mathbb{Z}$, define U_e^n to be the distinguished neighborhood such that $J_{np_e} = \text{Fr } U_e^n$ (respectively $J_{-np_e} = \text{Fr } U_e^n$).

Theorem 4.24 immediately implies,

Lemma 4.40. *For an end e , $\{U_e^i\}_{i=0}^{\infty}$ is a fundamental neighborhood system of e .*

Remark. By reindexing, we can choose the distinguished neighborhoods U_e so that, as e ranges over all ends, positive and negative, the distinguished neighborhoods U_e will be pairwise disjoint. We fix such a choice.

4.5. The cores K, K_i and the sets W^\pm, W_i^\pm .

Definition 4.41 (core and i^{th} core, K, K_i, W^\pm, W_i^\pm). For each $i \geq 0$, denote by W_i^+ the union of the U_e^i 's as e ranges over the positive ends. Similarly define W_i^- , for each $i \geq 0$. These choices of the distinguished neighborhoods U_e have been made so that $W_i^+ \cap W_i^- = \emptyset$, $i \geq 0$. The compact submanifold K_i complementary to $\text{int}(W_i^+ \cup W_i^-)$ is called the i^{th} core of L . We will write $K = K_0$ and call it simply the core. We will also write W^+ for W_0^+ and W^- for W_0^- .

Remark. We will often change the indexing and relabel K_i by K_0 , thus choosing a larger core $K = K_0$.

Remark. Note that $\text{Fr } K$ is a finite union of positive and negative junctures. Accordingly, we will write $\text{Fr } K = \text{Fr}_+ K \cup \text{Fr}_- K$.

The surface L has decomposition $L = W^- \cup K \cup W^+$ where,

$$W^- = \bigcup_{e \in \mathcal{E}_-(L)} U_e \quad \text{and} \quad W^+ = \bigcup_{e \in \mathcal{E}_+(L)} U_e.$$

Definition 4.42 (W, \mathcal{J}_W). Let $W = W^- \cup W^+$. Let \mathcal{J}_W be the set consisting of all components of negative junctures in W^- and all components of positive junctures in W^+ .

Lemma 4.43. *There exists an endperiodic automorphism g isotopic to f such that $g(\alpha)$ is a geodesic for every $\alpha \in \mathcal{J}_W$.*

Proof. Enumerate the elements of \mathcal{J}_W as $\{\tau^i\}_{i=1}^\infty$ in such a way that those meeting K_i are listed before those not meeting K_i , $i \geq 0$. Let $\psi_0 = \text{id}$. Using Theorem 3.1, Theorem 3.2 and the remark after Theorem 3.3, inductively find sequences $\{\Phi_i\}$ of isotopies and $\{\psi_i\}$ of homeomorphisms with $\psi_i = \Phi_i^1 \circ \psi_{i-1}$, $i \geq 1$, fixing ∂L pointwise and with Φ_i fixing $(f(\tau^j))^\mathfrak{g}$, $1 \leq j \leq i-1$, pointwise and moving $\psi_{i-1} \circ f(\tau^i)$ to its geodesic tightening. That is, $\psi_i(f(\tau^i)) = (f(\tau^i))^\mathfrak{g}$. By the remark after Theorem 3.3, the supports of at most finitely many Φ_i meet any compact set. Thus, $\psi_i \rightarrow \psi$, a well defined homeomorphism isotopic to the identity by an isotopy Φ and $g = \psi \circ f$ is such that $g(\alpha)$ is a geodesic for every $\alpha \in \mathcal{J}_W$. \square

Lemma 4.44. *For each $i \geq 0$ and each choice J of negative juncture, all but finitely many components of $\bigcup_{n \geq 0} J_n$ meet W_i^+ and, similarly, if J is a positive juncture, all but finitely many components of $\bigcup_{n \geq 0} J_{-n}$ meet W_i^- .*

Proof. If a component of the negative juncture J is a properly embedded geodesic arc, the assertion is immediate since the positive iterates of its endpoints escape. If the component is a simple closed geodesic σ , denote the corresponding component of J_n by σ_n . Suppose, by contradiction, that $\sigma_n \subset K_i$, for infinitely many values of $n > 0$. Since K_i is a compact surface, there exists a positive integer k such that if there exist more than k disjoint, simple closed curves in K_i two of them must bound an annulus. If σ_n and σ_m cobound an annulus, then, as homotopic geodesics, they must coincide. It follows that infinitely many of the σ_n coincide which is a contradiction. \square

Lemma 4.45. (1) *The arcs of $\mathcal{J}_\mp | K_i$ with endpoints on $\text{Fr}_\pm K_i$ are boundary incompressible and fall into finitely many isotopy classes in $(K_i, \text{Fr}_\pm K_i)$;*

- (2) For every such isotopy class \mathcal{A} , there exists a rectangle $R \subset K_i$ with a pair of opposite edges each intervals in not necessarily distinct components α and β of $\text{Fr}_\pm K_i$ such that for every pair of arcs $\sigma, \tau \in \mathcal{A}$, the track of the isotopy between σ and τ lies in R .

Proof. Item (1) is an elementary topological consequence of the fact that K_i is a compact surface. Let \mathcal{A} be such an isotopy class. There exist not necessarily distinct components α, β of $\text{Fr}_\pm K_i$ such that the endpoints of each arc in \mathcal{A} lie in α and β . If $\sigma, \tau \in \mathcal{A}$, then there exists a rectangle $R_{\sigma\tau}$ with one pair of opposite edges σ and τ and the other pair of opposite edges subintervals of α and β respectively such that the track of the isotopy between σ and τ lies in $R_{\sigma\tau}$.

If \mathcal{A} has one element, then (2) is obviously true. If \mathcal{A} has two elements σ, τ , then $R = R_{\sigma\tau}$ is the desired rectangle. Therefore we can assume \mathcal{A} has at least 3 elements τ_-, τ_0, τ_+ such that $\tau_0 \in R_{\tau_-\tau_+}$. Let R_- (respectively R_+) be the closure of $\bigcup\{R_{\tau_0\tau} \mid \tau \in \mathcal{A}, R_{\tau_0\tau_-} \subset R_{\tau_0\tau}\}$ (respectively $\bigcup\{R_{\tau_0\tau} \mid \tau \in \mathcal{A}, R_{\tau_0\tau_+} \subset R_{\tau_0\tau}\}$). If either R_- or R_+ is not a rectangle, then we have the contradiction that $K = R_-$ or $K = R_+$ is an annulus with boundary α and β . Similarly if $R = R_- \cup R_+$ is not a rectangle, we have the contradiction that $K = R$ is an annulus with boundary $\alpha \cup \beta$. Thus, R is the desired rectangle. \square

Lemma 4.46. *Every neighborhood of each end of a leaf of Λ_\pm meets W_i^\pm , $i \geq 0$.*

Proof. For definiteness, assume $\lambda \in \Lambda_+$. Choose $K = K_0$ large enough that λ meets $\text{int } K$, choose $x \in \lambda \cap \text{int } K$ and let $\{\sigma_k\}_{k=1}^\infty$ be a sequence of negative juncture components with $x_k \in \sigma_k$ such that $x_k \rightarrow x$ as $k \rightarrow \infty$. Fix $i \geq 0$. Using Lemma 4.44 and passing to a subsequence, we can assume that a subarc $\tau_k \subset \sigma_k \cap K_i$ contains x_k and has endpoints $a_k, b_k \in \text{Fr}_+ K_i$. By passing to a subsequence and using Lemma 4.45, assume that the arcs τ_k are all isotopic. Thus the points a_k belong to the same component α of $\text{Fr}_+ K_i$ and converge to $a \in \alpha$ and the points b_k belong to the same component β of $\text{Fr}_+ K_i$ and converge to $b \in \beta$. Without loss we can assume $a_k \rightarrow a$ and $b_k \rightarrow b$ monotonically. By Lemma 4.45 there exists a rectangle R with a pair of opposite edges in α and β and containing the sequence of arcs $\{\tau_k\}_{k \geq 0}$.

Choosing a lift \tilde{R} of R , where \tilde{R} is a rectangle with a pair of opposite edges in $\tilde{\alpha}$ a lift of α and $\tilde{\beta}$ a lift of β , determines lifts $\tilde{\tau}_k$ of τ_k for all $k \geq 0$ such that $\tilde{\tau}_k$ has endpoints $\tilde{a}_k \in [\tilde{a}, \tilde{a}_0] \subset \tilde{\alpha}$ and $\tilde{b}_k \in [\tilde{b}, \tilde{b}_0] \subset \tilde{\beta}$ where \tilde{a} is a lift of a , \tilde{b} is a lift of b , \tilde{a}_k is a lift of a_k , and \tilde{b}_k is a lift of b_k . The geodesic arcs $\tilde{\tau}_k$ nest in Δ on the geodesic arc $[\tilde{a}, \tilde{b}] \subset \ell \in \tilde{\Lambda}_+$ and $\tilde{a}_k \rightarrow \tilde{a}$, and $\tilde{b}_k \rightarrow \tilde{b}$. Since the lift $\tilde{x}_k \in \tilde{\tau}_k$ converges to a lift $\tilde{x} \in [\tilde{a}, \tilde{b}]$ of $x \in \lambda$, it follows that $\ell = \tilde{\lambda}$. The arc $[\tilde{a}, \tilde{b}]$ projects to a subarc $[a, b] = \gamma_i \subset \lambda$ containing the point x .

Then λ exits K_i at a and b and, varying i , $x \in \gamma_0 \subset \gamma_1 \subset \dots \subset \gamma_i \subset \dots$ forms an exhaustion of λ . The lemma follows. \square

Corollary 4.47. *No leaf of Λ_\pm is contained in a bounded region of L .*

Corollary 4.48. *Every leaf of Λ_\pm is a one-one immersed copy of \mathbb{R} .*

Proof. By Corollary 4.47, a leaf of Λ_\pm can not be homeomorphic to a circle. Hypothesis 3 assures that no leaf of Λ_\pm has an endpoint on ∂L . \square

Definition 4.49 (passes arbitrarily near). An end ε of a curve s in L passes arbitrarily near an end e of L if every neighborhood of e meets every neighborhood of ε . In this case we also say that s passes arbitrarily near e .

Notice that a curve can pass arbitrarily near e and still return repeatedly to some compact subset $X \subset L$. A careful analysis of Example 4.13 shows that only one end of only one leaf of Λ_+ escapes (cf. Definition 6.19) to the positive end. All other ends of all leaves repeatedly return to the same compact “core”. We will see that this second behavior is the typical behavior of the leaves of Λ_\pm .

The following corollary follows immediately from Lemma 4.46 since L has finitely many ends.

Corollary 4.50. *Both ends of every leaf of Λ_+ (respectively Λ_-) pass arbitrarily near at least one positive (respectively negative) end of L .*

Lemma 4.51. *No escaping component of a juncture meets $|\Lambda_+| \cup |\Lambda_-|$.*

Proof. For definiteness, suppose that σ is a negative escaping juncture component. Since each point of σ has a neighborhood that meets no other negative juncture component, it follows that $\sigma \cap |\Lambda_+| = \emptyset$. It remains to show that $\sigma \cap |\Lambda_-| = \emptyset$. Suppose the contrary that there exists $\lambda \in \Lambda_-$ meeting σ . Consider lifts $\tilde{\sigma}$ of σ with endpoints (finite or ideal) $a, b \in S_\infty$ and $\tilde{\lambda}$ of λ with endpoints $x, y \in E \subset S_\infty$ such that the pair $\{a, b\}$ separates the pair $\{x, y\}$ in S_∞ . The extension of a lift \tilde{f} to a homeomorphism $\hat{f} : S_\infty \rightarrow S_\infty$ [15, Theorem 2] either preserves or reverses cyclic order, hence $\{\hat{f}^k(a), \hat{f}^k(b)\}$ separates $\{\hat{f}^k(x), \hat{f}^k(y)\}$, $k \geq 0$. Thus, $\sigma_k = f^k(\sigma)$ meets a leaf of Λ_- , $k \geq 0$. For k sufficiently large, σ_k is disjoint from W^- . Since no leaf of Λ_- meets W^+ , it follows that, for all k is sufficiently large, σ_k meets the compact set K , contradicting the hypothesis that σ is an escaping juncture component. An analogous proof shows that a positive escaping juncture component is disjoint from $|\Lambda_+| \cup |\Lambda_-|$. \square

Define laminations $\Lambda_\pm|K$ by taking as leaves the path components of $|\Lambda_\pm| \cap K$.

Lemma 4.52. (1) *The leaves of $\Lambda_\pm|K$ are boundary incompressible arcs which have endpoints on $\text{Fr}_\pm K$ and which fall into finitely many isotopy classes in $(K, \text{Fr}_\pm K)$;*
 (2) *For every such isotopy class \mathcal{A} , there exists a rectangle $R \subset K_i$ with a pair of opposite edges each intervals in not necessarily distinct components α and β of $\text{Fr}_\pm K_i$ such that for every pair of arcs $\sigma, \tau \in \mathcal{A}$, the track of the isotopy between σ and τ lies in R .*

Proof. The leaves of $\Lambda_\pm|K$ have endpoints on $\text{Fr}_\pm K$ by Lemma 4.46. The other assertions of the lemma are proven exactly as in the proof of Lemma 4.46. \square

Lemma 4.53. *An isotopy class of leaves of $\Lambda_\pm|K$ either contains one arc or else contains two extreme arcs τ_1, τ_2 which, together with two arcs in $\text{Fr}_\pm K$, form a quadrilateral bounding a simply connected region in K containing all arcs in the isotopy class. The extreme arcs do not cut off another simply connected quadrilateral.*

Proof. Consider an isotopy class \mathcal{A} of leaves of $\Lambda_-|K$ containing more than one arc. The case of $\Lambda_+|K$ is similar. Let α and β be the components of $\text{Fr}_- K$ containing the endpoints of leaves in the isotopy class \mathcal{A} . By Lemma 4.52 (2), there exists a

rectangle R which contains every arc in \mathcal{A} and has a pair of opposite edges in α and β . Choosing a lift \tilde{R} of R , where \tilde{R} is a rectangle with a pair of opposite edges in $\tilde{\alpha}$ a lift of α and $\tilde{\beta}$ a lift of β , determines lifts $\tilde{\tau}$ lifts of τ for every $\tau \in \mathcal{A}$ such that $\tilde{\tau}$ has endpoints in $\tilde{\alpha}$ and $\tilde{\beta}$.

Because the set $\tilde{\Lambda}_+$ is closed, the set of arcs $\tilde{\mathcal{A}} = \{\tilde{\tau} \mid \tau \in \mathcal{A}\}$ in \tilde{R} contains two arcs σ_1, σ_2 which are extreme arcs of $\tilde{\mathcal{A}}$ and are thus lifts of arcs $\tau_1, \tau_2 \in \mathcal{A}$ which are extreme arcs of \mathcal{A} .

If the extreme arcs cut off another simply connected quadrilateral the union of the two quadrilateral would form an annulus bounded by α and β which would equal K contradicting the fact that K is not an annulus. \square

4.6. An endperiodic automorphism h preserving the laminations. In this subsection we prove the following theorem.

Theorem 4.54. *If $f : L \rightarrow L$ is an endperiodic automorphism, then there exists an endperiodic automorphism $h : L \rightarrow L$, isotopic to f permuting the elements of each of the sets $\Lambda_+, \Lambda_-, \mathcal{J}_+,$ and \mathcal{J}_- .*

Remark. Since h is isotopic to f and permutes the elements of each of the sets $\Lambda_+, \Lambda_-, \mathcal{J}_+,$ and \mathcal{J}_- , it follows that $h(\gamma) = (f(\gamma))^g = (h(\gamma))^g$ if $\gamma \in \Lambda_+ \cup \Lambda_- \cup \mathcal{J}_+ \cup \mathcal{J}_-$. In particular $J_n = h^n(J)$ if J is a juncture.

Remark. This homeomorphism h is not uniquely determined, but

$$h : |\Lambda_+| \cap |\Lambda_-| \rightarrow |\Lambda_+| \cap |\Lambda_-|$$

is unique. This will be called the core dynamical system and will be analyzed in Section 9.

We prove Theorems 4.54 in Section 4.6.3 after some preliminaries in Sections 4.6.1 and 4.6.2..

4.6.1. Sliding isotopies and other isotopies. Recall from page 17 that all isotopies are ambient isotopies. By Definition 4.19, the geodesic tightening τ^g of a pseudo-geodesic τ is the geodesic whose lifts are the geodesics in \tilde{L} sharing endpoints on S_∞ with the lifts of τ .

Suppose σ is a complete geodesic with ordered lift $\tilde{\sigma}$ and $\gamma_1, \dots, \gamma_k$ are pseudo-geodesics which are either disjoint or coincide, form no digons with σ , and meet σ at the points x_1, \dots, x_k with lifts $\tilde{\gamma}_1, \dots, \tilde{\gamma}_k$ meeting $\tilde{\sigma}$ at the points $\tilde{x}_1 < \dots < \tilde{x}_k$ so that the interval $[\tilde{x}_1, \tilde{x}_k] \subset \tilde{\sigma}$ meets no other lift of the $\tilde{\gamma}_i$. Let $\tilde{\gamma}_i^g$ be the lift of the geodesic tightening of γ_i sharing endpoints on S_∞ with $\tilde{\gamma}_i$, $1 \leq i \leq k$.

Lemma 4.55. *Under the above conditions, there exists an isotopy supported in a small neighborhood of σ with lift sliding \tilde{x}_i along $\tilde{\sigma}$ to the point $\tilde{x}_i^g = \tilde{\sigma} \cap \tilde{\gamma}_i^g$, $1 \leq i \leq k$.*

Definition 4.56 (sliding isotopy). We will refer to an isotopy as in Lemma 4.55 as a *sliding isotopy*.

Suppose σ_i , $i = 1, 2$, are complete geodesics which are either disjoint or coincide and have disjoint lifts $\tilde{\sigma}_i$, γ is a pseudo-geodesics forming no digons with the σ_i , and $\alpha = [x_1, x_2] \subset \gamma$ is an arc meeting the σ_i only at the points $x_1 \in \sigma_1$ and $x_2 \in \sigma_2$ with lift $\tilde{\alpha} = [\tilde{x}_1, \tilde{x}_2] \subset \tilde{\gamma}$ meeting $\tilde{\sigma}_1$ at the point \tilde{x}_1 and $\tilde{\sigma}_2$ at the point \tilde{x}_2 . Let $\tilde{\gamma}^g$ be the lift of the geodesic tightening of γ sharing endpoints on S_∞ with $\tilde{\gamma}$. The next lemma follows from Lemma 4.55 and Theorem 3.1.

Lemma 4.57. *Under the above conditions, there is an isotopy, fixing each σ_i , $i = 1, 2$, with lift moving the arc $\tilde{\alpha} = [\tilde{x}_1, \tilde{x}_2] \subset \tilde{\gamma}$ to the geodesic arc $\tilde{\alpha}^g = [\tilde{x}_1^g, \tilde{x}_2^g] \subset \tilde{\gamma}^g$, \tilde{x}_i sliding along $\tilde{\sigma}_i$ to \tilde{x}_i^g , $i = 1, 2$, as in Lemma 4.55.*

4.6.2. *The tilings used in inductive proofs.* Cutting L apart along the juncture components in \mathcal{J}_W (Definition 4.42) decomposes L into a set \mathfrak{T}^g of compact surfaces. Similarly, cutting L apart along the juncture components in $\{g(\gamma) \mid \gamma \in \mathcal{J}_W\}$ decomposes L into a set \mathfrak{T}_*^g of compact surfaces where g is the endperiodic automorphism of Lemma 4.43.

Definition 4.58 (\mathfrak{T}^g , \mathfrak{T}_*^g , tile, tiling). The sets \mathfrak{T}^g and \mathfrak{T}_*^g will be called *tilings* of L . The surfaces in \mathfrak{T}^g and \mathfrak{T}_*^g will be called *tiles*.

Remark. The superscript g on the symbols $\mathfrak{T}^g, \mathfrak{T}_*^g$ emphasizes that the juncture components which are frontiers of the tiles are geodesics.

Remark. The tilings $\mathfrak{T}^g, \mathfrak{T}_*^g$ will be used in the proofs of Theorems 4.54 and 8.1. The tiling \mathfrak{T}^g will be used in the proof of Theorem 10.15.

The following is immediate.

Lemma 4.59. *The tiles of \mathfrak{T}^g and of \mathfrak{T}_*^g have boundary consisting of arcs and circles in ∂L and juncture components.*

The next lemma is an immediate consequence of Lemma 4.43.

Lemma 4.60. *For $P \in \mathfrak{T}^g$, $g(P) \in \mathfrak{T}_*^g$ where g is the endperiodic automorphism of Lemma 4.43.*

4.6.3. *The construction of h .* Let g be the endperiodic automorphism of Lemma 4.43.

Strategy. We first define h on \overline{U} , where U is any component of $L \setminus (|\Lambda_+| \cup |\Lambda_-| \cup |\mathcal{J}_+| \cup |\mathcal{J}_-|)$ which is not a rectangle, in such a way that $h(\partial U)$ is contained in the union of $|g(\mathcal{J}_W)|$ and the circles and extreme arcs of Lemma 4.61. To define h on all such \overline{U} , we modify g by isotopies (Proposition 4.62 and Lemma 4.63) so that if we define $h = g$ on \overline{U} for all such components U , then the procedure of Casson-Bleiler [19, pp. 89-90] can be used to extend h over the rest of L .

The following lemma is proven like Lemmas 4.52 and 4.53.

Lemma 4.61. *If $P \in \mathfrak{T}_*^g$, then there are finitely many isotopy classes of leaves of each of the laminations $(\Lambda_+ \cup (\mathcal{J}_- \setminus g(\mathcal{J}_W)))|P$ and $(\Lambda_- \cup (\mathcal{J}_+ \setminus g(\mathcal{J}_W)))|P$. Each isotopy class either contains one circle in \mathcal{J}_\pm , one arc, or two extreme arcs α_1, α_2 which, together with two arcs in ∂P , form a quadrilateral bounding a simply connected region in P containing all arcs in the isotopy class.*

Remark. If an isotopy class contains only one arc we will also refer to that arc as an extreme arc.

Notation. For $P \in \mathfrak{T}_*^g$ denote by

$$\begin{aligned} \mathcal{G}_+(P) &= g(\Lambda_+ \cup (\mathcal{J}_- \setminus \mathcal{J}_W))|P \\ \mathcal{G}_-(P) &= g(\Lambda_- \cup (\mathcal{J}_+ \setminus \mathcal{J}_W))|P \end{aligned}$$

the laminations consisting of the arcs and circles which are the path components of $|g(\Lambda_+ \cup (\mathcal{J}_- \setminus \mathcal{J}_W))| \cap P$ and $|g(\Lambda_- \cup (\mathcal{J}_+ \setminus \mathcal{J}_W))| \cap P$ and by

$$\begin{aligned} \mathcal{G}_+^g(P) &= (\Lambda_+ \cup (\mathcal{J}_- \setminus g(\mathcal{J}_W)))|P \\ \mathcal{G}_-^g(P) &= (\Lambda_- \cup (\mathcal{J}_+ \setminus g(\mathcal{J}_W)))|P, \end{aligned}$$

the laminations consisting of the geodesic arcs and circles which are the path components of $|\Lambda_+ \cup (\mathcal{J}_- \setminus g(\mathcal{J}_W))| \cap P$ and $|\Lambda_- \cup (\mathcal{J}_+ \setminus g(\mathcal{J}_W))| \cap P$.

Choose lifts \tilde{L} of L and $\tilde{f}: \tilde{L} \rightarrow \tilde{L}$ of f . This determines a lift $\tilde{g}: \tilde{L} \rightarrow \tilde{L}$.

A leaf $\alpha \in \mathcal{G}_\pm(P)$ is either a circle in $g(\mathcal{J}_\pm)$ or an arc. If $\alpha = [x_1, x_2] \in \mathcal{G}_\pm(P)$ is an arc, then α is contained in a leaf $\gamma_\alpha \in g(\Lambda_+ \cup \mathcal{J}_- \cup \Lambda_- \cup \mathcal{J}_+)$ and has endpoints x_i in the geodesics σ_i which are either juncture components in $\text{Fr } P$ or components of ∂L , $i = 1, 2$. Let $\tilde{\alpha} = [\tilde{x}_1, \tilde{x}_2] \subset \tilde{\gamma}_\alpha$ be lifts of $\alpha = [x_1, x_2] \subset \gamma_\alpha$ with \tilde{x}_i in the lift $\tilde{\sigma}_i$ of σ_i , $i = 1, 2$. The geodesic tightening γ_α^g of γ_α has lift $\tilde{\gamma}_\alpha^g$ sharing endpoints on S_∞ with $\tilde{\gamma}_\alpha$ and is a leaf of the lamination $\Lambda_\pm \cup \mathcal{J}_\mp$. There is a unique arc $\tilde{\alpha}^g = [\tilde{x}_1^g, \tilde{x}_2^g] \subset \tilde{\gamma}_\alpha^g$ with $\tilde{x}_i^g \in \tilde{\sigma}_i$, $i = 1, 2$.

The projection of the arc $\tilde{\alpha}^g$ is in $\mathcal{G}_\pm^g(P)$ and will be denoted $\alpha^g = [x_1^g, x_2^g]$. If α is a circle, let α^g be the geodesic tightening of α .

Remark. The correspondence $\alpha \leftrightarrow \alpha^g$ induces a one-one correspondence between $\mathcal{G}_\pm(P)$ and $\mathcal{G}_\pm^g(P)$.

Remark. Suppose $P \in \mathfrak{T}_*^g$.

- (1) If $P \subset g(W_-)$, then $\mathcal{G}_+(P) = \emptyset$ and $\mathcal{G}_+^g(P) = \emptyset$;
- (2) If $P \subset g(W_+)$, then $\mathcal{G}_-(P) = \emptyset$ and $\mathcal{G}_-^g(P) = \emptyset$;
- (3) If $P = g(K)$, $\mathcal{G}_-(P) \neq \emptyset$, $\mathcal{G}_-^g(P) \neq \emptyset$, $\mathcal{G}_+(P) \neq \emptyset$, $\mathcal{G}_+^g(P) \neq \emptyset$.

It follows that an element of $\mathcal{G}_-(P)$ can meet an element of $\mathcal{G}_+(P)$ only if $P = g(K)$.

We will call $\alpha \in \mathcal{G}_\pm(P)$ an extreme arc of $\mathcal{G}_\pm(P)$ if the corresponding arc $\alpha^g \in \mathcal{G}_\pm^g(P)$ is an extreme arc of $\mathcal{G}_\pm^g(P)$.

Proposition 4.62. *There exists an isotopy Φ preserving each $P \in \mathfrak{T}_*^g$ and such that $\Phi^1(\alpha) = \alpha^g$ for each circle or extreme arc $\alpha \in \mathcal{G}_\pm(P)$, all $P \in \mathfrak{T}_*^g$.*

Proof. We define Φ inductively. Enumerate the circles and extreme arcs of $\mathcal{G}_\pm(P)$, $P \in \mathfrak{T}_*^g$, in a sequence $\{\alpha_n\}_{n=0}^\infty$ in such a way that every such arc or circle lying in the i^{th} -core K_i is listed before every such arc or circle meeting $L \setminus K_i$, $i \geq 0$. Let $\psi_0 = \text{id}$ and inductively find sequences $\{\Phi_n\}$ of isotopies and $\{\psi_n\}$ of homeomorphisms such that $\psi_n = \Phi_n^1 \circ \psi_{n-1}$, Φ_n preserves each $P \in \mathfrak{T}_*^g$, Φ_n fixes $\psi_{n-1}(\alpha_i)$ for $1 \leq i \leq n-1$, and $\psi_n(\alpha_n) = \alpha_n^g$.

In defining Φ_n there are three cases to consider. If $\alpha_n \in \mathcal{J}_- \cup \mathcal{J}_+$ is a circle in P not meeting $\alpha_1 \cup \dots \cup \alpha_{n-1}$, define Φ_n using Theorem 3.1.

If $\alpha_n \in \mathcal{G}_\pm(P)$, $P \in \mathfrak{T}_*^g$, is an extreme arc not meeting $\alpha_1 \cup \dots \cup \alpha_{n-1}$, then $\alpha_n = [x_1, x_2]$ has endpoints x_1, x_2 in geodesics σ_1, σ_2 which are juncture components in $\text{Fr } P$ or components of ∂L and $\alpha_n \subset \gamma_n \in g(\Lambda_+ \cup \mathcal{J}_- \cup \Lambda_- \cup \mathcal{J}_+)$. By Lemma 4.57, there is an isotopy, fixing each σ_i , $i = 1, 2$, with lift moving the arc $\tilde{\alpha} = [\tilde{x}_1, \tilde{x}_2] \subset \tilde{\gamma}_n$ to the geodesic arc $\tilde{\alpha}^g = [\tilde{x}_1^g, \tilde{x}_2^g] \subset \tilde{\gamma}_n^g$, \tilde{x}_i sliding along $\tilde{\sigma}_i$ to \tilde{x}_i^g , $i = 1, 2$, as in Lemma 4.55. Here $\tilde{\gamma}_n^g$ is the lift of γ_n^g sharing endpoints on S_∞ with $\tilde{\gamma}_n$.

If α_n is a circle in $\mathcal{J}_- \cup \mathcal{J}_+$ or extreme arc meeting $\alpha_1 \cup \dots \cup \alpha_{n-1}$ (which, by the Remark after the introduction of the notation $\mathcal{G}_\pm(P)$, $\mathcal{G}_\pm^g(P)$, can happen only when $P = g(K)$ where K is the core), then use Lemma 4.57 on each of the subarcs α_n is divided into by $\alpha_1, \dots, \alpha_{n-1}$. In all three cases apply the remark after Theorem 3.3 to the surface P .

Thus, $\psi_i \rightarrow \psi$, a well defined homeomorphism isotopic to the identity by an isotopy Φ and $\Phi^1(\alpha) = \alpha^g$ for each $\alpha \in \mathcal{G}_\pm(P)$. \square

Redefine g as $\Phi^1 \circ g$.

Since there are finitely many extreme arcs from Lemma 4.61 in each tile that can be edges of ∂U for some component U of $K \setminus (|\Lambda_+| \cup |\mathcal{J}_-| \cup |\Lambda_-| \cup |\mathcal{J}_+|)$ which is not a rectangle, these extreme arcs do not accumulate. The following is elementary.

Lemma 4.63. *After a further isotopy of g , we can assume that g is linear in the hyperbolic metric on each arc that is an edge of ∂U for each component U of $L \setminus (|\Lambda_+| \cup |\mathcal{J}_-| \cup |\Lambda_-| \cup |\mathcal{J}_+|)$ which is not a rectangle.*

At this stage we define $h|_{\bar{U}} = g|_{\bar{U}}$ if U is a component of $L \setminus (|\Lambda_+| \cup |\mathcal{J}_-| \cup |\Lambda_-| \cup |\mathcal{J}_+|)$ which is not a rectangle.

To finish the proof of Theorem 4.54, we must extend h over the rest of L . To do this, we mimic the proof of Lemma 6.1 of Casson-Bleiler [19, pp. 89-90]. Since Casson-Bleiler deal with an irreducible endperiodic automorphism of a closed surface they do not have to handle regions that are not simply connected.

Choose lifts \tilde{L} of L and $\tilde{g} : \tilde{L} \rightarrow \tilde{L}$ of g . This determines an extension $\tilde{g} : \tilde{L} \rightarrow \tilde{L}$. Let

$$X = \left((|\Lambda_+| \cup |\mathcal{J}_-|) \cap (|\Lambda_-| \cup |\mathcal{J}_+|) \right) \cup \left((|\mathcal{J}_-| \cup |\mathcal{J}_+|) \cap \partial \tilde{L} \right)$$

and \tilde{X} the set of lifts of the points of X to \tilde{L} . Define $\tilde{h} : \tilde{X} \rightarrow \tilde{X}$ by,

- (1) $\tilde{h}(\tilde{x}) = (\tilde{g}(\tilde{\gamma}_1))^{\mathfrak{s}} \cap (\tilde{g}(\tilde{\gamma}_2))^{\mathfrak{s}}$ if $\tilde{x} = \tilde{\gamma}_1 \cap \tilde{\gamma}_2$ with $\tilde{\gamma}_1 \in \tilde{\Lambda}_+ \cup \tilde{\mathcal{J}}_-$ and $\tilde{\gamma}_2 \in \tilde{\Lambda}_- \cup \tilde{\mathcal{J}}_+$;
- (2) $\tilde{h}(\tilde{x}) = \tilde{g}(\tilde{x})$ if $\tilde{x} \in (|\mathcal{J}_-| \cup |\mathcal{J}_+|) \cap \partial \tilde{L}$.

The fact that $\tilde{g} : \tilde{L} \rightarrow \tilde{L}$ is continuous implies that the map $\tilde{h} : \tilde{X} \rightarrow \tilde{X}$ is a homeomorphism. As in Casson-Bleiler [19, pp. 89-90], we extend \tilde{h} linearly and equivariantly over any lift of an arc of $|\Lambda_+| \cup |\mathcal{J}_-| \cup |\Lambda_-| \cup |\mathcal{J}_+|$ with both endpoints in X and interior disjoint from X and equivariantly over the lifts of rectangular components of $L \setminus (|\Lambda_+| \cup |\mathcal{J}_-| \cup |\Lambda_-| \cup |\mathcal{J}_+|)$ using the technique of Casson-Bleiler [19, pp. 90]. We have already defined h on the nonrectangular components of $L \setminus (|\Lambda_+| \cup |\mathcal{J}_-| \cup |\Lambda_-| \cup |\mathcal{J}_+|)$ to match the extensions over the rectangular components on shared boundary edges. These extensions lift to give $\tilde{h} : \tilde{L} \rightarrow \tilde{L}$.

Both $\tilde{f}, \tilde{h} : \tilde{L} \rightarrow \tilde{L}$ are defined and agree on S_∞ . Thus, Corollary 5 of [15] implies that the maps h and f are isotopic on L .

Theorem 4.54 is proven.

4.7. The escaping sets. From now on, $h : L \rightarrow L$ is an endperiodic automorphism, isotopic to f , and permuting the elements of each of the sets Λ_+ , Λ_- , \mathcal{J}_+ , and \mathcal{J}_- .

Definition 4.64 (positive/negative escaping set, \mathcal{U}_e , \mathcal{U}_\pm). For $e \in \mathcal{E}(L)$, set $\mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} U_e^n$. The union of the sets \mathcal{U}_e as e ranges over the negative (respectively positive) ends will be denoted by \mathcal{U}_- (respectively \mathcal{U}_+). We will call \mathcal{U}_- the *negative escaping set* and \mathcal{U}_+ the *positive escaping set*.

Lemma 4.65. *The set \mathcal{U}_+ (respectively \mathcal{U}_-) consists of the set of points $x \in L$ such that the sequence $\{h^n(x)\}_{n \geq 0}$ (respectively $\{h^n(x)\}_{n \leq 0}$) escapes (Definition 3.7).*

Lemma 4.66. *The positive escaping set \mathcal{U}_+ and the negative escaping set \mathcal{U}_- are each independent of the choice of the set of f -junctures.*

Proof. By the definition of the sets \mathcal{U}_\pm and the remark after Definition 4.37, it suffices to prove that the set $\mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} U_e^n$ is independent of the choice of

distinguished neighborhood U_e for every end $e \in \mathcal{E}(L)$. Let V_e be another choice of distinguished neighborhood of e and $\mathcal{V}_e = \bigcup_{n=-\infty}^{\infty} V_e^n$. By Lemma 4.40, $\{U_e^i\}_{i=0}^{\infty}$ is a fundamental system of neighborhoods of e . Therefore, there exists $i \geq 0$ such that $U_e^i \subset V_e$. It follows that $\mathcal{U}_e \subset \mathcal{V}_e$. The reverse inequality is proven in the same way. Thus, $\mathcal{U}_e = \mathcal{V}_e$ as desired. \square

Remark. The set \mathcal{U}_e is clearly open and connected. By Lemma 4.38, if $e \neq e'$ are both positive ends or both negative ends, $\mathcal{U}_e \cap \mathcal{U}_{e'} = \emptyset$, hence the \mathcal{U}_e 's are the connected components of \mathcal{U}_- , as e ranges over the negative ends, and the connected components of \mathcal{U}_+ , as e ranges over the positive ends. Evidently, every negative juncture lies in \mathcal{U}_- and every positive one in \mathcal{U}_+ .

Remark. $\mathcal{U}_- \cap \mathcal{U}_+ \neq \emptyset$.

Definition 4.67 (escaping set, \mathcal{U}). The *escaping set* is $\mathcal{U} = \mathcal{U}_- \cap \mathcal{U}_+$.

Lemma 4.68. *The set \mathcal{U} consists of the set of points $x \in L$ such that the sequence $\{h^n(x)\}_{n \in \mathbb{Z}}$ escapes.*

Lemma 4.69. *The leaves of Λ_{\pm} do not meet \mathcal{U}_{\mp} .*

Proof. If $x \in \mathcal{U}_-$ then, by Definition 4.64, $x \in \mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} U_e^n$ for some negative end e . Thus, x has a neighborhood that meets at most one negative juncture component. Thus, $x \notin |\Lambda_+|$. A parallel argument shows $\mathcal{U}_+ \cap |\Lambda_-| = \emptyset$. \square

Corollary 4.70. $\Lambda_+ \cap \Lambda_- \subset \text{int } K$.

Proof. Since $W^- \subset \mathcal{U}_-$, $W^+ \subset \mathcal{U}_+$, and $\text{int } K$ is the complement of $W^- \cup W^+$, the corollary follows immediately from Lemma 4.69. \square

Lemma 4.71. *The frontier of \mathcal{U}_{\mp} is $|\Lambda_{\pm}|$.*

Proof. Let $x \in |\Lambda_{\pm}|$. By construction, there is a sequence $\{x_n\}_{n \geq 0} \subset |\mathcal{X}_{\mp}|$ which converges to x . This sequence lies in \mathcal{U}_{\mp} and, by Lemma 4.69, $x \notin \mathcal{U}_{\mp}$, hence $x \in \text{Fr } \mathcal{U}_{\mp}$. For the reverse inclusion, let $x \in \text{Fr } \mathcal{U}_{\mp}$. Then $x \notin \mathcal{U}_{\mp}$ but every connected neighborhood of x meets \mathcal{U}_{\mp} , hence by Definition 4.64 meets a distinguished neighborhood, hence meets a juncture. It follows that $x \in |\Lambda_{\pm}|$. \square

Lemma 4.71 and Corollary 4.66 imply,

Corollary 4.72. *The laminations Λ_{\pm} are independent of the choice of the set of f -junctures.*

Proposition 4.73. *Leaves $\tilde{\lambda}_- \in \tilde{\Lambda}_-$ and $\tilde{\lambda}_+ \in \tilde{\Lambda}_+$ cannot have a common ideal endpoint.*

Proof. Suppose $a \in E$ is a common ideal endpoint of $\tilde{\lambda}_- \in \tilde{\Lambda}_-$ and $\tilde{\lambda}_+ \in \tilde{\Lambda}_+$. Let the end ε of λ_+ have a neighborhood $[x, \varepsilon) \subset \lambda_+$, whose lift approaches a . By Corollary 4.50 $[x, \varepsilon) \subset \lambda_+$ must pass arbitrarily near some positive end e . Let $\{U_e^n\}_{n=0}^{\infty}$ be a fundamental system of distinguished neighborhoods of e (Lemma 4.40). Thus, the neighborhood $[x, \varepsilon)$ of ε must cross $J_e^n = \text{Fr } U_e^n$ for all $n \geq 0$. Let σ_n be a component of J_e^n crossed by $[x, \varepsilon)$ and $\tilde{\sigma}_n$ be a lift of σ_n that meets $\tilde{\lambda}_+$. By Theorem 4.24, the $\tilde{\sigma}_n$ have endpoints on S_{∞} which nest on a . Since $\tilde{\lambda}_-$ has ideal endpoint a , it follows that λ_- meets a positive juncture which violates Lemma 4.69. \square

Proposition 4.74. *Every leaf of Λ_{\pm} meets at least one leaf of Λ_{\mp} .*

Proof. Let λ be a leaf of Λ_- . The proof is parallel for $\lambda \in \Lambda_+$. By Lemma 4.46, λ meets some negative juncture component $\sigma \in \mathcal{X}_-$. Choose a point $x \in \sigma \cap \lambda$. Let $\lambda_n = h^n(\lambda)$ and $\sigma_n = h^n(\sigma)$. Let $x_n = \lambda_n \cap \sigma_n$. The sequence $\{x_n\}$ accumulates at a point $y \in \text{int } K$. The point y lies in $|\Lambda_-|$ since the sequence $\{x_n\}$ lie in the closed set $|\Lambda_-|$. Since $x_n \in h^n(\sigma)$, the sequence $\{x_n\} \subset |\mathcal{X}_-|$. Thus, by the construction of Λ_+ , the point y lies in a leaf $\lambda_+ \in \Lambda_+$.

Let (V, X, Y, φ) be a bilamination chart for the bilamination (Λ_+, Λ_-) where V is an open neighborhood of y . Choose N so that $x_N \in V$. Then the plaque $P \subset \lambda_N \cap V$ containing x_N meets λ_+ . Thus, $\lambda = h^{-N}(\lambda_N)$ meets $h^{-N}(\lambda_+)$ which is a leaf of Λ_+ . The proposition is proven. \square

Corollary 4.75. *Each leaf of Λ_{\pm} meets $\text{int } K$.*

Proof. The corollary follows since, by Corollary 4.70, $|\Lambda_-| \cap |\Lambda_+| \subset \text{int } K$. \square

4.8. Translations. We next consider the possibility, not yet excluded, that the laminations Λ_{\pm} and Γ_{\pm} might be empty.

Proposition 4.76. *The following are equivalent,*

- (1) $\Lambda_+ = \emptyset = \Lambda_-$.
- (2) *All junctures J escape.*
- (3) *Some juncture J escapes.*
- (4) *f is isotopic to a translation g (Definition 2.34).*

Proof. Clearly, (1) \Rightarrow (2) \Rightarrow (3). We prove (3) \Rightarrow (4) \Rightarrow (1).

(3) \Rightarrow (4). Suppose that $J = \text{Fr } U_{e_-}$ is a juncture that escapes where U_{e_-} is a distinguished neighborhood (Definition 4.37) of a negative end e_- . The proof in the other case is analogous. By Definition 4.25, the juncture J escapes if the set $\{J_n\}_{n \in \mathbb{Z}}$ escapes. Exactly as in the proof of Lemma 4.43, f is isotopic to an endperiodic automorphism g such that $g^n(J) = J_n$, $n \in \mathbb{Z}$. Then there exists an integer $N \geq 0$ such that $J_N \subset W_+$. Since $L \setminus g^N(U_{e_-})$ is connected (by Definition 2.2) and $J_N = \text{Fr } g^N(U_{e_-}) \subset W_+$, it follows that $L \setminus g^N(U_{e_-})$ is contained in a component of W_+ . That is, $L \setminus g^N(U_{e_-}) \subset U_{e_+}$ where U_{e_+} is a distinguished neighborhood of a positive end e_+ . It follows that L has just two ends e_-, e_+ .

Let V be an arbitrarily small neighborhood of e_+ such that $\text{Fr } V$ separates L . Since J escapes, there exists an integer $n \geq 0$ such that $J_n \subset V$. Since $L \setminus g^n(U_{e_-})$ is connected and $\text{Fr } V$ separates L , it follows that $L \setminus g^n(U_{e_-}) \subset V$. Since V was an arbitrarily small neighborhood of e_+ , it follows that $\bigcup_{n=0}^{\infty} g^n(U_{e_-}) = L$. Since $\{J_n\}_{n \leq 0}$ escapes and $g^n(J) = J_n$, $n \leq 0$, it follows that U_{e_-} is g -neighborhood of e_- and that g is a translation.

(4) \Rightarrow (1). Let g be a translation isotopic to f . By a preliminary isotopy of g , we can assume that $f|_{\partial L} = g|_{\partial L}$. Since $\widehat{f}|_E = \widehat{g}|_E$ for suitable choices of completed lifts [15, Corollary 5], we see that $\widehat{f}|_{S_{\infty}} = \widehat{g}|_{S_{\infty}}$. Suppose $\sigma \in \mathcal{J}_-$. Since g is a translation, $\{g^n(\sigma)\}_{n \geq 0}$ escapes. Denote by σ_n , the geodesic tightening of $g^n(\sigma)$, $n \geq 0$. By Theorem 3.11, since $\{g^n(\sigma)\}_{n \geq 0}$ escapes, then $\{\sigma_n\}_{n \geq 0}$ escapes. Since $\widehat{f}|_{S_{\infty}} = \widehat{g}|_{S_{\infty}}$, σ_n is also the geodesic tightening of $f^n(\sigma)$, $n \geq 0$. Therefore, an arbitrary negative juncture component σ escapes under f . It follows that $\Lambda_+ = \emptyset$. Similarly $\Lambda_- = \emptyset$. \square

Corollary 4.77. *If f is not isotopic to a translation and e is a positive (respectively negative) end of L , then every neighborhood of e meets $|\Lambda_+|$ (respectively $|\Lambda_-|$).*

Proof. For definiteness, let e be a positive end of L and consider a juncture J which is the frontier of a distinguished neighborhood of e . The sequence $\{h^k(J)\}_{k \geq 0}$ escapes, but by Proposition 4.76, for some component σ of J , the sequence $\{h^k(\sigma)\}_{k \leq 0}$ does not escape and therefore accumulates locally uniformly on at least one leaf $\lambda_- \in \Lambda_-$. By Proposition 4.74, λ_- meets at least one leaf $\lambda_+ \in \Lambda_+$ and does so transversely. Thus, λ_+ meets $h^k(J)$, for some $k < 0$. Applying suitable arbitrarily large positive powers of h produces leaves of Λ_+ meeting $h^{npe}(J)$ for arbitrarily large values of n and the assertion follows. \square

Corollary 4.78. $\Lambda_+ = \emptyset$ if and only if f is isotopic to a translation if and only if $\Lambda_- = \emptyset$.

Proof. If either lamination is empty, some juncture escapes. Since (3) implies (4) in Lemma 4.76, f is isotopic to a translation. If f is isotopic to a translation, the implication (4) \Rightarrow (1) implies $\Lambda_+ = \emptyset = \Lambda_-$. \square

Thus, endperiodic automorphisms that are isotopic to translations are uninteresting from the point of view of Handel-Miller theory.

Hypothesis 6. Hereafter, we assume that f is not isotopic to a translation.

5. THE COMPLEMENTARY REGIONS TO THE LAMINATIONS

We identify the components of $L \setminus |\Lambda_{\pm}|$. They fall into two essentially different types: the positive and negative escaping sets (Definition 4.64) and the principal regions. First we need some technicalities about open sets, their internal completions and borders.

5.1. Internal completion and the border of open sets. Let $U \subseteq L$ be an open, connected subset and define the “path metric” d as follows. Given two points x, y in this open, connected set, there are piecewise geodesic paths connecting these points. Each such path has a well-defined length and we define $d(x, y)$ to be the greatest lower bound of these lengths. It is standard that this defines a topological metric on U compatible with the topology of this open set. Notice that two points can be very far apart in the metric d and very close together in the hyperbolic metric of L .

Definition 5.1 (internal completion, \ddot{U}, \dot{i}). Define the *internal completion* \ddot{U} to be the completion of U in the metric d and define the *boundary* of \ddot{U} to be

$$\partial\ddot{U} = (\ddot{U} \setminus U) \cup (U \cap \partial L).$$

As is standard, the metric d extends to a metric on \ddot{U} which we will again denote by d .

While \ddot{U} and $\partial\ddot{U}$ are not generally subsets of L , the inclusion map $\iota : U \hookrightarrow L$ is a topological embedding. It extends canonically to a continuous map $\dot{i} : \ddot{U} \rightarrow L$. This map may be very pathological on $\ddot{U} \setminus U$, but in the cases occurring in this paper, it will be an immersion on this set.

Definition 5.2 (border component). The image under i of a component of $\ddot{U} \setminus U$ will be called a *border component* of U . The set δU of border components of U will be called the *border* of U . The *support* $|\delta U| \subset L$ of the border is the union of the elements of δU as subsets of L .

Remark. Distinct components of δU can intersect as subsets of L .

Remark. We must be careful to distinguish $|\delta U|$ from the set-theoretic *frontier* $\text{Fr } U$. Clearly, $|\delta U| \subset \text{Fr } U$.

Example 5.3. Quite often a component U of the escaping set \mathcal{U} (Definition 4.67) has $|\delta U| = \text{Fr } U$, but not always. In Example 4.35, if U is the component of \mathcal{U} containing the bottom boundary component of L , then $\text{Fr } U$ properly contains $|\delta U|$ and \bar{U} properly contains $i(\ddot{U})$. That is, $i(\ddot{U})$ is not closed in L . For all other components U of \mathcal{U} in this example, $|\delta U| = \text{Fr } U$ and $\bar{U} = i(\ddot{U})$.

Remark. For the open sets U encountered in this paper, δU is a set of lines and circles, often pieced together from arcs in leaves of Γ_+ and Γ_- .

These notions extend to open sets that are not connected. If U is such, \ddot{U} denotes the disjoint union of the internal completions of each component. Correspondingly, $\partial\ddot{U}$ is defined separately for each component, $i : \ddot{U} \rightarrow L$ is defined componentwise, and δU is the set of images under i of components of $\ddot{U} \setminus U$.

Lemma 5.4. *For each point $x \in \partial\ddot{U}$, there exists a continuous map $s : [0, 1] \rightarrow \ddot{U}$ such that $s(1) = x$ and $s(t) \in U$, $0 \leq t < 1$.*

Proof. If $x \in \partial\ddot{U}$, there is a Cauchy sequence (relative to the metric d) $\{x_n\}_{n=1}^\infty \subset U$ that converges to x . Let $\varepsilon_n > d(x_n, x_{n+1})$ such that $\lim_{n \rightarrow \infty} \varepsilon_n = 0$. Then there is a piecewise geodesic path $s_n \subset U$ joining x_n to x_{n+1} of length $< \varepsilon_n$. Every point in this path is within ε_n of x_{n+1} . Joining these paths sequentially and suitably parametrizing produces a continuous map $s : [0, 1] \rightarrow U$ such that $\lim_{t \rightarrow 1} s(t) = x$. Thus, s is extended continuously to $[0, 1]$ so that $s(1) = x$. \square

Applying i to the above picture gives the following.

Corollary 5.5. *If x is a point of an element $\ell \in \delta U$, there exists a continuous map $s : [0, 1] \rightarrow L$ such that $s(1) = x$ and $s(t) \in U$, $0 \leq t < 1$.*

A leaf λ_\pm of Λ_\pm will either be *isolated* in Λ_\pm (i.e. approached by points of $|\Lambda_\pm|$ on neither side) or approached by points of $|\Lambda_\pm|$ on one or both sides.

Definition 5.6 (semi-isolated). If a leaf $\lambda_\pm \in \Lambda_\pm$ is approached by points of $|\Lambda_\pm|$ on at most one side, we say λ_\pm is *semi-isolated*.

Remark. Note that our definition of semi-isolated includes all isolated leaves. Recall that \mathcal{U} denotes the escaping set (Definition 4.64).

Corollary 5.7. *An element of $\delta\mathcal{U}$ is a subset of the union of the semi-isolated leaves in $\Lambda_+ \cup \Lambda_-$.*

Proof. If $x \in |\delta\mathcal{U}|$, then $x \notin \mathcal{U}$, but x is the limit of a sequence $\{x_n\}$ of points of \mathcal{U} . Thus, either $x \notin \mathcal{U}_+$ or $x \notin \mathcal{U}_-$. In either case, the sequence $\{x_n\} \subset \mathcal{U}_+ \cap \mathcal{U}_-$ and so either $x \in \text{Fr } \mathcal{U}_+$ or $x \in \text{Fr } \mathcal{U}_-$. By Lemma 4.71, $x \in |\Lambda_+| \cup |\Lambda_-|$. Then, by Corollary 5.5, the points of $|\delta\mathcal{U}|$ must lie on semi-isolated leaves of Λ_\pm . \square

The sets U that we will be considering have $|\delta U|$ made up piecewise of subarcs of $|\Gamma_+| \cup |\Gamma_-|$. Since each such subarc of $|\delta U|$ has two sides, the following lemma is obvious.

Lemma 5.8. *If the set U is such that $|\delta U|$ is made up piecewise of subarcs of $|\Gamma_+| \cup |\Gamma_-|$, then each such subarc is the image of at most two such subarcs in $\tilde{U} \setminus U$.*

Remark. Thus a point $x \in |\delta U|$ may be viewed as (one of possibly two preimages) living in $\tilde{U} \setminus U$.

5.2. The positive and negative escaping sets. Recall that $|\Lambda_{\pm}| = \text{Fr } \mathcal{U}_{\mp}$ (Lemma 4.71).

Lemma 5.9. *An element of $\delta \mathcal{U}_{\pm}$ is a semi-isolated leaf in Λ_{\mp} .*

Proof. We prove the lemma for $\delta \mathcal{U}_+$. The proof for $\delta \mathcal{U}_-$ is analogous. If $\gamma \in \delta \mathcal{U}_+$, then $\gamma \subset |\delta \mathcal{U}_+| \subset \text{Fr } \mathcal{U}_+ = |\Lambda_-|$. Since the components of $|\Lambda_-|$ are the leaves of Λ_- and γ is connected, it follows that γ is a subset of a leaf $\lambda \in \Lambda_-$. Let U_γ be the component of \mathcal{U}_+ such that $\gamma \in \delta U_\gamma$. By Corollary 5.5 applied to $x \in \gamma \subset \lambda$, it follows that λ borders U_γ and is semi-isolated on the side bordering U_γ . Thus, $\gamma = \lambda$. \square

Lemma 5.10. *If e is a negative (respectively, positive) end, then \mathcal{U}_e (Definition 4.64) is a component of $L \setminus |\Lambda_+|$ (respectively, of $L \setminus |\Lambda_-|$).*

Proof. Indeed, \mathcal{U}_e is connected and lies in the complement of $|\Lambda_+|$. But $\text{Fr } \mathcal{U}_e \subseteq \text{Fr } \mathcal{U}_- = |\Lambda_+|$. \square

Proposition 5.11. *If $\lambda \in \Lambda_+$ (respectively $\lambda \in \Lambda_-$) is a border leaf of a component \mathcal{U}_e of \mathcal{U}_- (respectively of \mathcal{U}_+), then the set of negative junctures (respectively positive junctures) accumulates locally uniformly on λ on any side that borders \mathcal{U}_e .*

Proof. Suppose $\lambda \in \Lambda_+$ is a border leaf of \mathcal{U}_e where \mathcal{U}_e is a component of \mathcal{U}_- . The proof in the alternate case is analogous. If $x \in \lambda$, then there exists a sequence $\{x_n\}_{n \geq 0} \subset \mathcal{U}_e \cap |\mathcal{X}_-|$ which converges to x . The proposition then follows since Γ_+ is strongly closed. \square

Proposition 5.12. *$|\delta \mathcal{U}_{\mp}|$ is dense in $|\Lambda_{\pm}|$.*

Proof. We show that $|\delta \mathcal{U}_-|$ is dense in $|\Lambda_+|$. The proof that $|\delta \mathcal{U}_+|$ is dense in $|\Lambda_-|$ is analogous. Let x be a point of a leaf of Λ_+ and α a transverse arc containing x in its interior. Since $|\mathcal{X}_-|$ accumulates on every point of $|\Lambda_+|$, there exists a point $y \in \alpha$ lying on a negative juncture and thus in \mathcal{U}_e for some negative end e . Thus α meets a border component of \mathcal{U}_- . Since α was arbitrary it follows that $|\delta \mathcal{U}_{\mp}|$ is dense in $|\Lambda_{\pm}|$. \square

5.3. Principal regions. In the Nielsen-Thurston theory, the connected components of the complements of the laminations are called principal regions. In the Handel-Miller theory, there are escaping regions which should not be thought of as principal regions. In fact, the appropriate analogues of principal regions in our situation do not even exist in many cases. When they do, they have a nucleus and finitely many arms, analogous to the principal regions of Nielsen-Thurston.

Definition 5.13 (positive/negative principal regions). The set \mathcal{P}_+ is the union of the components of the complement of $|\Lambda_+|$ that contain no points in $|\mathcal{X}_-|$. We define a *positive principal region* to be a component of \mathcal{P}_+ . Similarly, the set \mathcal{P}_- is the union of the components of the complement of $|\Lambda_-|$ that contain no points in $|\mathcal{X}_+|$ and we define a *negative principal region* to be a component of \mathcal{P}_- .

Lemma 5.14. $L = \mathcal{U}_- \cup |\Lambda_+| \cup \mathcal{P}_+$ and $L = \mathcal{U}_+ \cup |\Lambda_-| \cup \mathcal{P}_-$ where the unions are disjoint.

Proof. Suppose V is a component of $L \setminus |\Lambda_+|$ that contains a point $x \in |\mathcal{X}_-|$. Since some component \mathcal{U}_e of $L \setminus |\Lambda_+|$ contains x , $V = \mathcal{U}_e$. \square

Lemma 5.15. The escaping set $\mathcal{U} = L \setminus (|\Lambda_-| \cup \mathcal{P}_- \cup |\Lambda_+| \cup \mathcal{P}_+)$.

Proof.

$$\begin{aligned} \mathcal{U} &= \mathcal{U}_- \cap \mathcal{U}_+ \\ &= (L \setminus (|\Lambda_+| \cup \mathcal{P}_+)) \cap (L \setminus (|\Lambda_-| \cup \mathcal{P}_-)) \\ &= L \setminus (|\Lambda_-| \cup \mathcal{P}_- \cup |\Lambda_+| \cup \mathcal{P}_+). \end{aligned}$$

\square

Definition 5.16 (invariant set). The *invariant set* is $\mathcal{J} = L \setminus (\mathcal{U}_- \cup \mathcal{U}_+)$.

Of course, \mathcal{J} will be the set of points x such that neither the positive nor negative iterates under h of x escape.

Lemma 5.17. $\mathcal{J} = (|\Lambda_+| \cap |\Lambda_-|) \cup (|\Lambda_+| \cap \mathcal{P}_-) \cup (|\Lambda_-| \cap \mathcal{P}_+) \cup (\mathcal{P}_+ \cap \mathcal{P}_-)$

Proof.

$$\begin{aligned} \mathcal{J} &= L \setminus (\mathcal{U}_- \cup \mathcal{U}_+) \\ &= (L \setminus \mathcal{U}_-) \cap (L \setminus \mathcal{U}_+) \\ &= (|\Lambda_+| \cup \mathcal{P}_+) \cap (|\Lambda_-| \cup \mathcal{P}_-) \\ &= (|\Lambda_+| \cap |\Lambda_-|) \cup (|\Lambda_+| \cap \mathcal{P}_-) \cup (|\Lambda_-| \cap \mathcal{P}_+) \cup (\mathcal{P}_+ \cap \mathcal{P}_-) \end{aligned}$$

\square

Remark. If there is a finite set of compact components of ∂L which is permuted by h , then each of these components lies in $\mathcal{P}_+ \cap \mathcal{P}_-$.

Lemma 5.18. $\mathcal{P}_+ \subset \mathcal{J} \cup \mathcal{U}_+$ and $\mathcal{P}_- \subset \mathcal{J} \cup \mathcal{U}_-$.

Proof. If $x \in \mathcal{P}_+$ and $x \notin \mathcal{U}_+$, then $x \in L \setminus (\mathcal{U}_- \cup \mathcal{U}_+) = \mathcal{J}$. Thus $\mathcal{P}_+ \subset \mathcal{J} \cup \mathcal{U}_+$. Similarly, $\mathcal{P}_- \subset \mathcal{J} \cup \mathcal{U}_-$. \square

5.4. Action of h on semi-isolated leaves of Λ_\pm and components of \mathcal{U}_\pm .

Lemma 5.19. The map $h : \Lambda_\pm \rightarrow \Lambda_\pm$ carries semi-isolated leaves to semi-isolated leaves.

This is a consequence of the fact that $h : L \rightarrow L$ is a homeomorphism.

Since $\mathcal{U}_\mp \cup \mathcal{P}_\pm$ consists of all complementary regions of $|\Lambda_\pm|$, the semi-isolated leaves of Λ_\pm are exactly the border leaves of these regions. In the case of border leaves of \mathcal{U}_\pm , it is possible that both sides of the leaf borders this set, in which case the semi-isolated leaf is actually isolated and the natural map $\tilde{i} : \partial\tilde{\mathcal{U}}_\pm \rightarrow L$ will not be one-one, but rather will identify some boundary components pairwise. In the case of $\delta\mathcal{P}_\pm$, this cannot happen.

Lemma 5.20. *Each leaf ℓ of $\delta\mathcal{P}_\pm$ borders \mathcal{P}_\pm on only one side. Thus, $i : \partial\ddot{\mathcal{P}}_\pm \rightarrow L$ is one-one.*

Proof. Indeed, in the case of \mathcal{P}_+ , negative h -junctures cluster on ℓ on one side. An analogous argument holds for \mathcal{P}_- . \square

Lemma 5.21. *The sets of border leaves $\delta\mathcal{U}_\pm$ and $\delta\mathcal{P}_\pm$ are h -invariant.*

Proof. An isolated side of a leaf λ borders \mathcal{U}_\pm if and only if a sequence of leaves of \mathcal{X}_\mp accumulates on λ from that side. The lemma then follows since $h : L \rightarrow L$ is a homeomorphism. \square

Corollary 5.22. *The set $\delta\mathcal{U}$ is h -invariant.*

Lemma 5.23. $h(\mathcal{U}_e) = \mathcal{U}_{f(e)}$.

Proof. Recall that $\mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} U_e^n$ where $U_e^n = h^{np_e}(U_e)$ for U_e a distinguished neighborhood of e and that $\mathcal{U}_{f(e)} = \bigcup_{n=-\infty}^{\infty} U_{f(e)}^n$ where $U_{f(e)} = h(U_e)$ is a distinguished neighborhood of $f(e)$ and $U_{f(e)}^n = h^{np_e}(U_{f(e)}) = h^{np_e}(h(U_e)) = h(U_e^n)$ (clearly $p_e = p_{f(e)}$). Thus,

$$h(\mathcal{U}_e) = h\left(\bigcup_{n=-\infty}^{\infty} U_e^n\right) = \bigcup_{n=-\infty}^{\infty} h(U_e^n) = \bigcup_{n=-\infty}^{\infty} U_{f(e)}^n = \mathcal{U}_{f(e)}.$$

\square

Remark. We will show in Section 6.4 that there are finitely many principal regions and that the map h permutes the principal regions.

6. SEMI-ISOLATED LEAVES

It will turn out that there are only finitely many semi-isolated leaves and that each contains an h -periodic point. Our analysis will cast more light on \mathcal{U}_\pm and \mathcal{P}_\pm .

6.1. Counting the semi-isolated leaves. Recall that the core K is the complement of the union of the interiors of the disjoint distinguished neighborhoods U_e as e ranges over the set $\mathcal{E}(L)$ of ends of L , and that the choice of core is not unique. If $J_e = \text{Fr } U_e$, then $\text{Fr}_\pm K = \bigcup_{e \in \mathcal{E}_\pm(L)} J_e$.

Definition 6.1 (rectangle, rectangular). A *rectangle* is a four sided, simply connected set $R \subset L$ with one pair of opposite edges in $|\Gamma_+|$ and the other pair of opposite edges in $|\Gamma_-|$. The set R will also be described as “rectangular”.

Note that a rectangle can be open, closed, or neither.

Remark. We are not using “rectangle” in the geometric sense that the edges meet in right angles. In hyperbolic geometry, there are no such rectangles. Here, we are following Casson and Bleiler [19, p. 99] where rectangles had opposite edges in a pair of transverse foliations. Our rectangles are convex geodesic quadrilaterals.

Consider an arc α of $|\Lambda_\pm| \cap K$ with endpoints on components of $\text{Fr } K$ and consider the isotopy classes of such arcs *where the isotopy is to be through arcs with endpoints on components of $\text{Fr } K$* . By Lemma 4.53, this isotopy class contains two extreme arcs. These two extreme arcs together with a pair of arcs in $\text{Fr } K$ bound a rectangle R_α containing all the arcs in the isotopy class. Remark that the rectangle R_α may degenerate into a single arc.

Definition 6.2 (extreme rectangle, R_α). We will call R_α the *extreme rectangle* associated to the arc α .

Lemma 6.3. *Let α be an arc of $|\Lambda_+| \cap K$ (respectively $|\Lambda_-| \cap K$). The extreme rectangle R_α meets $|\Lambda_+|$ (respectively $|\Lambda_-|$) only in arcs isotopic to α .*

Proof. We consider the case α an arc of $|\Lambda_+| \cap K$. The case α an arc of $|\Lambda_-| \cap K$ is analogous. The leaves of Λ_+ cannot properly intersect one another, so any arc β of $|\Lambda_+| \cap K$ that meets R_α must be contained in that extreme rectangle. The ends of β cannot lie in the same edge of R_α because then β would form a digon with a component of $\text{Fr}_+ K$. But there are no geodesic digons in hyperbolic geometry. Thus β is isotopic to α . \square

The following lemma follows immediately from Lemmas 4.52 and 4.53.

Lemma 6.4. *The arcs α of $|\Lambda_+| \cap K$ (respectively $|\Lambda_-| \cap K$) fall into finitely many isotopy classes, determining finitely many disjoint extreme rectangles R_α .*

Theorem 6.5. *There are only finitely many semi-isolated leaves. Every semi-isolated leaf is h -periodic.*

Proof. We will prove the theorem for semi-isolated leaves in Λ_- . The proof in the other case is analogous. Let \mathcal{A} be the set of semi-isolated leaves $\lambda \in \Lambda_-$ such that there exists a component α of $\lambda \cap K$ which is an edge of a component U of $\mathcal{U}_+ \cap K$ which is not a rectangle. Since U is not a rectangle, it follows that α is an edge of the extreme rectangle R_α . Thus, by Lemma 6.4, the set \mathcal{A} is finite.

Note that if $\lambda \in \mathcal{A}$, then $h(\lambda) \in \mathcal{A}$. In fact, if $\lambda \in \mathcal{A}$, there exists a component α of $\lambda \cap K$ which is the edge of a component U of $\mathcal{U}_+ \cap K$ which is not a rectangle. If $h(\lambda) \notin \mathcal{A}$, then any component of $\mathcal{U}_+ \cap K$ that has $h(\alpha)$ on its boundary is a rectangular component R of $\mathcal{U}_+ \cap K$. If λ is an isolated leaf, there would be two such R . Otherwise R is unique. In either case it follows that the component U of $\mathcal{U}_+ \cap K$ with α an edge of U is a component of $h^{-1}(R) \cap K$ and thus a rectangle. This contradiction implies $h(\lambda) \in \mathcal{A}$.

Let λ be a semi-isolated leaf of Λ_- . We will prove that there are finitely many semi-isolated leaves by showing that λ lies in the finite set \mathcal{A} , proving that each semi-isolated leaf is h -periodic along the way. Let $\lambda_n = h^n(\lambda)$, $n \in \mathbb{Z}$. We first show that $\lambda_N \in \mathcal{A}$ for some $N \geq 0$ and therefore by the previous paragraph that $\lambda_n \in \mathcal{A}$ for all $n \geq N$. Assume on the contrary that $\lambda_N \notin \mathcal{A}$ for any $N \geq 0$. Since the leaf $\lambda \notin \mathcal{A}$ and meets the core K , it follows that λ contains an edge of a rectangular component R of $\mathcal{U}_+ \cap K$. Again since $\lambda_1 \notin \mathcal{A}$, the component of \mathcal{U}_+ meeting $h(R)$ contains a rectangular component R_1 of $\mathcal{U}_+ \cap K$ so that $h(R) \subset R_1$. By iterating this argument, we obtain an infinite increasing nest

$$R \subset h^{-1}(R_1) \subset h^{-2}(R_2) \subset \cdots \subset h^{-r}(R_r) \subset \cdots,$$

where each R_r is a rectangular component of $\mathcal{U}_+ \cap K$. Since R_r is rectangular with two edges segments of negative junctures, $h^{-r}(R_r)$ is rectangular with two edges segments of negative junctures that lie arbitrarily deep in neighborhoods of negative ends as $r \rightarrow \infty$. The increasing union of this nest of rectangles gives an infinite rectangle whose sides are leaves of Λ_- . By Theorem 4.24, a lift of this infinite rectangle has sides with the same ideal endpoints. But geodesics with lifts having the same endpoints coincide, so our rectangles all degenerate to arcs in λ , a contradiction.

Thus, $\lambda_n \in \mathcal{A}$ for $n \geq N$. Since the set \mathcal{A} is finite, so there exists integers $n \geq N$ and $p > 0$ such that $\lambda_n = \lambda_{n+p}$. Therefore,

$$\lambda_k = h^{k-n}(\lambda_n) = h^{k-n}(\lambda_{n+p}) = \lambda_{k+p}, \quad \text{all } k \in \mathbb{Z}.$$

Thus λ is h -periodic. It follows that $\lambda = \lambda_{ip} \in \mathcal{A}$ if $i \in \mathbb{Z}$ is chosen so that $ip \geq N$. \square

Corollary 6.6. *For every negative (respectively, positive) end e of L , there is a leaf of Λ_- (respectively, of Λ_+) with an end that passes arbitrarily near e (see Definition 4.49).*

Proof. Consider the case that e is a negative end, the other case is analogous. By Corollary 4.77, every neighborhood of e meets $|\Lambda_-|$, hence meets semi-isolated leaves of Λ_- . Since there are only finitely many of these leaves, the assertion follows. \square

Remark. Of course, every end of every leaf of the laminations passes arbitrarily near an end of L . The above corollary only points out that no end of L is left out.

6.2. Periodic leaves. We consider leaves $\lambda_{\pm} \in \Lambda_{\pm}$ which are periodic under h . For definiteness, assume that $\lambda_- \in \Lambda_-$ is h -periodic of period p . If h^p reverses orientation of λ_- , replace p by $2p$. Thus, we assume $h^p(\lambda_-) = \lambda_-$ and preserves the orientation of λ_- .

Notation. Set $g = h^p$.

We will study this situation in the universal cover $\tilde{L} \subset \Delta$. Recall that the ideal boundary of \tilde{L} is $E = \tilde{L} \cap S_{\infty}^1$ (Definition 3.4). In the universal cover \tilde{L} , we fix a lift $\tilde{\lambda}_-$ and choose the lift $\tilde{g} : \tilde{\Gamma}_{\pm} \rightarrow \tilde{\Gamma}_{\pm}$ so that $\tilde{g}(\tilde{\lambda}_-) = \tilde{\lambda}_-$. Set $\hat{g} = \tilde{h}^p$ and remark that \hat{g} fixes each endpoint c and d of $\tilde{\lambda}_-$ in E .

We get three possibilities by taking the completion $\hat{\sigma}$ of a lift $\tilde{\sigma}$ of a component σ of \mathcal{X}_- which meets $\tilde{\lambda}_-$ and iterating using \tilde{g} . Remark that σ cannot be an escaping component since it meets a leaf of Λ_- . Thus, depending on where we start, the strongly closed property (Definition 4.33) implies that the sequence $\{\hat{g}^n(\hat{\sigma})\}_{n=1}^{\infty}$ either strongly converges to (1) the completion $\hat{\lambda}_1$ of a leaf $\tilde{\lambda}_1$ of $\tilde{\Lambda}_+$ from the left

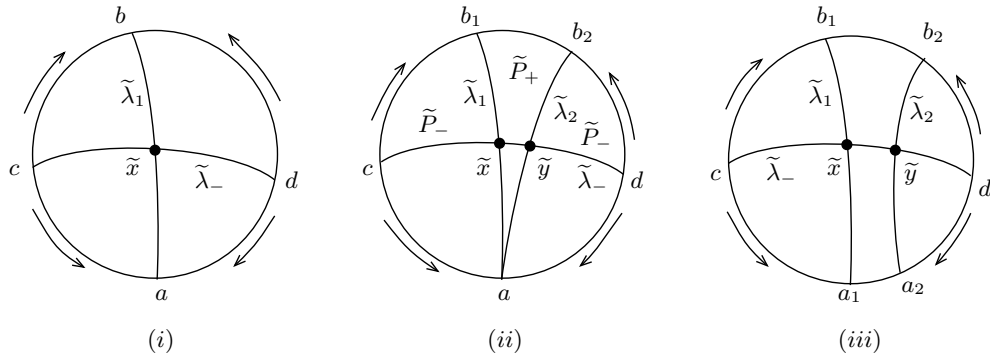


FIGURE 13. Three cases

or (2) the completion $\widehat{\lambda}_2$ of a leaf $\widetilde{\lambda}_2 \in \widetilde{\Lambda}_+$ from the right. (See Figure 13. It is drawn explicitly for the case that $\widetilde{L} = \Delta$, but works just as well when \widetilde{L} is a proper subset of the unit disk and $E = \widehat{L} \cap S_\infty^1$ is a Cantor set. In the latter case, the circles in the figure represent S_∞ .)

Lemma 6.7. *The sequence $\{\widetilde{g}^{-n}(\widetilde{\sigma})\}_{n=1}^\infty$ converges to c in case (1) and to d in case (2), hence the endpoints of $\widehat{\lambda}_-$ are repelling under $\widehat{g}: S_\infty \rightarrow S_\infty$.*

Indeed, $\{h^k(\sigma)\}_{k \leq 0}$ escapes (cf. Theorem 4.24). By abuse, we will often say that \widetilde{g} fixes the endpoints c and d . Following the endpoints (ideal or finite) of $\widetilde{g}^{-n}(\widetilde{\sigma})$, we see that the action of \widehat{g} on S_∞ is as indicated by the arrows in Figure 13.

The leaf $\widetilde{\lambda}_- \in \widetilde{\Lambda}_-$ must be approached on at least one side by lifts of components of \mathcal{X}_+ . Without loss, we may assume there exists a component τ of \mathcal{X}_+ so that the sequence $\{\widehat{g}^n(\tau)\}_{n \leq 0}$ strongly converges to $\widehat{\lambda}_-$ from below (in Figure 13 as drawn).

Lemma 6.8. *Case (iii) in Figure 13 cannot occur.*

Again, since $\{h^k(\tau)\}_{k \geq 0}$ escapes, Theorem 4.24 implies that the sequence $\{\widehat{g}^n(\tau)\}_{n=1}^\infty$ must converge to a point $a \in E$ as in Figure 13 (i) or (ii). This proves the impossibility of Figure 13 (iii). We now analyze Figure 13 (i) and (ii).

Proposition 6.9. *Let $\widetilde{\lambda}_-$ and its ends be fixed by \widetilde{g} . Then,*

- (i) *If $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$, the endpoints a, b of $\widetilde{\lambda}_1$ in E are attracting fixed points of \widetilde{g} and the endpoints c, d of $\widetilde{\lambda}_-$ in E are repelling fixed points and no point of $E \setminus \{a, b, c, d\}$ is fixed by \widetilde{g} . The point $\widetilde{x} = \widetilde{\lambda}_1 \cap \widetilde{\lambda}_-$ is fixed by \widetilde{g} .*
- (ii) *If $\widetilde{\lambda}_1 \neq \widetilde{\lambda}_2$, the endpoints c, d of $\widetilde{\lambda}_-$ are repelling fixed points of \widetilde{g} , there is a common endpoint a of $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ which is an attracting fixed point, and the other endpoints b_1, b_2 of $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ are attracting on the sides facing the endpoints of $\widetilde{\lambda}_-$. No point in the intervals $(c, b_1), (c, a), (d, b_2), (d, a)$ is fixed by \widetilde{g} . The points $\widetilde{x} = \widetilde{\lambda}_1 \cap \widetilde{\lambda}_-$ and $\widetilde{y} = \widetilde{\lambda}_2 \cap \widetilde{\lambda}_-$ are each fixed by \widetilde{g} .*

Parallel assertions hold when $\widetilde{\lambda}_-$ is replaced by a leaf $\widetilde{\lambda}_+$ of $\widetilde{\Lambda}_+$.

Proof. If $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$, we let $\widetilde{x} = \widetilde{\lambda}_- \cap \widetilde{\lambda}_1$. If $\widetilde{\lambda}_1 \neq \widetilde{\lambda}_2$, we let $\widetilde{x} = \widetilde{\lambda}_- \cap \widetilde{\lambda}_1$ and $\widetilde{y} = \widetilde{\lambda}_- \cap \widetilde{\lambda}_2$. If $\widetilde{\lambda}_1 = \widetilde{\lambda}_2$ we have case (i). If $\widetilde{\lambda}_1 \neq \widetilde{\lambda}_2$ but $\widetilde{\lambda}_1, \widetilde{\lambda}_2$ share the endpoint $a \in E$, then we have case (ii).

Parallel considerations hold when $\widetilde{\lambda}_-$ is replaced by a leaf $\widetilde{\lambda}_+$ of $\widetilde{\Lambda}_+$. □

Corollary 6.10. *Every semi-isolated leaf has an h -periodic point.*

Proof. By Theorem 6.5 every semi-isolated leaf is h -periodic and so Proposition 6.9 implies that every semi-isolated leaf has an h -periodic point. □

The set \widetilde{P}_+ (and a corresponding set \widetilde{P}_- when $\widetilde{\lambda}_-$ is replaced by $\widetilde{\lambda}_+$) in Figure 13 (ii) are lifts of *principal regions* and will be explained shortly.

Corollary 6.11. *In case (i), \widetilde{x} is an attracting (in $\widetilde{\lambda}_-$) \widetilde{g} -fixed point on both sides and $\widetilde{\lambda}_1$ can not border the lift of a principal region on either side. In case (ii), \widetilde{x} and \widetilde{y} are attracting in $\widetilde{\lambda}_-$ on the sides not meeting $(\widetilde{x}, \widetilde{y})$. If $\widetilde{\lambda}_-$ is replaced by $\widetilde{\lambda}_+ \in \widetilde{\Lambda}_+$, these points are repelling in $\widetilde{\lambda}_+$.*

Interchanging the roles of λ_- and λ_1 in the previous corollary one sees,

Corollary 6.12. *In case (i), $\tilde{\lambda}_-$ can not border the lift of a principal region on either side.*

The proof of the following corollary is similar to the proof of Lemma 6.8.

Corollary 6.13. *In case (ii), $\tilde{\lambda}_-$ is semi-isolated and $\tilde{\lambda}_-$ is bordered above by the lift of a negative principal region P_- .*

Proof. In case (ii) of Figure 13, if there exists a sequence $\{x_n\}$ of lifts of points of $|\mathcal{X}_+|$ converging to a point of $\tilde{\lambda}_-$ from above, then, by the strongly closed property, there exists a juncture component $\tau \in \mathcal{X}_+$ with lift $\tilde{\tau}$ above $\tilde{\lambda}_-$ with endpoints on S_∞ separated by both b_1 and b_2 . Since $\{h^k(\tau)\}_{k \geq 0}$ escapes, Theorem 4.24 implies that the sequence $\{\hat{g}^n(\tilde{\tau})\}_{n \geq 0}$ must converge to one point in E implying $b_1 = b_2$ and contradicting that we are in case (ii). The corollary follows. \square

Corollary 6.14. *If $\lambda_- \in \Lambda_-$ and p are as in the second paragraph of Section 6.2, then λ_- contains a periodic point of period p or $p/2$.*

Further, since the segment $[\tilde{x}, \tilde{y}]$ is contained in the invariant set, no lifts of components of positive junctures can meet the segment $[\tilde{x}, \tilde{y}]$. It follows that,

Corollary 6.15. *In case (ii), $\tilde{\lambda}_1, \tilde{\lambda}_2 \in \tilde{\Lambda}_+$ are semi-isolated and lifts of border leaves of a positive principal region P_+ .*

Lemma 6.16. *The leaf $\tilde{\lambda}_-$ can either be approached from below by leaves of $\tilde{\Lambda}_-$ or be bordered from below by a lift of \mathcal{U}_+ . Similarly, $\tilde{\lambda}_1$ (respectively $\tilde{\lambda}_2$) can be approached from the left (respectively from the right) by leaves of $\tilde{\Lambda}_+$ or be the border from the left (respectively from the right) of a lift of \mathcal{U}_- .*

Let \tilde{A} be the portion of the cusp in case (ii) of Figure 13 bordered by the half-infinite segments $[\tilde{x}, a) \subset \tilde{\lambda}_1$ and $[\tilde{y}, a) \subset \tilde{\lambda}_2$ and the segment $[\tilde{x}, \tilde{y}]$.

Definition 6.17 (arm). The projection $A \subset L$ of \tilde{A} under the covering map is called an *arm* of the principal region P_+ .

Lemma 6.18. *The projection of \tilde{A} onto A is one-one. Thus any arm A of a principal region is simply connected.*

Proof. We must show that if T is any nontrivial covering transformation, then $T(\tilde{A}) \cap \tilde{A} = \emptyset$. If $T(a) \neq a$ and if $T(\tilde{A}) \cap \tilde{A} \neq \emptyset$ then one of $T(\tilde{\lambda}_1), T(\tilde{\lambda}_2)$ must meet one of $\tilde{\lambda}_1, \tilde{\lambda}_2$ which is a contradiction. Thus we can assume $T(a) = a$ but in that case, if $T(\tilde{A}) \cap \tilde{A} \neq \emptyset$, we must have $T(\tilde{\lambda}_1) = \tilde{\lambda}_1, T(\tilde{\lambda}_2) = \tilde{\lambda}_2$. This is not possible since T must have exactly two fixed points on E . \square

6.3. Escaping ends. Let λ be a leaf of Λ_\pm and ε an end of λ .

Definition 6.19 (escaping end of leaf). A ray $[x, \varepsilon) \subset \lambda$ represents an escaping end ε of λ if there is an end e of L such that, for every neighborhood U of e in L , there is $z \in (x, \varepsilon)$ such that $(z, \varepsilon) \subset U$.

Remark. By abuse of language, we often call the ray $[x, \varepsilon)$ itself an escaping end. Note that this is a much stronger property than passing arbitrarily near an end e of L . The latter allows return to the core infinitely often which the above definition does not. By Proposition 4.74, we can and do require that $x \in |\Lambda_+| \cap |\Lambda_-|$.

Let e be a positive end of L and consider the component \mathcal{U}_e of the positive escaping set. Since L has only finitely many ends and, since there are only finitely many border leaves of \mathcal{U}_e (Theorem 6.5), we can assume that there is $p \geq 1$ such that h^p takes each such border leaf to itself, preserving its ends.

Remark. As before, the use of the terms “below” and “above” in this subsection always refers to Figure 13.

Fix a connected lift $\tilde{\mathcal{U}}_e$ of \mathcal{U}_e . The lift \tilde{g} of h^p that we consider can be chosen to take $\tilde{\mathcal{U}}_e$, hence $\delta\tilde{\mathcal{U}}_e$, to itself. In addition, we can assume that \tilde{g} fixes a lift $\tilde{\lambda}_- \subset \delta\tilde{\mathcal{U}}_e$ of a specific border leaf λ_- of \mathcal{U}_e . View $\tilde{\lambda}_-$ as in Figure 13, (i) or (ii). In case (i), interchange the roles of a and b , if necessary, to assume that $\tilde{\mathcal{U}}_e$ lies below $\tilde{\lambda}_-$. In case (ii), $\tilde{\mathcal{U}}_e$ also borders $\tilde{\lambda}_-$ from below (Corollary 6.13). Thus every point $z \in \tilde{\lambda}_-$ has the property that a small enough transverse arc $[z, \eta]$ issuing from z into the region below $\tilde{\lambda}_-$ in Figure 13 has $(z, \eta) \subset \tilde{\mathcal{U}}_e$. Thus,

Lemma 6.20. *Under the above assumptions, leaves of $\tilde{\Lambda}_-$ cannot accumulate on $\tilde{\lambda}_-$ from below.*

Lemma 6.21. *The ray(s) (\tilde{x}, a) (and (\tilde{y}, a)) lie in $\tilde{\mathcal{U}}_e$.*

Proof. Indeed, no lift of a leaf of $\tilde{\Lambda}_-$ can have one endpoint in the arc of S_∞ between a and c and the other in the arc between a and d . Otherwise leaves of $\tilde{\Lambda}_-$ would accumulate on $\tilde{\lambda}_-$ from below, contradicting Lemma 6.20. Since the rays in question start out in $\tilde{\mathcal{U}}_e$, they can never exit. \square

By Proposition 5.11 and the strongly closed property, we conclude the following.

Lemma 6.22. *Completions of components of $\tilde{\mathcal{X}}_+$ accumulate on $\hat{\lambda}_-$ from below, becoming uniformly close in the Euclidean metric on $\hat{L} \subseteq \mathbb{D}^2$.*

By this lemma, let a component $\tilde{\sigma}$ of $\tilde{\mathcal{X}}_+$ meet (\tilde{x}, a) in the single point u . In case (ii), $\tilde{\sigma}$ also meets (\tilde{y}, a) in a singleton u' . Set $v, v' = \tilde{g}(u), \tilde{g}(u')$.

Lemma 6.23. *Only finitely many components of $\tilde{\mathcal{X}}_+$ meet the arc $[u, v]$ (and $[u', v']$).*

Proof. Otherwise the intersections of these curves with $[u, v]$ or $[u', v']$ cluster there, implying that $(\tilde{x}, a) \cap |\tilde{\Lambda}_-| \neq \emptyset$. This contradicts $(\tilde{x}, a) \subset \tilde{\mathcal{U}}_e$. \square

Corollary 6.24. *There exist an integer $k > 0$ so that exactly k components of $\tilde{\mathcal{X}}_+$ meet $\tilde{h}^{np}([u, v])$ (and, if pertinent, $\tilde{h}^{np}([u', v'])$), $n \geq 0$.*

Corollary 6.25. *If λ_- is a border leaf of \mathcal{U}_+ , then one of the following holds:*

- (1) *There is a unique h -periodic point $x \in \lambda_-$ and the ray $[x, \varepsilon) \subset |\Lambda_+|$ issuing from x into \mathcal{U}_+ represents an escaping end ε of a leaf $\lambda \in \Lambda_+$;*
- (2) *There is a unique maximal, compact, h -periodic interval $I \subset \lambda_-$, with x either endpoint of I , and the ray $[x, \varepsilon) \subset |\Lambda_+|$ issuing from x into \mathcal{U}_+ represents an escaping end ε of a leaf $\lambda \in \Lambda_+$.*

In case (2), the rays issuing from the endpoints of I determine an escaping cusp. The corresponding assertions hold for a border leaf λ_+ of \mathcal{U}_- .

Proof. Let $\mathcal{U}_e = \bigcup_{n=-\infty}^{\infty} U_e^n$ be the component of \mathcal{U}_+ bordered by λ_- where U_e is any distinguished neighborhood of e (Definition 4.64). As usual set $g = h^p$ where $p = kp_e$. By Corollary 4.50, the ray $[x, \varepsilon)$ meets infinitely many positive junctures, necessarily in nonescaping components. Choose a point $s \in [x, \varepsilon) \cap |\mathcal{X}_+|$ and let $t = g(s)$. Only finitely many components of positive junctures meet the interval $[s, t]$. Let N be the least integer such that a component of the juncture $J_e^N = \text{Fr } U_e^N$ meets the interval $[s, t)$. Since $[t, \varepsilon) = \bigcup_{n=1}^{\infty} g^n([s, t))$, the interval $[t, \varepsilon)$ does not meet the juncture J_e^N and thus lies in U_e^N . Since the interval $[t, \varepsilon)$ does not meet the juncture J_e^N , it follows that the interval $g^i([t, \varepsilon))$ does not meet the juncture $J_e^{N+ik} = \text{Fr } U_e^{N+ik}$ and thus lies in U_e^{N+ik} . Thus, $[x, \varepsilon)$ ultimately enters and remains in any neighborhood of e in L . \square

This corollary has the following converse.

Lemma 6.26. *If the end ε of $\ell \in \Lambda_+$ is escaping, it has a neighborhood $[x, \varepsilon)$ satisfying either (1) or (2) in Corollary 6.25. The analogous assertion holds for escaping cusps and the corresponding assertions hold for negative escaping ends and cusps.*

Proof. Let $\lambda_- \in \delta\mathcal{U}_e$ be the leaf such that there is a point $z \in \lambda_- \cap |\Lambda_+|$ with the open ray $(z, \varepsilon) \subset \mathcal{U}_e$. The leaf λ_- is semi-isolated and thus has a periodic point. If z is the unique h^p -fixed point $x \in \lambda_-$ or an endpoint of the unique maximal h^p -invariant interval $[x, y] \subset \lambda_-$, we are done. If not, assume that, in the lifted picture in Figure 13, Case (i), $\tilde{z} \in (\tilde{x}, d)$ and deduce a contradiction. (A similar contradiction occurs if $\tilde{z} \in (c, \tilde{x})$ in Figure 13, Case (i) or if $\tilde{z} \in (c, \tilde{x})$ or $\tilde{z} \in (\tilde{y}, d)$ in Figure 13, Case (ii).) Note that by Corollary 6.15, in the case of an escaping cusp, (\tilde{x}, \tilde{y}) is contained in a principal region. Then the points $\tilde{z}_n = \tilde{h}^{-np}(z)$ converge to d as $n \rightarrow \infty$ and the lift of a positive escaping ray issues from each \tilde{z}_n . Projecting down to L by the covering map, we obtain escaping rays $[z_n, \varepsilon_n)$ issuing into \mathcal{U}_e from the points $z_n \in \lambda_-$. These points do not converge in the intrinsic real line topology of λ_- . But, as points of intersection $z_n \in |\Lambda_+| \cap |\Lambda_-|$, these are points of the compact invariant set and cluster in L at a point z_* of that set. Fix a neighborhood V of z_* in L which is a product neighborhood for both laminations. A subsequence z_{n_k} consists of points that lie on distinct components (plaques) Q_k of $V \cap \lambda_-$ and converge to z_* . Clearly, for all but at most one Q_k , the ray $[z_{n_k}, \varepsilon_{n_k})$ must cross at least one other $Q_{k'}$, contrary to hypothesis. \square

We have completely characterized the escaping ends and escaping cusps.

Theorem 6.27. *The escaping ends of leaves $\lambda_{\pm} \in \Lambda_{\pm}$ are exactly those represented by rays (z, ε) lying in \mathcal{U}_{\pm} where z is either the unique periodic point on a leaf of $\delta\mathcal{U}_{\pm}$ or an endpoint of the unique maximal periodic compact interval on such a leaf. The escaping cusps are similarly characterized where the unique periodic point is replaced by the unique maximal compact periodic interval.*

Remark. There are only finitely many escaping ends and escaping cusps. Some ends and cusps may escape and some may not.

Corollary 6.28. *If a leaf λ of Λ_{\pm} has an escaping end, it is a periodic leaf.*

The converse, of course, is false as there are generally infinitely many periodic leaves.

6.4. The structure of principal regions and their crown sets. We consider \mathcal{P}_+ and its components P_+ , but all arguments and results have parallels for \mathcal{P}_- and P_- . These components are the principal regions and, by Theorem 6.5, there are only finitely many of them.

Fix a choice of $P \subset \mathcal{P}_+$. The components of $P \cap K$ will be rectangles or regions with frontier finitely many simple closed curves

$$\varepsilon_1 \cup \beta_1 \cup \varepsilon_2 \cup \cdots \cup \varepsilon_r \cup \beta_r,$$

where the $\beta_i \subset \ell_i \in \Lambda_+$ are extreme arcs of $|\Lambda_+| \cap K$, alternating with proper subarcs ε_i of positive junctures in $\text{Fr } K$. There is a least integer $p > 0$ such that $h^{-p}(\beta_i) \subset \beta_i$. Then $h^{-p}(\varepsilon_i)$ is a segment of positive juncture with endpoints in β_{i-1} and β_i , respectively. Infinite iteration gives a sequence of segments of positive junctures converging to a segment α_i of a leaf ℓ'_i of Λ_- having endpoints x_i and y_i on β_{i-1} and β_i , respectively. Shorten the arcs β_i to have endpoints y_i and x_{i+1} , defining a simple closed curve

$$\gamma = \alpha_1 \cup \beta_1 \cup \alpha_2 \cup \cdots \cup \alpha_r \cup \beta_r.$$

Note that the indices are taken mod r . The lifts of each of the curves ℓ_i (resp. ℓ'_i) can play the role of λ_- in case (ii) of Proposition 6.9 with lifts of the arcs β_i (resp. α_i) playing the role of the segments $[\tilde{x}, \tilde{y}]$ in that proposition. Thus, the β_i lie in the invariant set and cut off $r \geq 1$ arms A_i of the principal region P . Similarly, the α_i lie in the invariant set and cut off r arms A'_i of the principal region $P' \subset \mathcal{P}_-$.

Definition 6.29 (dual principal regions). The principal regions P and P' are called dual principal regions.

Definition 6.30 (nucleus). The closure of the intersection $P \cap P'$ is the *nucleus* of the principal region P and of its dual P'

Thus a principal region is the union of the interior of its nucleus and arms. This nucleus may be bounded by several polygonal curves γ_i as above and to each is attached a set of arms for P and a set of arms for P' .

Lemma 6.31. *The nucleus N of a principal region is compact.*

Proof. Indeed, $\partial N \setminus \partial L$ is the union of all the curves γ associated to the principal region P and the dual P' and N lies in both P and P' . If N were noncompact then, since its boundary is compact, it would be a neighborhood of at least one end of L , hence would contain positive and/or negative junctures, contradicting the fact that $N = P \cap P'$. \square

Remark. There are four cases to consider:

- (1) N has negative Euler characteristic. In this case, γ can be tightened in its homotopy class to a unique simple closed geodesic $\rho_\gamma \subset \text{int } N$ and the correspondence $\gamma \leftrightarrow \rho_\gamma$ is one-one.
- (2) N is an annulus. If there are two piecewise geodesic boundary curves γ_1 and γ_2 , both will be homotopic to the same geodesic $\rho_{\gamma_1} = \rho_{\gamma_2} \subset \text{int } N$. It may happen that one boundary curve δ of the annulus is a component of ∂L . In this case, denote the other component of ∂N by γ and note that $\rho_\gamma = \delta$.

- (3) N is a Möbius strip with one boundary curve γ and center circle a geodesic $\sigma \subset \text{int } N$. The curve γ is homotopic to an immersed geodesic ρ_γ that is a two-to-one cover of σ .
- (4) N is a disk. There is one boundary curve γ , but it is not homotopic to a closed geodesic. In this case, we set $\rho_\gamma = \emptyset$.

In any event, the arcs making up γ are isolated on the side facing the nucleus of the principal region P .

Definition 6.32 (dual crown sets). The *crown set* C_γ is the closure of the component of $P \setminus \rho_\gamma$ that contains the curve γ . The *crown set* C'_γ is the closure of the component of $P' \setminus \rho_\gamma$ that contains the curve γ . The crown sets C_γ and C'_γ are called *dual crown sets*.

Thus, in all cases except (4) in the above remark, the crown sets are annuli with finitely many cusps. In case (4), the crown sets are disks with finitely many cusps.

Definition 6.33 (rim). The closed curve $\rho_\gamma \subset P$ that cuts off a crown set is called the *rim* of the crown set. The closed curves γ and ρ_γ will be said to be associated.

Since h permutes the borders of the principal regions, there is a corresponding permutation of the crown sets themselves. Thus, we get cycles $C = C_0, C_1 = h(C), \dots, C_n = h^n(C) = C_0$, and a corresponding cycle $\rho = \rho_0, \rho_1, \dots, \rho_n = \rho_0$ of rims. If n is the minimal period, then h^n induces a permutation in the arms of each crown set.

6.5. The set \mathfrak{S} of reducing curves. As in Nielsen-Thurston theory [19], we split L into simpler pieces using *reducing curves* [24, page 5]. The reducing curves will be geodesics, either homeomorphic to S^1 or \mathbb{R} and will lie in $\text{int } L$. They will be constructed as we develop our theory. The reducing curves will be *nonperipheral* in the sense that none cobounds an annulus or infinite strip with a component of ∂L and they will be disjoint from Λ_\pm .

Definition 6.34 (\mathfrak{S}). We will let \mathfrak{S} denote the set of reducing curves.

The first set of reducing curves we construct are the geodesic rims ρ_γ of crown sets for the case of nuclei which are neither discs nor Möbius strips. (The rim ρ_γ for the Möbius strip case could be counted as a reducing curve, but we choose not to because doing so would make some statements awkward.) If the nucleus is peripheral, ρ_γ is a component of ∂L and is not taken as a reducing curve. If the nucleus N is an annulus with boundary $\gamma_1 \cup \gamma_2$, the two rims ρ_{γ_1} and ρ_{γ_2} are identical and this will be a reducing curve. Otherwise, the rims ρ_γ correspond one-one with γ , are disjoint from each other and disjoint from the geodesic laminations Λ_\pm and ∂L .

Every γ associated to a reducing curve in \mathfrak{S} will be the union of segments of semi-isolated leaves (Definition 5.6) of Λ_\pm . As usual, we will distinguish \mathfrak{S} from the *support* (i.e., union) $|\mathfrak{S}|$.

In Section 8, we will construct an endperiodic automorphism g which is isotopic to f and permutes both the sets Λ_+ and Λ_- and the set \mathfrak{S} of reducing curves.

6.6. Examples of principal regions. Principal regions occur for an endperiodic automorphism when there is some topology that remains in the core or when there are more than two repelling or attracting ends. There can not be simply connected principal regions with one or two arms.

Example 6.35. In this example, the principal regions have only one arm. Let L be a surface with two nonplanar ends and one disk removed as in Figure 14. Then L has one circle boundary component C . If g is the endperiodic automorphism that moves each handle to the right one unit near both ends but leaves C invariant, then the laminations Λ_{\pm} each contain one leaf as in Figure 14. Both the positive and negative principal regions have one arm. If one composes g with Dehn twists in the three dotted curves in Figure 14 to get an endperiodic automorphism f , the new laminations Λ_{\pm} both have uncountably many leaves. The one arm of the positive principal region is bordered by an isolated leaf λ_+ and no longer is escaping but returns infinitely often to the core. The negative principal region is bordered by a semi-isolated but not isolated leaf λ_- and has one escaping arm.

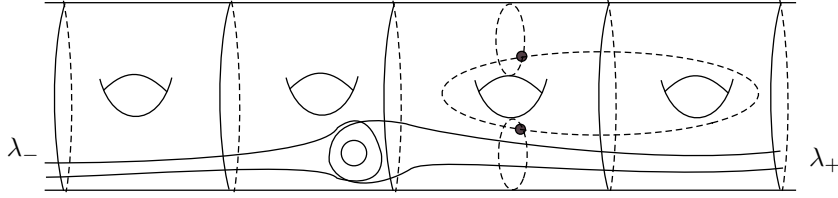


FIGURE 14. An example whose principal regions have one arm

Example 6.36. The double of the surface L of Example 6.35 along the circle boundary component C of L has dual positive and negative principal regions whose nucleus is an annulus. Each principal region has two crown sets.

Example 6.37. Let S be a pair of pants with boundary C_0, C_1, C_2 . Attach three copies of L along C to each of C_0, C_1, C_2 . This example has dual positive and negative principal region whose nucleus is a pair of pants. The rims of the three pairs of dual crown sets are C_0, C_1, C_2 .

Example 6.38. We give examples in which the principal regions have $r \geq 3$ arms. Let L be a surface similar to the surface in Figure 1 but with $r \geq 3$ negative ends and one positive end and let f be an endperiodic automorphism that permutes the negative ends and moves the handles along in a way similar to the example illustrated in Figure 1. The laminations Λ_{\pm} each contain r leaves and there are positive and negative principal regions, each with r arms. By composing f with suitable Dehn twists the laminations can be made more complicated. The principal regions are simply connected but topology can be added to them as in Example 6.39.

Example 6.39. Let L be the surface in Figure 1 but with a disk centered at the saddle point removed. Thus, L has one circle boundary component C . Let f be the map indicated in Figure 1 modified to send C to itself. The laminations Λ_{\pm} each contain two leaves. There are positive and negative principal regions each with two arms. By composing f with suitable Dehn twists one can obtain similar examples with more complicated laminations.

6.7. The induced laminations on the compact surface F . Let e be an end of L and denote its full h -cycle by $c = \{e = e_1, e_2, \dots, e_{p_e}\}$. Recall that,

$$\mathcal{U}_c = \bigcup_{n=-\infty}^{\infty} h^n(U_e) = \mathcal{U}_{e_1} \cup \mathcal{U}_{e_2} \cup \dots \cup \mathcal{U}_{e_{p_e}},$$

where U_e is any h -neighborhood (Definition 2.3) of e . Remark that \mathcal{U}_c is an open, h -invariant set with no periodic points. The connected components \mathcal{U}_{e_i} of \mathcal{U}_c are permuted cyclically by h . As in Section 2.1, one has an infinite cyclic covering

$$q : \mathcal{U}_c \rightarrow F.$$

The group of deck transformations for q is generated by $h|_{\mathcal{U}_c}$.

For definiteness, we consider the case that c is an h -cycle of negative ends of L . Since \mathcal{U}_c and Λ_- are h -invariant, it follows that the induced lamination $\Lambda_-|_{\mathcal{U}_c}$ is invariant under the group of deck transformations of $q : \mathcal{U}_c \rightarrow F$. Similarly, the set \mathcal{J}_- of negative juncture components is h -invariant and transverse to Λ_- . Thus,

Lemma 6.40 (Λ_F, J_κ). *The lamination $\Lambda_-|_{\mathcal{U}_c}$ descends to a well defined closed lamination Λ_F of F and the set of negative juncture components in \mathcal{U}_c descends to a compact, transversely oriented, properly embedded 1-manifold J_κ that is transverse to Λ_F .*

Remark. The 1-manifold J_κ is the 1-manifold J_κ of Definition 2.15.

We analyze the structure of Λ_F using the properties of $\Lambda_-|_{\mathcal{U}_c}$.

The border $\delta\mathcal{U}_-$ consists of semi-isolated leaves in Λ_+ and is invariant under h . Thus, $\delta\mathcal{U}_-$ and $\delta\mathcal{U}_c$ each consist of h -cycles of semi-isolated leaves of Λ_+ . Recall that there are only finitely many semi-isolated leaves of Λ_\pm (Theorem 6.5) and that each contains either a unique h -periodic point or a unique maximal, compact, nondegenerate h -periodic interval. For the semi-isolated leaves of Λ_+ , the isolated periodic point is repelling under applications of h and the endpoints of the periodic interval are each repelling on the side not meeting the interval (Corollary 6.11). For Λ_- , these points are attracting.

Evidently, if $[a, \infty)$ is the neighborhood of an escaping end issuing from a periodic point a on the isolated side of a leaf λ of Λ_+ in $\delta\mathcal{U}_c$, the leaf (a, ∞) of $\Lambda_-|_{\mathcal{U}_c}$ descends to a circle leaf $C_a \subset F$ of Λ_F . Either $a \in \lambda$ is an isolated, repelling, h -periodic point, or it is an endpoint of a compact, nondegenerate h -periodic arc $[a, a'] \subset \lambda$. In this case there are two escaping ends $[a, \infty)$ and $[a', \infty)$ and, by the structure theory of principal regions, these cobound an arm A of a principal region. Since A is simply connected (Lemma 6.18), (a, ∞) and (a', ∞) descend to a pair of circle leaves C_a and $C_{a'}$ of Λ_F , cobounding an annulus in F which meets no other leaves of Λ_F . The points a, a' are both repelling on the sides opposite to $[a, a']$. In all cases, the natural orientation of the escaping ends toward ∞ induces an orientation on these circle leaves of Λ_F .

Lemma 6.41. *Every leaf of $\Lambda_-|_{\mathcal{U}_c}$ issues from either one or two points of $|\delta\mathcal{U}_-|$.*

Proof. We need to show that no leaf of Λ_- lies entirely in \mathcal{U}_c . Otherwise, by Lemma 4.69, that leaf would meet no leaf of Λ_+ , contrary to Proposition 4.74. \square

Lemma 6.42. *The only leaves of $\Lambda_-|_{\mathcal{U}_c}$ which do not complete to compact arcs with endpoints in $\delta\mathcal{U}_c$ are the escaping ends with completion $[a, \infty)$*

Proof. Suppose there is a leaf $\ell = (b, \infty)$ of $\Lambda_-|_{\mathcal{U}_c}$ with completion $\widehat{\ell} = [b, \infty)$, $b \in |\delta\mathcal{U}_c|$, which is not an escaping end. Then ℓ either crosses some fixed juncture J infinitely often or eventually remains in some compact part of \mathcal{U}_c . In either case, since Λ_- is closed, there is a leaf $\ell' \in \Lambda_-$ in the asymptote of the end of $\ell = (b, \infty)$. Since $\ell \subset \mathcal{U}_c$, Lemma 4.69 implies that $\ell \cap |\Lambda_+| = \emptyset$. Therefore, $\ell' \cap |\Lambda_+| = \emptyset$ contradicting Proposition 4.74. \square

Definition 6.43 (parallel packets). The compact completions $\widehat{\ell}_1$ and $\widehat{\ell}_2$ of two bounded leaves of $\Lambda_-|\mathcal{U}_c$ are parallel if they are a pair of opposite sides of a rectangle in $\check{\mathcal{U}}_c$, the other two sides being compact arcs in $|\delta\mathcal{U}_c|$. This is an equivalence relation on the set of compact completions of leaves of $\Lambda_-|\mathcal{U}_c$ and the equivalence classes will be called *parallel packets*. Each parallel packet contains two *extreme leaves*.

Evidently, the parallel packets are permuted by h .

Lemma 6.44. *There are only finitely many h -orbits of parallel packets.*

Proof. Since $\delta\mathcal{U}_c$ has only finitely many elements, there is a least integer $p \geq 0$ such that h^{-p} carries each onto itself, preserving orientation. It is enough to show that there are only finitely many h^{-p} -orbits of parallel packets. If not, there is a leaf λ of $\delta\mathcal{U}_c$ and a compact subarc $[x, h^{-p}(x)] \subset \lambda$ containing no periodic point, and infinitely many points $x_n \in [x, h^{-p}(x)]$ out of which issue completions pertaining to distinct packets. Let $y \in [x, h^{-p}(x)]$ be a cluster point of $\{x_n\}$. Since Λ_- is a closed lamination, there exists a leaf of Λ_- issuing from y . Since $y \in [x, h^{-p}(x)]$, y is not a periodic point. Thus, by Theorem 6.27 and Lemma 6.42, the completion of the leaf of $\Lambda_-|\mathcal{U}_c$ issuing from y is a compact arc $\check{\ell}$. Let $\check{\ell}_n$ be the completion of the leaf of $\Lambda_-|\mathcal{U}_c$ issuing from x_n . Then the $\check{\ell}_n$'s cluster locally uniformly on the compact arc $\check{\ell}$, proving that infinitely many of them are parallel to $\check{\ell}$. This is contrary to hypothesis, completing the proof. \square

Remark (Properties of the lamination Λ_F). Putting these lemmas together, we see that the escaping rays descend under the covering projection $q : \mathcal{U}_c \rightarrow F$ to finitely many circle leaves of Λ_F and the parallel packets descend under the covering projection to finitely many packets, homeomorphic to $X \times \mathbb{R}$, of parallel noncompact leaves of Λ_F . Here X is compact and totally disconnected. One end of such a packet spirals in on a circle leaf, as does the other end. Indeed, if an end of a leaf of $\Lambda_-|\mathcal{U}_c$ issues from a point $x_0 \in \delta\mathcal{U}_c$, then the image of the that end under h^{np} issues from a point $x_n \in \delta\mathcal{U}_c$, the sequence $\{x_n\}_{n \geq 0}$ converges to a periodic point $x \in \delta\mathcal{U}_c$, and the leaf of $\Lambda_-|\mathcal{U}_c$ issuing from x is an escaping ray. The leaves (and packets) upstairs accumulate locally uniformly on two (not necessarily distinct) h -orbits of escaping rays which descend to (one or two) circle leaves in F .

Remark (continued). The circle leaves of Λ_F have a preferred orientation obtained from orienting the escaping ends towards the end. The noncompact leaves of Λ_F can not be oriented.

Remark (continued). By Lemma 6.18 and Theorem 6.27, an arm of a principal region is an infinite strip bounded by escaping rays (a, ∞) and (a', ∞) , where $[a, a']$ is an h -periodic interval and thus descends under the covering projection $q : \mathcal{U}_c \rightarrow F$ to an annulus bounded by two circle leaves of Λ_F . This is the only way two parallel circles leaves can occur in Λ_F . As a subset of a principal region, an arm contains no leaves of $\Lambda|\mathcal{U}_c$. Thus, it is not possible to have three mutually parallel circle leaves in Λ_F nor is it possible that Λ_F contains a ‘‘Reeb’’ annuli.

The above development can be used to define a traintrack T carrying the lamination Λ_F . We have,

Proposition 6.45 (T). *The traintrack T in F carrying the lamination Λ_F consists of finitely many oriented circles, one for each single oriented circle in Λ_F or pair*

of oriented circles in Λ_F bounding an annulus, and finitely many compact arcs, one for each packets of parallel noncompact leaves of Λ_F . Each end of each arc meets a circle in a point called a switch so that the arc makes an acute angle with the outgoing arc of the circle.

Remark. Arcs can not have a preferred orientation. Either choice of orientation will be coherent with that of the circle at the one switch and will be opposed at the other.

We view T as a branched 1-manifold. As a graph T is 3-valent.

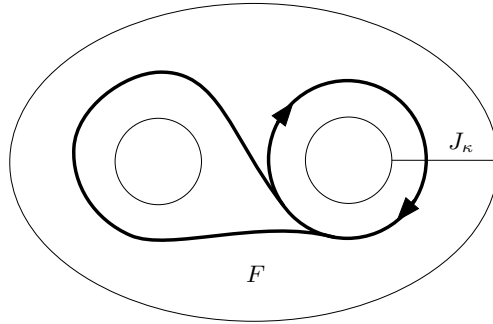


FIGURE 15. Traintrack T for negative end of Example 4.35

Example 6.46. Figure 15 gives the traintrack T , juncture J_κ , and surface F for the negative end of Example 4.35.

Remark. In [24], the third author induces a hyperbolic structure and geodesic laminations on F with only finitely many leaves. The laminations in [24] are the “geodesic tightening” of ours and are carried by the same traintrack. Our packets $X \times \mathbb{R}$ of leaves winding in on the circle leaves generally contain infinitely many leaves, but each packet lifts to the universal cover \tilde{F} to curves having the same ideal endpoints. Hence, tightening our circle leaves to geodesics and each of our finitely many packets to a single geodesic, each end of which winds in on one of the circles, gives the geodesic lamination of F produced in [24].

7. THE ESCAPING SET \mathcal{U}

The main purpose of this section is to construct reducing curves associated to certain border components of \mathcal{U} . This involves a detailed, technical study of $\delta\mathcal{U}$. The reader should review in Section 5.1 the definition of the metric completion \check{U} of a component U of \mathcal{U} , the metric completion of the set \mathcal{U} (which is not generally connected), the map $\check{i} : \check{\mathcal{U}} \rightarrow L$, and the definition of the set $\delta\mathcal{U}$ of border components of \mathcal{U} .

Recall (Lemma 5.15) that the escaping set satisfies,

$$\mathcal{U} = \mathcal{U}_+ \cap \mathcal{U}_- = L \setminus (|\Lambda_-| \cup \mathcal{P}_- \cup |\Lambda_+| \cup \mathcal{P}_+)$$

and that \mathcal{U} is thus disjoint from the principal regions. Further, the set $|\mathcal{X}_\pm|$ will meet each component U of \mathcal{U} but will cluster in U only on points of $|\delta U|$ (Definition 5.2). An element of \mathcal{X}_\pm can not meet \mathcal{P}_\mp but will meet \mathcal{P}_\pm in the arms.

Lemma 7.1. *If U is a component of \mathcal{U} , then $\delta U \neq \emptyset$.*

Proof. By our assumption that f is not isotopic to a translation (Hypothesis 6) and Proposition 4.76, U is a proper subset of L . Choose $x \notin U$ and $y \in U$ and denote by $[x, y]$ a closed geodesic arc with endpoints x and y . There is a point $z \in [x, y]$ such that $(z, y) \subset U$ and $z \notin U$. Then clearly $z \in |\delta U|$ so $\delta U \neq \emptyset$. \square

Remark. The components U of \mathcal{U} that are rectangles play a special role. If a component U of \mathcal{U} is a rectangle, the sequence of iterates under h of a point of U escapes. The sequence of iterates under h of a vertex of U remains in the core and cannot escape while only the sequence of positive (respectively negative) iterates of an interior point of the two edges in $|\Lambda_+|$ (respectively $|\Lambda_-|$) escape.

7.1. The border of the escaping set. The next lemma is an immediate consequence of Proposition 4.74.

Lemma 7.2. *An entire leaf $\lambda \in \Lambda_\pm$ cannot be an element of $\delta\mathcal{U}$.*

The following lemma formally defines what we mean by vertices and edges and is clear by Lemma 5.7.

Lemma 7.3 (vertex/edge). *If $x \in \gamma \in \delta\mathcal{U}$, then x lies in some semi-isolated leaf λ of one of the laminations and either,*

- (1) $x \in |\Lambda_+| \cap |\Lambda_-|$ and x also lies in a semi-isolated leaf λ' of the other lamination and there are maximal, nondegenerate subarcs $[x, y]$ and $[x, z]$ of λ and λ' , respectively, that meet no other points of $|\Lambda_+| \cap |\Lambda_-|$. If y is an end of λ , then $[x, y) \subset \gamma$ and otherwise $[x, y] \subset \gamma$ and $y \in |\Lambda_+| \cap |\Lambda_-|$. In either case, the resulting arc is called an edge of γ and x is called a vertex of γ . Similar considerations hold for $[x, z)$.
- (2) $x \notin |\Lambda_+| \cap |\Lambda_-|$ and there is a maximal open subarc $(y, z) \subset \lambda$ containing x and not meeting $|\Lambda_+| \cap |\Lambda_-|$. By Lemma 7.2, one or both of y, z is finite, lies in $|\Lambda_+| \cap |\Lambda_-|$, and is again called a vertex of γ , the resulting closed subarc or infinite ray in λ being an edge of γ .

Lemma 7.4. *Each $\gamma \in \delta\mathcal{U}$ is either an immersed copy of the real line or an immersed circle with an even number of edges.*

Proof. The lemma follows since the components of $\ddot{U} \setminus U$ are homeomorphic to a line or circle for every component U of \mathcal{U} and every $\gamma \in \delta\mathcal{U}$ is the image under \ddot{i} of a component of $\ddot{U}_\gamma \setminus U_\gamma$ for some component U_γ of \mathcal{U} . \square

Remark. We will show in Corollary 7.11 that every border component of \mathcal{U} is embedded.

Lemma 7.5. *If $\gamma \in \delta\mathcal{U}$ is an immersed line then, either its sequence of vertices $\dots, x_i, x_{i+1}, \dots$ is bi-infinite or γ has only one vertex x_0 connecting two unbounded edges α_1 and β_1 . In the latter case, x_0 is the unique h -periodic point on the semi-isolated leaves of Λ_\pm passing through it.*

Proof. Suppose, for definiteness, that the sequence is not infinite to the left and denote its initial vertex by x_0 . Thus, its initial edge α_1 must be a ray in a semi-isolated leaf λ of one of the laminations. By Theorem 6.5, we know that, for some integer $k \geq 1$, $h^k(\lambda) = \lambda$. Without loss of generality, we can suppose that h^k fixes the ends of λ . Thus, orienting λ so that its initial end is the end of α_1 , we see that

either $h^k(x_0) < x_0$, $h^{-k}(x_0) < x_0$, or $h^k(x_0) = x_0$. The first two cases imply that $\text{int } \alpha_1$ meets the other lamination, hence contains a vertex. Hence the third case holds. Now suppose that the edge $\beta_1 = [x_0, x_1]$ is bounded. It must be fixed by h^k and so by (2) of Corollary 6.15, (x_0, x_1) lies in a principal region P which then meets \mathcal{U} which is a contradiction. The final assertion is clear. \square

Definition 7.6 (first/second kind). A real line border component γ of \mathcal{U} with just one vertex is said to be of the *first kind*. Otherwise, the border component is of the second kind.

Notation. The border components γ of \mathcal{U} of the first kind are of the form $\gamma = \alpha_0 \cup \beta_0$ with $\alpha_0 \subset \Lambda_-$ and $\beta_0 \subset \Lambda_+$ and with one vertex $y_0 = \alpha_0 \cap \beta_0$. Otherwise a border γ of \mathcal{U} is of the form $\gamma = \bigcup_{i=-\infty}^{\infty} \alpha_i \cup \beta_i$ with $\alpha_i \subset |\Lambda_-|$, $\beta_i \subset |\Lambda_+|$, $\beta_{i-1} \cap \alpha_i = x_i$, and $\alpha_i \cap \beta_i = y_i$, $i \in \mathbb{Z}$. If γ is of the second kind, then the α_i, β_i form a bi-infinite sequence. If γ is compact, then there exists an integer $r > 0$ such that $\alpha_{i+r} = \alpha_i$ and $\beta_{i+r} = \beta_i$, all $i \in \mathbb{Z}$.

The next lemma follows immediately from the definitions of \mathcal{U} and $\delta\mathcal{U}$.

Lemma 7.7. *If $\gamma \in \delta\mathcal{U}$, then there exist a component U_γ of \mathcal{U} , $e_- \in \mathcal{E}_-(L)$ and $e_+ \in \mathcal{E}_+(L)$ such that $\gamma \in \delta U_\gamma$ and $U_\gamma \subset \mathcal{U}_{e_-} \cap \mathcal{U}_{e_+}$.*

From now on, we focus on the end $e_- \in \mathcal{E}_-(L)$. The discussion for $e_+ \in \mathcal{E}_+(L)$ is analogous. Let $c = \{e_- = e_1, e_2, \dots, e_{p_{e_-}}\}$ be the full h -cycle containing e_- . Recall the projection $q : \mathcal{U}_c \rightarrow F$, the lamination Λ_F , and the traintrack T for Λ_F (Section 6.7). Note that the surface F depends on the h -cycle of ends containing e_- .

Lemma 7.8. *If $\gamma \in \delta\mathcal{U}$ is not of the first kind and is not the border component of a rectangle and $\alpha \subset |\Lambda_-|$ is an edge of γ , then α is an extreme leaf of a parallel packet of leaves of $\Lambda_-|_{\mathcal{U}_{e_-}}$.*

Proof. If α is an interior leaf of a parallel packet of leaves, then γ is the border component of a rectangle contrary to assumption. \square

Notation. Denote by T^* the compact surface obtained by fattening up T in F .

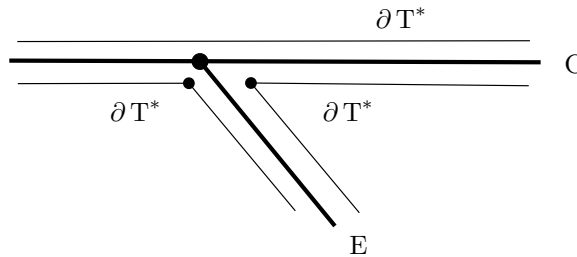


FIGURE 16. Inserting vertices into ∂T^* at a switch

A switch (Proposition 6.45) on the train track T comes from the end of a parallel packet of leaves winding in on a circle leaf. If one considers T as a 3-valent graph, a switch is a vertex of degree three. Two of the edges coming out of a switch are subsets of the circle leaf. The third edge represents the parallel packet of leaves and is thus distinguished from the other two edges. In a natural way, the structure of T as a 3-valent graph divides certain components of ∂T^* into vertices and edges

by inserting two vertices into ∂T^* at each switch as indicated in Figure 16. In Figure 16, the two horizontal edges at the switch belong to a circle leaf C while the third edge E represents an end of a parallel packet of leaves winding in on C .

Remark. Some components of ∂T^* may contain no vertices. In Theorem 7.9, we show that these components correspond to border components of \mathcal{U} of the first kind.

Remark. The surface T^* may be non-orientable.

- Theorem 7.9.** (1) *There is a ι^*/p_{e_-} -to-one correspondence between the set of border components γ of \mathcal{U} of the first kind that meet \mathcal{U}_{e_-} and the set of components C^* of ∂T^* not containing a vertex where ι^* is the intersection number of C^* and J_κ ;*
- (2) *There is a one-one correspondence between the set of infinite families of compact border components γ of \mathcal{U} that meet \mathcal{U}_{e_-} and are not rectangles and the set of components C^* of ∂T^* containing a vertex with zero intersection number with the juncture J_κ .*
- (3) *There is a ι^*/p_{e_-} -to-one correspondence between the set of border components γ of \mathcal{U} of the second kind that meet \mathcal{U}_{e_-} and the set of components C^* of ∂T^* containing a vertex such that C^* has non-zero intersection number ι^* with the juncture J_κ ;*

Proof. First, suppose $\gamma = \alpha_0 \cup \beta_0$ with $\alpha_0 \subset \Lambda_-$ and $\beta_0 \subset \Lambda_+$ and with one vertex $y_0 = \alpha_0 \cap \beta_0$ is a border component of \mathcal{U} of the first kind that meets \mathcal{U}_{e_-} , then $C_\gamma = q(\text{int } \alpha_0)$ is a circle leaf of Λ_F such that there are no edges of the traintrack on one side of the circle C_γ . In T^* , C_γ gives rise to a component C_γ^* of ∂T^* containing no vertices. Thus, every border component γ of the first kind that meets \mathcal{U}_{e_-} corresponds to a unique component C_γ^* of ∂T^* .

Conversely, if C^* is a component of ∂T^* containing no vertices, then there exists a corresponding circle leaf C of Λ_F such that there are no edges of the traintrack on one side of the circle C . It is then easy to see that $C = q(\text{int } \alpha_0)$ where $\gamma = \alpha_0 \cup \beta_0$ is a border leaf of \mathcal{U} of the first kind that meets \mathcal{U}_{e_-} . Thus, every component $C^* \subset \partial T^*$ containing no vertices corresponds to at least one border component of \mathcal{U} of the first kind that meets \mathcal{U}_{e_-} .

Let ι^* be the intersection number of C^* and J_κ . Since $h^{\iota^*}(\gamma) = \gamma$ and the curves $h^j(\gamma)$, $0 \leq j < \iota^*$, are distinct and there are p_{e_-} ends in the cycle of ends containing e_- , it follows that there are ι^*/p_{e_-} border components of \mathcal{U} of the first kind that meet \mathcal{U}_{e_-} corresponding to the component C^* .

Next, suppose $\gamma = \bigcup_{i=-\infty}^{\infty} \alpha_i \cup \beta_i$ (in the notation of page 63) is a border component of a component U_γ of \mathcal{U} such that $U_\gamma \subset \mathcal{U}_{e_-}$ is not a rectangle and γ is not a border component of \mathcal{U} of the first kind. The leaf α_i , $i \in \mathbb{Z}$, of $\Lambda_- | \ddot{\mathcal{U}}$ is extreme in a parallel packet of leaves (Lemma 7.8) and, since $q(\alpha_i) \subset q(U_\gamma)$, determines an edge of a component C_γ^* of ∂T^* which we denote by α_i^* . The edge α_i^* inherits an orientation from α_i and we denote the initial vertex of α_i^* by x_i^* and the terminal vertex by y_i^* .

The terminal vertex y_{i-1}^* of α_{i-1}^* and the initial vertex x_i^* of α_i^* determine a unique interval $(y_{i-1}^*, x_i^*) \subset C_\gamma^*$ which contains no vertices. This follows since there exists an edge $\beta_i \subset |\Lambda_+|$ such that $\alpha_{i-1} \cap \beta_i = y_i$, $\beta_i \cap \alpha_i = x_{i+1}$, and $\alpha_{i-1} \cup \beta_i \cup \alpha_i \subset \delta U_\gamma$ for some component U_γ of \mathcal{U} .

We denote the interval $[y_{i-1}^*, x_i^*]$ by β_i^* . Then $\bigcup_{i=-\infty}^{\infty} \alpha_i^* \cup \beta_i^*$ where $\alpha_{i-1}^* \cap \beta_i^* = y_i^*$ and $\beta_i^* \cap \alpha_i^* = x_{i+1}^*$ equals C_γ^* . The union is, in fact, a finite union $\bigcup_{i=0}^{p-1} \alpha_i^* \cup \beta_i^*$ with $\alpha_0^* = \alpha_p^*$, for a least integer $p > 0$, since each component of ∂T^* has finitely many edges. Thus γ corresponds to $C_\gamma^* \subset \partial T^*$.

If γ is compact, it follows that $\alpha_p = \alpha_0$ and $\alpha_j \neq \alpha_0$ for $0 < j < p$. Thus the intersection number of C_γ^* and the juncture J_κ is zero. Similarly, if γ is of the second kind, then $\alpha_0 \neq \alpha_p$. In this case $h^{\iota^*}(\alpha_0) = \alpha_p$ where ι^* is the intersection number of C_γ^* and J_κ .

Conversely, if C^* is a component of ∂T^* containing vertices, choose an edge $\alpha_0^* = [x_0^*, y_0^*]$ corresponding to an extreme leaf α_0 of a parallel packet of leaves of $\Lambda_- | \ddot{U}_{e_-}$. Then $\alpha_0 \subset \gamma$ where γ is a border component of \mathcal{U} that meets \mathcal{U}_{e_-} which is not a rectangle or a border component of \mathcal{U} of the first kind. Exactly as above, γ corresponds to C^* . Thus for each $C^* \subset \partial T^*$ containing a vertex there is at least one γ not a rectangle and not of the first kind that corresponds to it.

If γ is compact, $\{h^{np_{e_-}}(\gamma) \mid n \in \mathbb{Z}\}$ is a bi-infinite family of disjoint compact border components that meet \mathcal{U}_{e_-} . Thus, in this case, there is one bi-infinite family of border components corresponding to C_γ^* .

If γ is not compact, then γ must be of the second kind. Let ι^* be the intersection number of C^* and J_κ . Since $h^{\iota^*}(\gamma) = \gamma$ and the curves $h^j(\gamma)$, $0 \leq j < \iota^*$, are distinct and there are p_{e_-} ends in the cycle of ends containing e_- , it follows that there are ι^*/p_{e_-} border components of \mathcal{U} of the second kind that meet \mathcal{U}_{e_-} corresponding to the component C^* . \square

In the case that γ is not a rectangle, the next two corollaries follow because a component of ∂T^* is a simple curve. They are clear if γ is a rectangle.

Corollary 7.10. *If $\gamma \in \delta\mathcal{U}$, then γ is embedded in L .*

Corollary 7.11. *If $\gamma \in \delta\mathcal{U}$ is compact, then $h^m(\gamma)$ is disjoint from $h^n(\gamma)$ for all $m \neq n \in \mathbb{Z}$.*

The next corollary follows because ∂T^* has finitely many components and L has finitely many ends.

Corollary 7.12. (1) *There are finitely many infinite families $\{h^n(\gamma) \mid n \in \mathbb{Z}\}$ of disjoint compact border components of \mathcal{U} that are not rectangles where γ is a compact border component;*
 (2) *There are finitely many noncompact border components of \mathcal{U} .*

The next corollary follows because there are finitely many noncompact border components of \mathcal{U} .

Corollary 7.13. *If $\gamma \in \delta\mathcal{U}$ is noncompact, then there exists a least integer $r > 0$ such that $h^r(\gamma) = \gamma$.*

Remark. The integer r is necessarily a multiple of both p_{e_-} and p_{e_+} .

Remark. If $\gamma \in \delta\mathcal{U}$ is of the first kind, then h^r has one fixed point on γ . If $\gamma \in \delta\mathcal{U}$ is of the second kind, then h^r is fixed point free on γ .

Denote by ε_- the negative end of γ . That is, if $\gamma = \alpha_0 \cup \beta_0$ is of the first kind then $\alpha_0 = (\varepsilon_-, y_0]$ and if γ is of the second kind and $x \in \gamma$, then $h^{nr}(x) \rightarrow \varepsilon_-$ as $n \rightarrow -\infty$.

Lemma 7.14. *If $\gamma \in \delta\mathcal{U}$ is noncompact and $(\varepsilon_-, x_0] \subset \gamma$ is a neighborhood of ε_- , then there exists a neighborhood of e_+ (see Lemma 7.7) disjoint from $(\varepsilon_-, x_0]$.*

Proof. The lemma is clear if γ is of the first kind. Suppose γ is of the second kind. Choose $z \in \gamma$ and let $z_n = h^{nr}(z)$, $n \in \mathbb{Z}$. Since $[z_{-1}, z_0] \subset \gamma$ is compact, there exists a distinguished neighborhood U_{e_+} of e_+ disjoint from $[z_{-1}, z_0]$. Since $U_{e_+} \supset U_{e_+}^k = h^{kp_{e_+}}(U_{e_+})$, all $k \geq 0$, it follows that $U_{e_+}^k$ is disjoint from $[z_{-1}, z_0]$, all $k \geq 0$. Thus $U_{e_+} = h^{-kr}(U_{e_+}^{kr/p_{e_+}})$ is disjoint from $h^{-kr}([z_{-1}, z_0]) = [z_{-(k+1)}, z_{-k}]$. Thus, U_{e_+} is disjoint from $\bigcup_{k=0}^{\infty} [z_{-(k+1)}, z_{-k}] = (\varepsilon_-, z_0]$. \square

Corollary 7.15. *If $\gamma \in \delta\mathcal{U}$ is noncompact, then any lift $\tilde{\gamma}$ of γ has two endpoints $a \neq b \in S_{\infty}^1$.*

Proof. Let σ be a positive juncture component meeting γ at a point z . Let $\sigma_n = h^{nr}(\sigma)$ and $z_n = h^{nr}(z)$. Let $\tilde{\gamma}$ be a lift of γ , $\tilde{z}_n \in \tilde{\gamma}$ lifts of z_n , and $\tilde{\sigma}_n$ lifts of σ_n containing \tilde{z}_n . Then the geodesics $\tilde{\sigma}_n$ nest on a point $b \in S_{\infty}^1$ as $n \rightarrow \infty$. An analogous argument shows that the negative end of $\tilde{\gamma}$ limits on a well defined point $a \in S_{\infty}^1$.

If $a = b$ then every neighborhood of the negative end of $\tilde{\gamma}$ meets $\tilde{\sigma}_n$ for n sufficiently large. Since the sequence $\{\sigma_n\}_{n \geq 0}$ escapes to e_+ , this contradicts Lemma 7.14. \square

Example 7.16. In Example 4.35, there is one semi-isolated leaf $\lambda_- \in \Lambda_-$ and one semi-isolated leaf $\lambda_+ \in \Lambda_+$. This example has one real line border component of the first kind and one real line border component of the second kind, both clearly visible in Figure 11.

Example 7.17. Example 4.35 has one bi-infinite sequence \mathcal{C} of compact border components of \mathcal{U} that are not borders of rectangles. Each $\gamma_n \in \mathcal{C}$ has one edge in the semi-isolated leaf $\lambda_- \in \Lambda_-$ and one edge in the semi-isolated leaf $\lambda_+ \in \Lambda_+$. The component $U_n \subset \mathcal{U}$ with this γ_n as border is a (stretched out) annulus and has one of the boundary circles of L in its boundary. In this example (and most examples) there are infinitely many bi-infinite sequences of compact border components of \mathcal{U} that are the borders of rectangles. Example 2.10 is similar, each γ_n again having two edges and two vertices and all belonging to the same bi-infinite sequence \mathcal{C} , but now they are the elements of δU , where U is a single unbounded component of \mathcal{U} .

Definition 7.18 (peripheral border component). A border component γ of \mathcal{U} is *peripheral* if there is a component $C \subset \mathcal{U}$ of ∂L such that γ and C cobound a component U_{γ} of \mathcal{U} homeomorphic either to an open annulus or an open infinite strip $\mathbb{R} \times (0, 1)$.

7.2. Reducing circles. Let $\gamma \in \delta U_{\gamma}$ be a circle border component of \mathcal{U} , where U_{γ} is a component of \mathcal{U} that is not an open disk nor an open Möbius strip. Then there is a unique simple closed geodesic σ_{γ} in the free homotopy class of γ . By the convexity of U_{γ} , $\sigma_{\gamma} \subset U_{\gamma}$. This geodesic σ_{γ} is 2-sided and cobounds an open annulus $A \subset U_{\gamma}$ with γ . If γ is peripheral, then σ_{γ} is a component of ∂L and is not taken as a reducing curve. Otherwise σ_{γ} will be included in the set \mathfrak{S} of reducing curves. If the set U_{γ} is an open annulus with boundary the two curves γ_1 and γ_2 , then $\sigma_{\gamma_1} = \sigma_{\gamma_2}$. Otherwise, the various σ_{γ} 's will be disjoint and disjoint from Λ_{\pm} , ∂L , and the previously constructed reducing curves.

Lemma 7.19. *The sequence of reducing circles $\{\sigma_{h^n(\gamma)}\}_{n \in \mathbb{Z}}$ escapes.*

Proof. Note that $\sigma_{h^n(\gamma)} = (h^n(\sigma_\gamma))^{\mathfrak{g}}$. Since the circle σ_γ is compact and contained in $\mathcal{U}_{e_-} \cap \mathcal{U}_{e_+}$, it follows that $\sigma_\gamma \subset U_{e_-} \cap U_{e_+}$ for a distinguished neighborhood U_{e_-} of e_- and distinguished neighborhood U_{e_+} of e_+ . We will use the fact that $\sigma_\gamma \subset U_{e_+}$ to show that $\{\sigma_{h^n(\gamma)}\}_{n \geq 0}$ escapes. In an analogous way the fact that $\sigma_\gamma \subset U_{e_-}$ shows that $\{\sigma_{h^n(\gamma)}\}_{n \leq 0}$ escapes.

Let $J = \text{Fr } U_{e_+}$. By the construction of h in Section 4.6, $J_n = h^n(J)$, $n \in \mathbb{Z}$. Thus $J_n = \text{Fr } h^n(U_{e_+})$ and $h^n(U_{e_+})$ is the distinguished neighborhood of a positive end in the cycle of ends containing the end e_+ . Since $\text{Fr } h^n(U_{e_+})$ consists of the geodesics forming J_n and $h^n(\sigma_\gamma) \subset h^n(U_{e_+})$, it follows that $\sigma_{h^n(\gamma)} = (h^n(\sigma_\gamma))^{\mathfrak{g}} \subset h^n(U_{e_+})$, $n \in \mathbb{Z}$. Since only finitely many of the distinguished neighborhoods $h^n(U_{e_+})$, $n \geq 0$, meet any compact set, it follows that $\{\sigma_{h^n(\gamma)}\}_{n \geq 0}$ escapes. \square

Remark. As one forwardly iterates applications of h to γ , the vertices remain in K and the edges $\beta_i \subset |\Lambda_+|$ stretch without bound. Similarly, under iterates of h^{-1} , the edges $\alpha_i \subset |\Lambda_-|$ become unbounded. The open annulus cobounded by σ_γ and γ also stretches without bound. A simple modification of Example 4.35 illustrates this behavior. Alter L by gluing a punctured torus to each boundary circle. Those circles now become reducing curves σ_γ and one easily sees γ and the annulus that it cobounds with σ_γ .

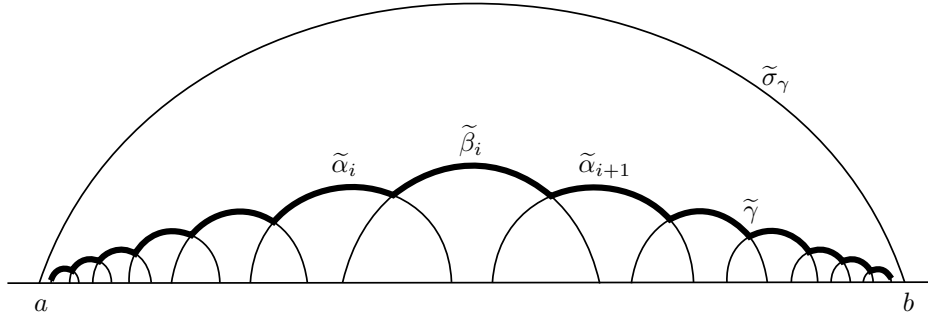


FIGURE 17. The lifts $\tilde{\sigma}_\gamma$ and $\tilde{\gamma}$ (boldface) for $\gamma \in \delta\mathcal{U}$ of second kind

7.3. Reducing lines. Let $\gamma \in \delta\mathcal{U}_\gamma$ be a border component of \mathcal{U} homeomorphic to the reals. The associated curve σ_γ is the geodesic whose lift has endpoints $a \neq b$ of Corollary 7.15. Figure 17 (respectively Figure 18) illustrates $\tilde{\gamma}$ and $\tilde{\sigma}_\gamma$ when γ is of the second kind (respectively first kind). Here $\tilde{\gamma}$ is given in boldface and the bottom line represents a subarc of S_∞ (Definition 3.12).

Lemma 7.20. *The curve σ_γ is simple, homeomorphic to \mathbb{R} , and disjoint from γ .*

Proof. If σ_γ is not simple, then some lift of σ_γ intersects the lift $\tilde{\sigma}_\gamma$ with endpoints a, b . It follows that some lift of γ intersects the lift $\tilde{\gamma}$ with endpoints a, b which contradicts the fact that γ is simple. Thus, σ_γ is simple.

As in the proof of Corollary 7.15, there exists lifts σ_n of positive juncture components that lie in arbitrarily small neighborhoods of the end e_+ for n sufficiently

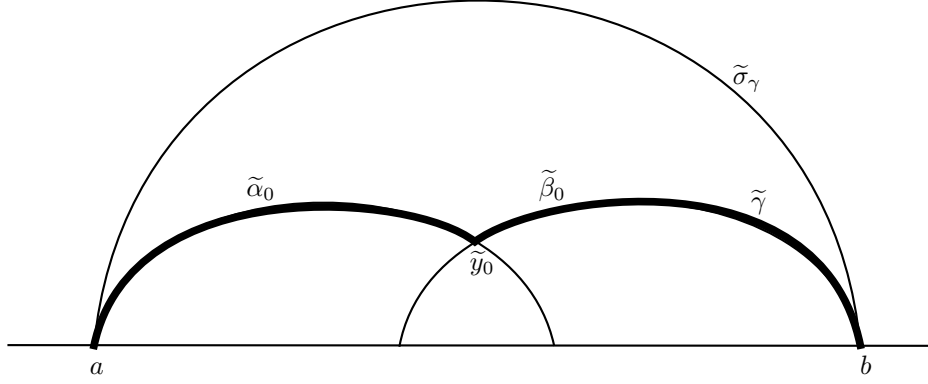


FIGURE 18. The lifts $\tilde{\sigma}_\gamma$ and $\tilde{\gamma}$ (boldface) for $\gamma \in \delta\mathcal{U}$ of first kind

large such that the σ_n nest on b . Since σ_γ has a lift with endpoint b , this implies that σ_γ is unbounded. If σ_γ is homeomorphic to a circle then it is compact and thus bounded. This contradiction implies that σ_γ is homeomorphic to the reals.

If σ_γ meets γ , then some lift $\tilde{\sigma}'_\gamma$ of σ_γ with endpoints $a', b' \in S_\infty^1$ meets the lift $\tilde{\gamma}$ of γ with endpoints a, b . If γ is a border component of \mathcal{U} of the first kind, then $\tilde{\sigma}'_\gamma$ meets one of the rays $\tilde{\alpha}_0$ or $\tilde{\beta}_0$ (see Figure 18). If γ is a border component of \mathcal{U} of the second kind, then $\tilde{\sigma}'_\gamma$ meets one of the compact arcs $\tilde{\alpha}_i$ or $\tilde{\beta}_i$, $-\infty < i < \infty$ (see Figure 17). Let α with lift $\tilde{\alpha}$ be such a ray or compact arc that $\tilde{\sigma}'_\gamma$ meets $\tilde{\alpha}$ and let $\lambda \in \Lambda_\pm$ be such that $\lambda \supset \alpha$ has lift $\tilde{\lambda} \supset \tilde{\alpha}$. Then the endpoints of $\tilde{\lambda}$ on S_∞^1 separate a' and b' so $\tilde{\lambda}$ intersects the lift $\tilde{\gamma}$ of γ with endpoints a', b' on S_∞^1 transversely contradicting the fact that $\gamma \in \delta\mathcal{U}$. Thus, σ_γ and γ are disjoint. \square

Corollary 7.21. *The curves γ and σ_γ cobound a region $O_\gamma \subset L$ homeomorphic to $(0, 1) \times \mathbb{R}$ and contained in the component U_γ of \mathcal{U} .*

Proof. Let O_γ be the projection of the open region $\tilde{O}_\gamma \subset \Delta$ bounded by the curves $\tilde{\gamma}$ and $\tilde{\sigma}_\gamma$. We must show that if T is any nontrivial covering transformation, then $T(\tilde{O}_\gamma) \cap \tilde{O}_\gamma = \emptyset$. Suppose to the contrary that $T(\tilde{O}_\gamma) \cap \tilde{O}_\gamma \neq \emptyset$. If $T(a) \neq a$ and/or $T(b) \neq b$, then one of $T(\tilde{\gamma}), T(\tilde{\sigma}_\gamma)$ must meet one of $\tilde{\gamma}, \tilde{\sigma}_\gamma$ which either contradicts the fact that γ and σ_γ are each simple or contradicts the fact that γ and σ_γ are disjoint. Thus we can assume $T(a) = a$ and $T(b) = b$ but in that case, we must have $T(\tilde{O}_\gamma) = \tilde{O}_\gamma$ and $T(\tilde{\gamma}) = \tilde{\gamma}$. Since γ is homeomorphic to the reals, T must be the identity transformation.

By the convexity of \tilde{O}_γ , any geodesic in $\tilde{\Lambda}_\pm$ that meets \tilde{O}_γ must also meet $\tilde{\gamma}$. It follows that \tilde{O}_γ is disjoint from $\tilde{\Lambda}_\pm$ and that $O_\gamma \subset U_\gamma$. \square

Corollary 7.22. *One end of the curve σ_γ approaches the positive end e_+ while the other end of σ_γ approaches the negative end e_- .*

If γ is peripheral, then $\sigma_\gamma \subset \partial L$ and is not taken as a reducing curve. If the set U_γ is a doubly infinite open strip with boundary two curves γ_1 and γ_2 , then $\sigma_{\gamma_1} = \sigma_{\gamma_2}$. In this case, both of the γ_i may be of the second kind or one of the γ_i

may be of the first kind and the other of the second kind. It can not happen that both of the γ_i are of the first kind.

Indeed, exactly as in Example 4.36, if the set U_γ is a doubly infinite open strip with boundary two curves γ_1 and γ_2 both of the first kind, then there would exist a leaf of Λ_+ and a leaf of Λ_- each meeting U_γ contradicting the fact that U_γ is contained in the escaping set.

Otherwise, the various σ_γ 's, associated to real line border components γ of \mathcal{U} , will be disjoint and disjoint from $\Lambda_\pm, \partial L$. The curves σ_γ will be included in the set \mathfrak{S} of reducing curves.

Remark. If γ is of the second kind, the edges $\beta_i \subset |\Lambda_+|$ of γ stretch unboundedly as $i \rightarrow \infty$ and the edges $\alpha_i \subset |\Lambda_-|$ stretch unboundedly as $i \rightarrow -\infty$. The vertices of γ remain in the core. Thus, these border components of the second kind appear quite bizarre in L and do not directly connect a negative end of L to a positive one but rather return infinitely often to the core. By Corollary 7.22, σ_γ does directly connect a negative end to a positive end.

The set \mathfrak{S} of reducing curves is now complete.

Remark. The reducing curves in \mathfrak{S} are of following kinds,

- (1) The rims of crown sets defined in Section 6.5.
- (2) The infinite families of reducing circles defined in Section 7.2.
- (3) The reducing lines corresponding to border components of \mathcal{U} of the first and second kind defined in Section 7.3.

Remark. A reducing curve is always a geodesic.

8. REDUCTION

By a reduced piece, we mean the internal completion $Q = \ddot{U}$ of a component U of $L \setminus |\mathfrak{S}|$. Recall that $\ddot{i} : Q \rightarrow \overline{U}$ is not necessarily one-one but possibly identifies border components of Q (Section 5.1). As we define g on \overline{U} , in abuse of notation, we will also consider g to be defined on Q . If $A \subset L$ we will abuse notation and denote $\ddot{i}^{-1}(A)$ by $A \cap Q$ and say A meets Q if $A \cap Q \neq \emptyset$.

Theorem 8.1. *If $f : L \rightarrow L$ is an endperiodic automorphism, then there exists an endperiodic automorphism $g : L \rightarrow L$, isotopic to f such that,*

- (1) $g|_{|\Lambda_+|} = h|_{|\Lambda_+|}$ and $g|_{|\Lambda_-|} = h|_{|\Lambda_-|}$ where h is the endperiodic automorphism of Theorem 4.54;
- (2) g permutes the elements of the set \mathfrak{S} of reducing curves;
- (3) *There are at least one and at most finitely many components U of $L \setminus |\mathfrak{S}|$ with \overline{U} noncompact. For such U , if $g^m(Q) = Q$, then either,*
 - (a) $g^m : Q \rightarrow g^m(Q)$ is isotopic to a translation;
 - (b) $g^m : Q \rightarrow g^m(Q)$ is a pseudo-anosov automorphism (Definition 8.13).

Remark. The dynamics of g on the compact reduced pieces is described in Propositions 8.9 and 8.10. Briefly, if Q is a compact reduced piece in a principal region, the dynamics of g on Q is given by Nielsen-Thurston theory. If Q is a compact reduced piece contained in the escaping set \mathcal{U} , then the dynamics of g on Q is trivial.

Remark. If σ_γ is a reducing curve associated to the border component $\gamma \in \delta\mathcal{U}$ then $g(\sigma_\gamma) = \sigma_{h(\gamma)}$. If ρ_γ is a reducing curve which is a rim of a crown set associated to a boundary component γ of the nucleus of a principal region, then $g(\rho_\gamma) = \rho_{h(\gamma)}$.

8.1. Proof of first part of Theorem 8.1. Let h be the endperiodic automorphism of Theorem 4.54. Note that the isotopies in Lemma 8.2 will move some juncture components but leave invariant the juncture components in $\{h(\sigma) \mid \sigma \in \mathcal{J}_W\}$ (\mathcal{J}_W is defined in Definition 4.42).

Lemma 8.2. *There exists an endperiodic automorphism g isotopic to h by an isotopy with support disjoint from $\Lambda_+ \cup \Lambda_- \cup \partial L$ such that $g(\mathfrak{S}) = \mathfrak{S}$.*

Sketch of proof. Enumerate the elements of $h(\mathfrak{S})$ as $\{\gamma_i\}_{i=1}^\infty$ (or $\{\gamma_i\}_{i=1}^N$ if there are finitely many reducing curves). Let $\psi_0 = \text{id}$. We inductively find sequences $\{\Phi_i\}$ of isotopies and $\{\psi_i\}$ of homeomorphisms with $\psi_i = \Phi_i^1 \circ \psi_{i-1}$, $i \geq 1$, with support disjoint from $\Lambda_+ \cup \Lambda_- \cup \partial L$, and with Φ_i fixing γ_j^g pointwise, $1 \leq j \leq i-1$, and moving $\psi_{i-1} \circ \gamma_i$ to its geodesic tightening. That is, $\psi_i(\gamma_i) = \gamma_i^g$.

Suppose Φ_i, ψ_i , $1 \leq i \leq n-1$, have been defined satisfying these properties and let $\tilde{\gamma}_n$ be a lift of γ_n and $\tilde{\gamma}_n^g$ be the lift of γ_n^g sharing endpoints on S_∞^1 with $\tilde{\gamma}_n$.

If σ is a juncture component in $h(\mathcal{J}_W)$ and $x \in \sigma \cap \gamma_i$ has lift $\tilde{x} \in \tilde{\sigma} \cap \tilde{\gamma}_i$ where $\tilde{\sigma}$ is a lift of σ , then by Lemma 4.55 there is a sliding isotopy supported in a small neighborhood of σ with lift sliding \tilde{x} along $\tilde{\sigma}$ to the point of intersection of $\tilde{\sigma}$ and $\tilde{\gamma}_i^g$. After a sequence of such sliding isotopies, we can assume that $\tilde{\sigma} \cap \tilde{\gamma}_i = \tilde{\sigma} \cap \tilde{\gamma}_i^g$ for every juncture component $\sigma \in h(\mathcal{J}_W)$ and lift $\tilde{\sigma}$ meeting $\tilde{\gamma}_i$.

Consider the set of all points of intersection of $\sigma \cap \gamma_n$, all $\sigma \in h(\mathcal{J}_W)$. If $a, b \in \gamma_n$ are two such points of intersection so that the interval $[a, b] \subset \gamma_n$ contains no other such point of intersection, then, by Theorem 3.2, there is an isotopy moving the interval $[a, b] \subset \gamma_n$ to the corresponding interval in γ_n^g with endpoints a, b . After a sequence of such isotopies we can assume γ_n has been moved to γ_n^g . Let Φ_n be the composition of the isotopies of this and the previous paragraph and $\psi_n = \Phi_n^1 \circ \psi_{n-1}$. By the remark after Theorem 3.3, we can assume Φ_n fixes γ_j^g pointwise, $1 \leq j \leq n-1$.

The supports of the Φ_n do not accumulate. Thus, $\psi_n \rightarrow \psi$, a well defined homeomorphism isotopic to the identity by an isotopy Φ and $g = \psi \circ h$ is such that $g(\alpha)$ is a geodesic for every $\alpha \in \mathcal{J}_W$. Since γ_n and γ_n^g both lie in the same component of $L \setminus (\Lambda_+ \cup \Lambda_-)$ for each $i \geq 1$, the isotopy Φ can be defined to have support disjoint from $\Lambda_+ \cup \Lambda_-$. The lemma follows. \square

This g is the required endperiodic automorphism of Theorem 8.1.

Corollary 8.3. *The endperiodic automorphism g permutes the components U of $L \setminus |\mathfrak{S}|$.*

The endperiodic automorphism g of Theorem 8.1 permutes the elements of each of the sets Λ_+ , Λ_- , and \mathfrak{S} . It is not possible to construct g isotopic to f so that g also permutes the elements of each of the sets \mathcal{J}_+ and \mathcal{J}_- without additional restriction on the choice of f -junctures. The problem one encounters in trying to do this is illustrated in the following example.

Example 8.4. If three pseudo-geodesics in Δ are such that any two of them intersect, then there might not exist an ambient isotopy of Δ that moves each of them to a geodesic. Figure 19 illustrates such a bad triple of pseudo-geodesics. In Figure 19, α and β are geodesics and σ is a pseudo-geodesic with geodesic tightening σ^g . There clearly does not exist an ambient isotopy that moves each of α , β , and σ to geodesics.

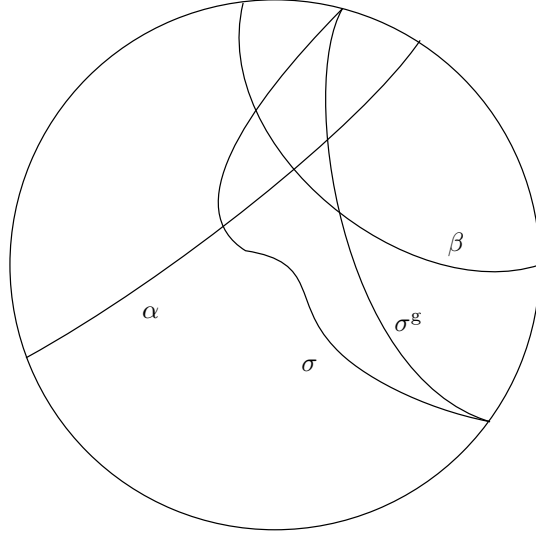


FIGURE 19. A bad configuration of three pseudo-geodesics in Δ

It is possible to construct an endperiodic automorphism f of a surface L with positive juncture component τ^+ , negative juncture component τ^- , and reducing curve γ such that the lifts of the pseudo-geodesics $f(\tau^+), f(\tau^-), f(\gamma)$ are configured as α, β, σ in Figure 19. For such an example it would be impossible to find an ambient isotopy of L permutting each of the sets $\mathcal{J}_+, \mathcal{J}_-$, and \mathfrak{S} .

8.2. The reduced pieces. By a reduced piece Q , we mean the internal completion $Q = \bar{U}$ of a component U of $L \setminus |\mathfrak{S}|$.

Lemma 8.5. *Suppose U is a component of $L \setminus \mathfrak{S}$.*

- (1) *If U does not meet $|\Lambda_+| \cup |\Lambda_-|$, then $\bar{i} : Q \rightarrow \bar{U}$ is a homeomorphism;*
- (2) *If U does meets $|\Lambda_+| \cup |\Lambda_-|$, then either $\bar{i} : Q \rightarrow \bar{U}$ is a homeomorphism or Q is homeomorphic to \bar{U} cut apart along some reducing curves.*

Proof. If σ_γ is a reducing curve, then on one or both sides of σ_γ , there is a curve $\gamma \subset |\Lambda_+| \cup |\Lambda_-|$ such that γ and σ_γ cobound an annulus or doubly infinite strip with interior disjoint from $|\mathfrak{S}|$. Thus, one or both sides of σ_γ borders a component of $L \setminus |\mathfrak{S}|$ which meets $|\Lambda_+| \cup |\Lambda_-|$ and (1) follows. In case (2), it is easy to construct examples in which $\bar{i} : Q \rightarrow \bar{U}$ is not a homeomorphism to \bar{U} but rather identifies pairs of components of δQ to form reducing curves. \square

8.2.1. The noncompact reduced pieces.

Lemma 8.6. *A noncompact reduced piece Q is standard in the hyperbolic metric induced by the metric of L .*

Proof. If $Q = \bar{U}$, then Q is a standard hyperbolic surface as a subsurface with geodesic boundary of the standard hyperbolic surface L . Otherwise, Q is \bar{U} cut apart along embedded geodesics and so is a standard hyperbolic surface. \square

Since there are finitely many reducing curves of the first and second kind, there are finitely many noncompact reduced pieces. Therefore, for any noncompact reduced piece Q , there exists an integer $m > 0$ such that $g^m(Q) = Q$.

Proposition 8.7. *Suppose Q is a noncompact reduced piece. Then,*

- (1) *The immersion $i : Q \looparrowright L$ induces a map $\iota_Q : \mathcal{E}(Q) \rightarrow \mathcal{E}(L)$ and $\mathcal{E}(Q)$ is finite;*
- (2) *If $m > 0$ is an integer such that $g^m(Q) = Q$, then $g^m : Q \rightarrow Q$ is endperiodic.*

Proof. Fix an integer $k > 0$ which is a multiple of m and of p_e for every end e of L . Also fix an exhaustion $K = K_0 \subset K_1 \subset \cdots \subset K_n \subset \cdots$ of L by larger and larger choices of cores. Then $\{K_n \cap Q\}_{n \geq 0}$ is an exhaustion of Q . Let $e \in \mathcal{E}(Q)$ and let U_n be the connected component of the complement in Q of $K_n \cap Q$ which is a neighborhood of e . Then $e = [\{U_n\}]$ (see the definition of “end” on page 3). Let V_n be the component of $L \setminus K_n$ which contains U_n . Thus, $e' = [\{V_n\}]$ is a uniquely defined end in $\mathcal{E}(L)$ and the map $\iota_Q : \mathcal{E}(Q) \rightarrow \mathcal{E}(L)$ defined by $\iota_Q(e) = e'$ is well defined, proving the first assertion of (1).

Next we prove that $\mathcal{E}(Q)$ is finite. Since k is divisible by $p_{e'}$, for every $e' \in \mathcal{E}(L)$, $g^k : L \rightarrow L$ fixes $\mathcal{E}(L)$ pointwise. The ends $e \in \mathcal{E}(Q)$ can arise in one of two ways. Let $\iota_Q(e) = e'$. In the first case, $\mathcal{U}_{e'} \subset Q$ and e is naturally identified to e' as an attracting or repelling end of Q relative to g^k . There can be at most finitely many ends of $\mathcal{E}(Q)$ obtained in this way. All ends of $\mathcal{E}(Q)$ obtained in this way are attracting or repelling ends for $g^k : Q \rightarrow Q$.

Otherwise, $\mathcal{U}_{e'}$ meets and therefore contains a reducing curve γ of the first or second kind or an infinite family of compact reducing curves which can be thought of as boundary component(s) of Q . Since there are only finitely many reducing curves of the first or second kind or infinite families of compact reducing curves, at most finitely many ends of Q can be obtained in this way.

Let $e \in \mathcal{E}(Q)$ be of either of these types, $e' = i_Q(e)$, and let $U_{e'} \subset L$ be a g -neighborhood of the end e' as in Definition 2.3. Suppose $U_e \subset Q$ is the component X of $(U_{e'} \cap Q)$ which is a neighborhood of the end e in Q with any contiguous compact components of $Q \setminus X$ added on. Then U_e satisfies the conditions of Definition 2.2 so e is an attracting or repelling end for $g^k : Q \rightarrow Q$. Since all ends in $\mathcal{E}(Q)$ are attracting or repelling, it follows that g^k is endperiodic on Q . By Lemma 2.7, it follows that g^m is endperiodic on Q . \square

Recall that the leaves of the laminations Λ_{\pm} do not intersect the reducing curves in \mathfrak{S} .

Lemma 8.8. *Suppose Q is a noncompact reduced piece and $g^m(Q) = Q$.*

- (1) *$\mathcal{U}_{\pm} \cap Q$ are the positive/negative escaping sets for the endperiodic automorphism $g^m : Q \rightarrow Q$.*
- (2) *$(\Lambda_- \cap Q, \Lambda_+ \cap Q)$ is the Handel-Miller bilamination for the endperiodic automorphism $g^m : Q \rightarrow Q$.*

Proof. For $x \in Q$, the sequence $\{g^n(x)\}_{n \geq 0}$ escapes in L if and only if the sequence $\{(g|Q)^{km}(x)\}_{k \geq 0}$ escapes in Q and (1) follows. By Lemma 4.71, $|\Lambda_{\pm}| = \text{Fr } \mathcal{U}_{\mp}$. Let Λ_{\pm}^Q denote the positive/negative Handel-Miller laminations for the endperiodic automorphism $g^m : Q \rightarrow Q$. By Lemma 4.71 and (1),

$$|\Lambda_{\pm}^Q| = \text{Fr}(Q \cap \mathcal{U}_{\mp}) = Q \cap \text{Fr } \mathcal{U}_{\mp} = Q \cap |\Lambda_{\pm}|$$

and (2) follows. \square

8.3. Description of the action of g on the reduced pieces.

8.3.1. *The Nielsen-Thurston case.* Suppose U is a component of $L \setminus |\mathfrak{S}|$ in the nucleus of a principal region. Then \bar{U} is compact and $(|\Lambda_+| \cup |\Lambda_-|) \cap U = \emptyset$. By Lemma 8.5, $\bar{i} : Q \rightarrow \bar{U}$ is a homeomorphism so Q can be identified with \bar{U} . By Corollary 8.3, g permutes the components U of $L \setminus |\mathfrak{S}|$. Since there are only finitely many such U , there exists a least integer m so that $g^m(Q) = Q$. We have,

Proposition 8.9. *Suppose Q is a compact reduced piece lying in a principal region and $g^m(Q) = Q$. Then the dynamics of the homeomorphism $g^m : Q \rightarrow h^m(Q)$ is given by Nielsen-Thurston theory.*

8.3.2. *The trivial case.* Suppose $U \subset \mathcal{U}$ is a component of $L \setminus |\mathfrak{S}|$ with \bar{U} compact. By Lemma 8.5, $\bar{i} : Q \rightarrow \bar{U}$ is a homeomorphism so Q can be identified with \bar{U} . By Corollary 8.3, g permutes the components U of $L \setminus |\mathfrak{S}|$. Since the sets $g^n(\bar{U})$, $n \in \mathbb{Z}$, are disjoint, the dynamics of g on Q is trivial in this case.

Proposition 8.10. *Suppose Q is a compact reduced piece lying in \mathcal{U} . Then for all nonzero $n \in \mathbb{Z}$ the map $g^n : Q \rightarrow g^n(Q)$ is a homeomorphism between disjoint compact sets.*

8.3.3. *The translation case.* Suppose U is a component of $L \setminus |\mathfrak{S}|$ with \bar{U} not compact that does not meet $|\Lambda_+| \cup |\Lambda_-|$. Then $U \subset \mathcal{U}$.

By Lemma 8.5, $\bar{i} : Q \rightarrow \bar{U}$ is a homeomorphism so Q can be identified with \bar{U} . By Corollary 8.3, g permutes the components U of $L \setminus |\mathfrak{S}|$. Since the number of noncompact reduced pieces is finite, it follows that there exists a least integer $m > 0$ so that $g^m(Q) = Q$. By Lemma 8.8 (2), $\Lambda_{\pm} \cap Q$ are the Handel-Miller laminations for the endperiodic automorphism $g^m : Q \rightarrow Q$. Since $\Lambda_{\pm} \cap Q = \emptyset$, by Lemma 4.76, g^m is isotopic to a translation. We have,

Proposition 8.11. *If Q is a noncompact reduced piece which does not meet $|\Lambda_+| \cup |\Lambda_-|$ and $g^m(Q) = Q$, then $g^m : Q \rightarrow Q$ is isotopic to a translation.*

8.3.4. *The pseudo-anosov case.* Suppose U is a components of $L \setminus |\mathfrak{S}|$ with noncompact closure meeting $|\Lambda_+| \cup |\Lambda_-|$. In this case the map $\bar{i} : Q \rightarrow \bar{U}$ is not always one-one but rather sometimes identifies border components of $Q = \bar{U}$. In addition, \bar{U} may share border components in \mathfrak{S} with other components of $L \setminus |\mathfrak{S}|$.

Since the leaves of the laminations Λ_{\pm} do not intersect the reducing curves in \mathfrak{S} we have,

Lemma 8.12. *For a component U of $L \setminus |\mathfrak{S}|$, the following two statements are equivalent,*

- (1) U meets $|\Lambda_+| \cup |\Lambda_-|$;
- (2) U meets both $|\Lambda_+|$ and $|\Lambda_-|$. If $x \in U$ and $x \in \lambda \in \Lambda_+ \cup \Lambda_-$, then $\lambda \subset U$.

Definition 8.13 (pseudo-anosov). An endperiodic automorphism $h : L \rightarrow L$ is a *pseudo-anosov automorphism* if,

- (1) h is not a translation;
- (2) L contains no reducing curves;
- (3) h preserves the Handel-Miller bilamination (Λ_-, Λ_+) .

Remark. Recall that by Lemma 4.76, an endperiodic automorphism is a translation if and only if $\Lambda_- = \emptyset = \Lambda_+$.

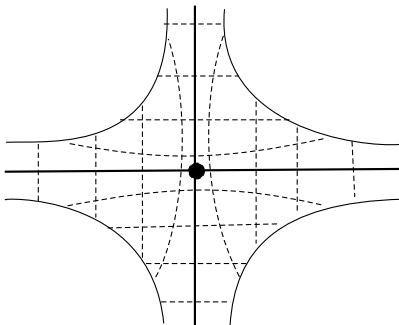


FIGURE 20. A pseudo-anosov component that is not admissible

Remark. We use the lower case “a” because the analogy with the pseudo-Anosov components for automorphisms of compact surfaces is weak. Also, some of these reduced pieces might not even be admissible, although they will be standard. A pseudo-anosov reduced piece as in Figure 20 occurs, for instance, in Figure 1. Here, because it inherits a complete hyperbolic metric from L , the simple ends are cusps. The laminations are still the locally uniform limits of the geodesic junctures. This endperiodic automorphism occurs as the monodromy of the depth one leaf in Gabai’s “stack of chairs” foliation of a sutured solid torus ([27], but also cf. [5, Section 11.1]).

By Corollary 8.3, g permutes the components U of $L \setminus |\mathfrak{G}|$. Since the number of noncompact reduced pieces is finite, it follows that if there exists a least integer $m > 0$ so that $g^m(Q) = Q$. We have,

Proposition 8.14. *If Q is a noncompact reduced piece which meets $|\Lambda_+| \cup |\Lambda_-|$ and $g^m(Q) = Q$, then $g^m : Q \rightarrow Q$ is a pseudo-anosov automorphism.*

9. DYNAMICS IN THE CORE

In the Nielsen-Thurston theory, the pseudo-Anosov homeomorphism is semi-conjugate to a two-ended Markov shift of finite type. The analogous result here concerns the core dynamical system.

Definition 9.1 (core dynamical system). Let $f : L \rightarrow L$ be an endperiodic automorphism and h an endperiodic automorphism isotopic to f and preserving the bilamination (Λ_+, Λ_-) . Then the restriction of h to $\mathcal{K} = |\Lambda_+| \cap |\Lambda_-|$ defines the core dynamical system $h : \mathcal{K} \rightarrow \mathcal{K}$.

The main theorem of this section is,

Theorem 9.2. *The core dynamical system $h : \mathcal{K} \rightarrow \mathcal{K}$ is topologically conjugate to a two-ended Markov shift of finite type.*

We allow $\partial L \neq \emptyset$ and we do not reduce. Theorem 9.2 will follow immediately from Proposition 9.12.

We fix a core K . The pre-Markov and Markov rectangles $R \subset K$ we consider below will have a pair of opposite edges α_R, β_R that are subarcs of Λ_+ (called the positive or vertical edges) and a pair of opposite edges δ_R, γ_R that are subarcs of Λ_- (called the negative or horizontal edges). As remarked earlier (on page 49), we are using the term “rectangle” as it is used in the literature on Markov partitions. Our rectangles will be convex geodesic quadrilaterals, not geometric rectangles, as the latter are impossible in hyperbolic geometry.

Remark. We allow degenerate rectangles. If $\alpha_R = \beta_R$, then $R = \alpha_R$ degenerates to an arc in Λ_+ and, if $\delta_R = \gamma_R$, $R = \delta_R$ degenerates to an arc in Λ_- . These degenerate possibilities should be kept in mind in what follows.

Definition 9.3 (\mathcal{Q}^+). Let $Q^\dagger \subset K$ be the *extreme rectangle* (Definition 6.2) with two edges $\alpha_{Q^\dagger}, \beta_{Q^\dagger}$ that are extreme arcs of $\Lambda_+ \cap K$ in their isotopy class. The other two edges $\delta_{Q^\dagger}, \gamma_{Q^\dagger}$ of Q^\dagger are arcs in positive junctures in $\partial_+ K$. Let $Q \subset Q^\dagger$ be the largest rectangular subset of Q^\dagger with two edges subsets of $\alpha_{Q^\dagger}, \beta_{Q^\dagger} \subset |\Lambda_+|$ and the other two edges subsets of leaves of Λ_- . The finite set of all such rectangles will be denoted by \mathcal{Q}^+ .

Remark. The rectangle $Q \subset Q^\dagger$ has as positive edges subarcs of the positive edges of Q^\dagger . Its negative edges are arcs of $\Lambda_- \cap Q^\dagger$. These arcs may be identical, in which case Q degenerates to an arc in Λ_- . Conceivably, $\Lambda_- \cap Q^\dagger = \emptyset$, hence $Q = \emptyset$. In this case Q^\dagger lies in the positive escaping set \mathcal{U}_+ and contributes nothing to the dynamical system $h : \mathcal{K} \rightarrow \mathcal{K}$.

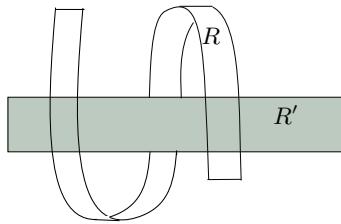
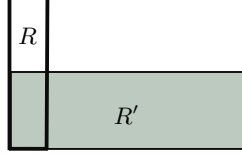


FIGURE 21. R completely crosses R' twice

Definition 9.4 (completely crosses). Let R and R' be nondegenerate rectangles with geodesic sides. Then R completely crosses R' in the positive direction if each component (if any) of $R \cap R'$ is a nondegenerate rectangle with horizontal boundary components subarcs of $|\Lambda_-| \cap R'$ and vertical boundary components subarcs of $|\Lambda_+| \cap R$. If R and/or R' degenerates, the definition is modified in the obvious way. We say that R completely crosses R' in the negative direction precisely if R' completely crosses R in the positive direction.

This definition allows R to cross R' multiple times (see Figure 21) or never. It also allows the situation pictured in Figure 22.

FIGURE 22. R completely crosses R' with overlapping edges

Proposition 9.5. *If $Q \in \mathcal{Q}^+$, then $h(Q)$ completely crosses (in the positive direction) any rectangle $Q' \in \mathcal{Q}^+$ that it meets.*

Proof. Let $Q \subset Q^\dagger$ and $Q' \subset (Q')^\dagger$ as in Definition 9.3. Suppose $h(Q)$ meets Q' . Since the vertical edges of $(Q')^\dagger$ are extremals of their isotopy class in $\Lambda_+ \cap K$ and the horizontal edges of $h(Q^\dagger)$ lie in positive junctures disjoint from K , we see that $h(Q^\dagger)$ completely crosses $(Q')^\dagger$, hence completely crosses Q' . If $h(Q)$ does not completely cross Q' then, since it meets Q' , at least one horizontal edge of $h(Q)$, say $\delta_{h(Q)} = h(\delta_Q)$, lies in $Q' \setminus \delta_{Q'}$. By applying h^{-1} , we see that this contradicts the fact that Q is the largest rectangular subset of Q^\dagger with pairs of opposite edges in Λ_+ and Λ_- respectively. \square

Definition 9.6 (Markov family, pre-Markov family). A finite family $\{R_1, \dots, R_n\}$ of disjoint rectangles is called a *Markov family* (in the literature frequently called a *Markov partition*) if it satisfies Properties I – IV below,

- I. One pair of opposite edges of each R_i is contained in $|\Lambda_+|$ and the other pair of opposite edges of each R_i is contained in $|\Lambda_-|$.
- II. $R_i \cap R_j = \emptyset$ if $i \neq j$.
- III. $h(R_i)$ completely crosses each R_j .
- IV. $h(R_i) \cap R_j$ has at most one component.
- V. For each bi-infinite sequence,

$$(\dots, i_{-k}, \dots, i_{-1}, i_0, i_1, \dots, i_k, \dots) \in \{1, 2, \dots, n\}^{\mathbb{Z}}$$

the intersection, $\bigcap_{k=-\infty}^{\infty} h^k(R_{i_k})$, is either empty or exactly one point.

If the family of rectangles satisfies Properties I – III, we call it a *pre-Markov family* (or partition).

Example 9.7. We will show that the family

$$\mathcal{M}^+ = \{\text{components of } h(Q) \cap Q' \mid Q, Q' \in \mathcal{Q}^+\} = \{R_1^+, \dots, R_n^+\}.$$

is a Markov family. We refer to \mathcal{M}^+ as the family of *positive Markov rectangles*. The family \mathcal{Q}^+ satisfying Proposition 9.5 is a pre-Markov family.

Remark. In the literature, a pre-Markov partition is sometimes called a Markov partition.

Remark. There are infinitely many different possible choices for the family of Markov rectangles. The family \mathcal{M}^+ satisfies Property V if and only if there are

no principal regions. Property V is required of a family used to prove Proposition 9.12 and Theorem 9.2. Below we will give a Markov family \mathcal{M} that does satisfy Property V.

Remark. Often a Markov family is allowed to have contiguous elements as in Figure 23. Our families do not have contiguous elements. This is because the rectangles Q^\dagger that we started with could not be contiguous.

Remark. It is not standard to allow degenerate rectangles in Markov families, but there is no real problem in doing so. In our situation, Markov rectangles that degenerate to arcs are forced, for example, if there are principal regions with isolated border leaves. It is possible that an isolated leaf of either lamination might contribute such a degenerate rectangle and that the components of $h(Q) \cap Q'$ might degenerate to points. In general, we allow Markov families to have some rectangles that degenerate to a point. The reader should keep such possible degeneracies in mind throughout the following discussion.

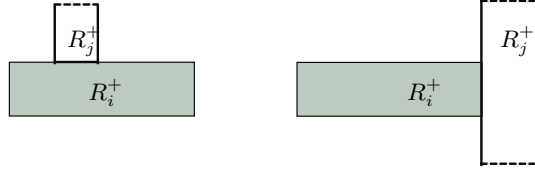


FIGURE 23. Contiguous rectangles

Corollary 9.8. *If $R_i^+ \in \mathcal{M}^+$ is a component of $h(Q) \cap Q'$, then R_i^+ completely crosses Q' in the positive direction.*

Proposition 9.9. *The family \mathcal{M}^+ has the property that $h(R_i^+) \cap R_j^+$ is either empty or $h(R_i^+)$ completely crosses R_j^+ in the positive direction and their intersection has a single component, $1 \leq i, j \leq n$. Consequently, \mathcal{M}^+ is a Markov family.*

Proof. Suppose that R_i^+ is a component of $h(Q) \cap Q'$ with $Q, Q' \in \mathcal{Q}^+$ and suppose $R_j^+ \in \mathcal{M}^+$. Since the sets in \mathcal{Q}^+ are disjoint, if $h(R_i^+) \subset h(Q')$ meets $R_j^+ \in \mathcal{M}^+$, it follows from the definition of \mathcal{M}^+ that R_j^+ is a component of $h(Q'') \cap Q''$ some $Q'' \in \mathcal{Q}^+$. By Corollary 9.8, R_i^+ completely crosses Q' in the positive sense so $h(R_i^+)$ completely crosses R_j^+ exactly once in the positive sense. “Exactly once” is due to the fact that R_j^+ is not Q'' , but a component of $h(Q'') \cap Q''$ and also $h(R_i^+) \subset h(Q')$. \square

Let $\{R_1, R_2, \dots, R_n\}$ be a Markov family. One sets up an $n \times n$ Markov matrix A with entries

$$A_{ij} = \begin{cases} 1, & \text{if } R_i \cap h(R_j) \neq \emptyset, \\ 0, & \text{if } R_i \cap h(R_j) = \emptyset. \end{cases}$$

The corresponding set \mathcal{S} of *symbols* consists of all bi-infinite sequences

$$\iota = (\dots, i_{-k}, \dots, i_{-1}, i_0, i_1, \dots, i_k, \dots) \in \{1, 2, \dots, n\}^{\mathbb{Z}}$$

such that $A_{i_k, i_{k+1}} = 1$, $-\infty < k < \infty$.

Remark. There are always periodic symbols in \mathcal{S} , usually a countable infinity of them.

For each bi-infinite sequence in the symbol set \mathcal{S} for \mathcal{M}^+ (and only for such),

$$\dots \cap h^{-k}(R_{i_{-k}}^+) \cap \dots \cap h^{-1}(R_{i_{-1}}^+) \cap R_{i_0}^+ \cap h(R_{i_1}^+) \cap \dots \cap h^k(R_{i_k}^+) \cap \dots = \zeta_\iota \neq \emptyset.$$

We would like to have that ζ_ι ranges exactly over \mathcal{K} as ι ranges over \mathcal{S} , in which case the right-shift operator $\sigma : \mathcal{S} \rightarrow \mathcal{S}$ will be exactly conjugate to $h : \mathcal{K} \rightarrow \mathcal{K}$. This will be true if there are no principal regions, but generally many ζ_ι may be whole arcs of intersection of Λ_+ with arms of negative principal regions. We will leave it to the reader to see this, remarking only that it is due to a basic asymmetry in the definition of \mathcal{M}^+ which favors the role of Λ_+ . In studying particular examples, and even for the applications of symbolic dynamics in [12, 16], it is always adequate to use \mathcal{M}^+ . The following discussion, which does not explicitly mention the principal regions, is motivated partly by aesthetics, but is also useful.

Example 9.10. We first define the negative pre-Markov rectangles. Let $Q^\dagger \subset K$ be the *extreme rectangle* (Definition 6.2) with two edges $\delta_{Q^\dagger}, \gamma_{Q^\dagger}$ that are extreme arcs of $\Lambda_- \cap K$ in their isotopy class. The other two edges $\alpha_{Q^\dagger}, \beta_{Q^\dagger}$ of Q^\dagger are arcs in negative junctures in ∂K . Let $Q \subset Q^\dagger$ be the largest rectangular subset of Q^\dagger with two edges subsets of $\delta_{Q^\dagger}, \gamma_{Q^\dagger} \subset |\Lambda_-|$ and the other two edges subsets of leaves of Λ_+ . The finite set of all such rectangles will be denoted by \mathcal{Q}^- . Define

$$\mathcal{M}^- = \{\text{components of } h^{-1}(Q) \cap Q' \mid Q, Q' \in \mathcal{Q}^-\} = \{R_1^-, \dots, R_m^-\}.$$

In analogy with \mathcal{M}^+ , this is shown to be a Markov family.

Remark. If there are no principal regions one expects \mathcal{M}^- to be closely related to \mathcal{M}^+ . For example, in Examples 4.13 and 4.35, if one takes the core K to be as in Figure 12, then $\mathcal{M}^+ = h(\mathcal{M}^-)$.

Example 9.11. The symmetric family of *Markov rectangles* is given by,

$$\mathcal{M} = \{\text{components of } R_i^+ \cap R_j^- \mid R_i^+ \in \mathcal{M}^+, R_j^- \in \mathcal{M}^-\} = \{R_1, \dots, R_q\}$$

The top and bottom edges of R_ℓ will be arcs in Λ_- and the left and right edges will be arcs in Λ_+ . The family \mathcal{M} (again Markov) satisfies Property V and will be used to prove Theorem 9.2.

It should be remarked that some of these new Markov rectangles might degenerate to a point, a component of intersection of a degenerate rectangle R_i^+ with a degenerate R_j^- . This finite set of points will be permuted by h . There may also be arcs from one or another of the laminations among these rectangles.

As we will see, the symbol set for this Markov family will exactly encode \mathcal{K} and provide the desired conjugacy of $h|_{\mathcal{K}}$ to the resulting shift map.

We will leave it to the reader to check that \mathcal{M} is again a Markov family for h .

Remark. Our situation in which distinct Markov rectangles cannot be contiguous is a bit stronger than the usual requirement that they merely not overlap. This eliminates the usual ambiguity in the coding, insuring that $\sigma : \mathcal{S} \rightarrow \mathcal{S}$ is conjugate to $h : \mathcal{K} \rightarrow \mathcal{K}$ and not merely semi-conjugate.

We are ready to prove the key result.

Proposition 9.12. *If $\iota = (\dots, i_{-k}, \dots, i_{-1}, i_0, i_1, \dots, i_k, \dots) \in \{1, 2, \dots, q\}^{\mathbb{Z}}$ is a symbol for \mathcal{M} , then the infinite intersection*

$$I_\iota^+ = R_{i_0} \cap h(R_{i_1}) \cap h^2(R_{i_2}) \cap \dots \cap h^k(R_{i_k}) \cap \dots$$

is an arc of $\Lambda_+ \cap R_{i_0}$ and

$$I_\iota^- = R_{i_0} \cap h^{-1}(R_{i_{-1}}) \cap h^{-2}(R_{i_{-2}}) \cap \dots \cap h^{-k}(R_{i_{-k}}) \cap \dots$$

is an arc of $\Lambda_- \cap R_{i_0}$. Furthermore, all such arcs are obtained in this way. Consequently, $\zeta_\iota = I_\iota^- \cap I_\iota^+ \in \mathcal{K}$ and every point of \mathcal{K} is of the form ζ_ι for a unique symbol ι .

Proof. One easily sees that \mathcal{M}^+ and \mathcal{M}^- each covers \mathcal{K} , hence so does \mathcal{M} . Consequently, the assertions about I_ι^+ and I_ι^- imply the assertion about ζ_ι . In fact, $\zeta_\iota = I_\iota^- \cap I_\iota^+$ will be one point as the intersection of two arcs, one in a leaf $\lambda_- \in \Lambda_-$ joining a pair of opposite edges of R_{i_0} and the other in a leaf $\lambda_+ \in \Lambda_+$ joining the other pair of opposite edges of R_{i_0} . As remarked above, the fact that each point of \mathcal{K} uniquely determines its symbol is due to the fact that our Markov rectangles are disjoint. Thus, it remains to prove the assertions about I_ι^+ and I_ι^- .

If I_ι^+ is not as asserted, it must be a rectangle with nonempty interior. Assume this and deduce a contradiction as follows. By the construction of \mathcal{M} , the sides of R_{i_k} in Λ_+ extend to the sides in Λ_+ of a rectangle $C_{i_k} \subset K$, the other two sides of which are arcs δ_k, γ_k in positive junctures in ∂K . Consider the set

$$P_k = h^k(C_{i_k}) \cap h^{k+1}(C_{i_{k+1}}) \cap \dots$$

By our hypothesis, P_k is a nondegenerate rectangle with two sides δ'_k, γ'_k subarcs of $h^k(\delta_k)$ and $h^k(\gamma_k)$, respectively. Thus δ'_k, γ'_k are subarcs of positive junctures in fundamental neighborhoods of positive ends. Furthermore,

$$P_0 \subset P_1 \subset \dots \subset P_k \subset \dots$$

and the edges in Λ_+ of each P_k are subarcs of the corresponding sides of P_{k+1} , $0 \leq k < \infty$. The increasing union of these rectangles is an infinite strip P bounded by *distinct* leaves $\lambda, \mu \in \Lambda_+$. Any lift of this strip to the universal cover is a strip with distinct boundary components $\tilde{\lambda}, \tilde{\mu} \in \tilde{\Lambda}_+$ covering $\lambda, \mu \in \Lambda_+$ and limiting on two pairs $\{x, y\}, \{z, w\} \subset S_\infty^1$. The notation is chosen so that the lifts $\tilde{\delta}'_k$ lying in this strip have endpoints x_k, y_k on $\tilde{\lambda}$ and $\tilde{\mu}$, respectively, and $\{x_k\}_{k \geq 0}$ converges to x and $\{y_k\}_{k \geq 0}$ to y . If $x \neq y$, then $\{\tilde{\delta}'_k\}_{k \geq 0}$ converges to the geodesic in \tilde{L} with endpoints x and y . Consequently the sequence $\{\delta'_k\}_{k \geq 0}$ accumulates locally uniformly on a geodesic in L . But by Theorem 4.24, this sequence escapes, hence $x = y$. Similarly, $z = w$, hence $\tilde{\lambda} = \tilde{\mu}$. This is the desired contradiction.

The assertion about I_ι^- is proven in the same way. Finally, the fact that \mathcal{M} covers \mathcal{K} implies that the union of all I_ι^+ 's also covers \mathcal{K} , as does the union of all I_ι^- 's. Thus, all arcs of $\Lambda_+ \cap R_{i_0}$ and all arcs of $\Lambda_- \cap R_{i_0}$ are obtained as asserted, $i_0 = 1, 2, \dots, q$. \square

Remark. It is possible to use the pre-Markov family $\mathcal{Q}^+ = \{Q_1, Q_2, \dots, Q_p\}$ of rectangles to produce projectively invariant measures for h on Λ_+ (and Λ_-) much as in the case of pseudo-Anosov automorphisms of compact surfaces. The following sketch follows the lead of [19, pages 95-102] and we refer the reader there for more details.

Let $B = (B_{k,j})$ be the incidence matrix, where $B_{k,j}$ is the number of components of $h(Q_k) \cap Q_j$. By the Brouwer fixed point theorem, this matrix has an eigenvector $y \neq \mathbf{0}$ with all entries nonnegative and with eigenvalue $\kappa \geq 1$. In the (typical) case that $\kappa > 1$, one obtains a transverse, projectively invariant measure μ_+ for h on Λ_+ with projective constant κ .

Now Ω^+ is also pre-Markov for h^{-1} with intersection matrix the transpose B^T and (left) eigenvector y^T . This gives a transverse, projectively invariant measure μ_- for h^{-1} on Λ_- with projective constant κ . Viewed as a projectively invariant measure for h , it has projective constant $\kappa^{-1} < 1$.

Since the eigenvectors y and y^T may have some zero entries, these measures will not generally have full support. See [24] for a simple example. If, however, Λ_+ is a minimal h -invariant lamination, the measures will evidently have full support.

10. PSEUDO-GEODESIC LAMINATIONS AND THE ISOTOPY THEOREM

The theory developed so far is for geodesic laminations, but for applications to foliation theory this is much too restrictive. In this section we give an axiomatic approach to endperiodic automorphisms using pseudo-geodesic laminations.

10.1. The axioms. We fix a choice of standard hyperbolic metric g on the admissible surface L .

Lemma 10.1. *Any two standard hyperbolic metrics on L have the same pseudo-geodesics.*

Proof. Assume first that $\partial L = \emptyset$ and let g and g' be two standard hyperbolic metrics on L . Let $\pi_0 : \Delta \rightarrow L$ and $\pi_1 : \Delta \rightarrow L$ be the universal covers corresponding to these respective metrics. It is well known that there is a commutative diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\varphi} & \Delta \\ \pi_0 \downarrow & & \downarrow \pi_1 \\ L & \xrightarrow{\text{id}} & L \end{array}$$

where φ is a homeomorphism completely determined by its value on a single point $x_0 \in \Delta$. Since the hyperbolic metrics are standard, as in the proof of [15, Theorem 2], φ extends canonically to a homeomorphism $\widehat{\varphi} : D^2 \rightarrow D^2$ which clearly takes geodesics to pseudo-geodesics.

If $\partial L \neq \emptyset$, the proof is similar. □

Remark. Since L has a standard hyperbolic metric, [15, Corollary 4] implies that an arbitrary homeomorphism $\varphi : L \rightarrow L$ takes pseudo-geodesics to pseudo-geodesics.

Recall from Definition 4.19 that, if γ is a pseudo-geodesic, then γ^g denotes its *geodesic tightening*, the unique geodesic the lifts of which have the same ideal endpoints as the corresponding lifts of γ . Note that $\widetilde{\gamma}^g \subset \widetilde{L}$ since ∂L , hence $\partial \widetilde{L}$, is geodesic. If we allow the metric g to vary, we will refer to the g -geodesic tightening.

Beginning with an endperiodic automorphism $f : L \rightarrow L$, we want to define an associated pair of pseudo-geodesic laminations satisfying four axioms.

The notations Λ_{\pm} and \mathcal{J}_{\pm} , which have been used up to now for the geodesic Handel-Miller laminations and the geodesic juncture components, will from now on

be used for the pseudo-geodesic laminations and juncture components given in the axioms.

Axiom 1. (Λ_+, Λ_-) is a bilamination, Λ_+ and Λ_- being closed, pseudo-geodesic laminations with all leaves disjoint from ∂L .

Hereafter, for the sake of economy, every assertion about “the lamination Λ_\pm ” is really two assertions, one about Λ_+ and one about Λ_- .

Axiom 2. A leaf of $\tilde{\Lambda}_\pm$ can meet a leaf of $\tilde{\Lambda}_\mp$ in at most one point.

Equivalently, the leaves of Λ_+ cannot intersect the leaves of Λ_- so as to form digons.

Definition 10.2 (endpoint correspondence property). The pseudo-geodesic lamination Λ_\pm has the endpoint correspondence property with respect to f if the correspondence $\lambda \mapsto \lambda^g$ sends Λ_\pm one-one onto the positive/negative geodesic Handel-Miller laminations associated to f (Definition 4.30).

When the context makes it clear, we sometimes shorten the language, saying that Λ_\pm has the endpoint correspondence property. We also say that the bilamination has the endpoint correspondence property (with respect to f).

The final two axioms will tie (Λ_+, Λ_-) to the endperiodic automorphism f .

Axiom 3. The bilamination (Λ_+, Λ_-) has the endpoint correspondence property with respect to f .

Notation. If X is a set of pseudo-geodesics, we use the notation

$$X^g = \{\gamma^g \mid \gamma \in X\}.$$

Remark. By Axiom 3, $(\Lambda_+^g, \Lambda_-^g)$ are the Handel-Miller geodesic bilamination associated to f .

The next two lemmas follow immediately from Axiom 3 since Λ_\pm^g has the same properties.

Lemma 10.3. *Every leaf of Λ_\pm meets at least one leaf of Λ_\mp .*

Lemma 10.4. *Each leaf of the lifted lamination $\tilde{\Lambda}_\pm$ of \tilde{L} is determined by its endpoints in S_∞^1 .*

Fix a set of f -junctures and corresponding set \mathcal{N} of f -juncture components (Definition 4.17). We define a set of pseudo-geodesic junctures and corresponding set \mathcal{J} of juncture components in analogy to the sets of geodesic junctures and juncture components we introduced in Definition 4.21.

Definition 10.5 (junctures, \mathcal{J} , \mathcal{J}_+ , \mathcal{J}_- , ι). A set \mathcal{J} of juncture components associated to a set \mathcal{N} of f -juncture components is a set of compact, properly embedded arcs and circles in L such that

- (1) there is a bijection $\iota : \mathcal{N} \rightarrow \mathcal{J}$;
- (2) for each $\gamma \in \mathcal{N}$, $\iota(\gamma)$ is homotopic to γ with endpoints (if any) fixed;
- (3) if $\gamma \neq \gamma'$ are both in \mathcal{N}_+ or both in \mathcal{N}_- , then $\iota(\gamma)$ is disjoint from $\iota(\gamma')$.

The set $\mathcal{J}_+ = \iota(\mathcal{N}_+)$ is called the set of *positive juncture components*. The set $\mathcal{J}_- = \iota(\mathcal{N}_-)$ is called the set of *negative juncture components*. Further the map ι extends in a natural way to the set of fixed f -junctures to define a fixed set of *junctures* of the form $J = \iota(N)$ where N is one of the fixed f -junctures.

The map $\iota : \mathcal{N} \rightarrow \mathcal{J}$ serves the role of the geodesic tightening map of Definition 4.20 and can be extended to a map $\iota : \mathcal{N}^\dagger \rightarrow \mathcal{J}$ where \mathcal{N}^\dagger is the set of pseudo-geodesics γ such that there exists an f -junction component $\sigma \in \mathcal{N}$ whose lifts have the same endpoints on S_∞ as the lifts of γ . Then define $\iota(\gamma) = \iota(\sigma)$. This definition then extends in a natural way to a function ι with domain the finite unions of elements of \mathcal{N}^\dagger . Then the last equality in the following definition makes sense. Compare this definition with Definition 4.22.

Remark. Notice that, by item (3) in Definition 10.5, the set of junctures has the juncture intersection property.

Definition 10.6 (J_n). Given a juncture $J = \iota(N)$, let $J_n = \iota(f^n(N)) = \iota(f^n(J))$, $n \in \mathbb{Z}$.

In what follows, we assume a choice of \mathcal{N} and \mathcal{J} , f -junction components and juncture components, and thus a choice of set of f -junctions and set of junctures. Axiom 4 will then claim the existence of choices with suitable properties.

Axiom 4. There is a choice of the families \mathcal{N}_\pm of positive/negative f -junction components and $\mathcal{J}_\pm = \iota(\mathcal{N}_\pm)$ of positive/negative juncture components such that:

- (1) $\Lambda_+ \cup \mathcal{J}_-$ and $\Lambda_- \cup \mathcal{J}_+$ are each sets of disjoint pseudo-geodesics;
- (2) Λ_+ is transverse to \mathcal{J}_+ and Λ_- is transverse to \mathcal{J}_- ;
- (3) no leaf of Λ_\pm can meet an element of \mathcal{J}_\pm so as to form digons.

This completes the list of axioms.

Definition 10.7 (Handel-Miller pseudo-geodesic bilamination associated to f). If (Λ_+, Λ_-) satisfies the four axioms, it will be called a Handel-Miller pseudo-geodesic bilamination associated to f . The individual laminations Λ_\pm will be called Handel-Miller pseudo-geodesic laminations associated to f .

Remark. The image of a Handel-Miller pseudo-geodesic bilamination associated to f under an isotopy of L will be a Handel-Miller pseudo-geodesic bilamination associated to f . Thus, a Handel-Miller pseudo-geodesic bilamination associated to f is not unique but, by Theorem 10.13, is unique up to isotopy.

Lemma 10.8. *The Handel-Miller pseudo-geodesic laminations associated to f are transversely totally disconnected* (Definition 4.4).

Proof. We will show that Λ_+ is transversely totally disconnected. The proof that Λ_- is transversely totally disconnected is similar. Suppose that $|\Lambda_+|$ has nonempty interior. Then $|\tilde{\Lambda}_+|$ has nonempty interior. Let $x \in \lambda_+ \in \tilde{\Lambda}_+$ have a neighborhood contained in $|\tilde{\Lambda}_+|$. Then there exists a leaf $\lambda_- \in \tilde{\Lambda}_-$ and a point $y \in \lambda_+ \cap \lambda_-$ such that y has a neighborhood $V \subset |\tilde{\Lambda}_+|$ which is an open disk. This follows since one can find a bilamination chart of the form $P \times (-\varepsilon, \varepsilon)$ with $x \in P \times \{0\}$ and $P \subset \lambda_+$ as long as desired (see Remark page 24) and since, by Lemma 10.3, every leaf in $\tilde{\Lambda}_+$ meets some leaf in $\tilde{\Lambda}_-$. Here, of necessity, $\varepsilon > 0$ will be small.

Thus, there exists an arc $[a, b] \subset \lambda_- \cap |\tilde{\Lambda}_+|$ with $y \in (a, b)$. Let $\lambda_a, \lambda_b \in \tilde{\Lambda}_+$ be such that $\lambda_a \cap \lambda_- = \{a\}$ and $\lambda_b \cap \lambda_- = \{b\}$. Consider the geodesics $\lambda_a^g, \lambda_b^g, \lambda_-^g$ in \tilde{L} with the same endpoints on S_∞^1 as $\lambda_a, \lambda_b, \lambda_-$ respectively. By Axiom 3, $\lambda_a^g, \lambda_b^g \in \tilde{\Lambda}_+^g$ and $\lambda_-^g \in \tilde{\Lambda}_-^g$. Let $\{a^g\} = \lambda_a^g \cap \lambda_-^g$, $\{b^g\} = \lambda_b^g \cap \lambda_-^g$, and $[a^g, b^g] \subset \lambda_-^g$. Since the semi-proper leaves are dense in Λ_+^g , there exists a pair of semi-proper leaves $\lambda_1^g, \lambda_2^g \in \tilde{\Lambda}_+^g$

meeting the interval $[a^g, b^g]$ at a pair of points x_1^g, x_2^g with $(x_1^g, x_2^g) \cap |\tilde{\Lambda}_+^g| = \emptyset$. Let $\lambda_1, \lambda_2 \in \tilde{\Lambda}_+$ have the same endpoints on S_∞^1 as λ_1^g, λ_2^g respectively (Axiom 3). By Axiom 3, if $x_i = \lambda_i \cap \lambda_-, i = 1, 2$, then $(x_1, x_2) \cap |\tilde{\Lambda}_+| = \emptyset$ which is a contradiction. Thus, $\text{int} |\Lambda_+| = \emptyset$. The lemma follows. \square

Corollary 10.9. *The union of the semi-isolated leaves of Λ_\pm is dense in $|\Lambda_\pm|$.*

Lemma 10.10. *No leaf $\lambda \in \Lambda_\pm$ is contained in a bounded region of L .*

Proof. Let $X_0 \subset X_1 \subset \dots \subset X_i \subset \dots$ be an exhaustion of L by compact subsets. Without loss we can assume $\text{Fr } X_i$ is a geodesic 1-manifold. By Lemma 4.46, λ^g meets $\text{Fr } X_i$, for i sufficiently large. Suppose α is a geodesic component of $\text{Fr } X_i$ with $\alpha \cap \lambda^g \neq \emptyset$. Let $\tilde{\alpha}$ and $\tilde{\lambda}^g$ be lifts of α and λ^g with $\tilde{\alpha} \cap \tilde{\lambda}^g \neq \emptyset$. If $\tilde{\lambda}$ is lift of λ sharing endpoints with $\tilde{\lambda}^g$, then $\tilde{\lambda} \cap \tilde{\alpha} \neq \emptyset$. Thus, $\lambda \cap \text{Fr } X_i \neq \emptyset$, for all i sufficiently large, and the lemma follows. \square

Corollary 10.11. *No neighborhood of any end of a leaf of Λ_\pm is contained in a bounded region of L .*

Proof. If a neighborhood of an end ε of a leaf of Λ_\pm is contained in a bounded region of L , then the asymptote of ε (see definition in the proof of Lemma 4.50) would contain a leaf of Λ_\pm in a bounded region of L which contradicts the lemma. \square

In particular, the leaves are noncompact and one-one immersions of \mathbb{R} in $\text{int } L$. The endpoints of the lifted leaves are ideal. That is they are in E .

Theorem 10.12. *The Handel-Miller geodesic bilamination $(\Lambda_+^g, \Lambda_-^g)$ associated to f satisfies the four axioms, where $\iota : \mathcal{N} \rightarrow \mathcal{J}^g$ is defined by $\iota(\gamma) = \gamma^g, \forall \gamma \in \mathcal{N}$.*

Proof. **Axiom 1.** By Proposition 4.28, $(\Lambda_+^g, \Lambda_-^g)$ are bilaminations. By construction they are closed. Clearly every geodesic is a pseudo-geodesic. If $x \in \partial L$, then x can not lie in either Λ_+^g or Λ_-^g because $\{h^n(x)\}_{n \in \mathbb{Z}}$ escapes if $x \in \partial L$ but not if $x \in \Lambda_\pm^g$.

Axiom 2. Since there cannot be geodesic digons in hyperbolic geometry, this axiom is immediate.

Axiom 3. This axiom is tautologically true.

Axiom 4. This follows from Proposition 4.28 and the fact that, in hyperbolic surfaces, there are no geodesic digons. \square

10.2. Uniqueness. An axiomatization needs to satisfy two conditions, consistency and completeness. The consistency of our axioms is given by Theorem 10.12. For completeness, one needs to show that the system defined by the axioms is unique up to a reasonable equivalence relation. In this section we will prove the following.

Theorem 10.13 (Uniqueness Theorem). *The Handel-Miller pseudo-geodesic bilamination associated to an endperiodic automorphism f is uniquely determined by f up to ambient isotopy.*

This is an immediate consequence of Theorem 10.15. The following is useful in applications.

Corollary 10.14. *The core dynamical system $h : \mathcal{K} \rightarrow \mathcal{K}$ is uniquely determined by the endperiodic automorphism f up to topological conjugacy $\psi \circ h | \mathcal{K} \circ \psi^{-1}$ by a homeomorphism $\psi : L \rightarrow L$ isotopic to the identity.*

Theorem 10.15 (Isotopy Theorem). *If (Λ_+, Λ_-) is a bilamination satisfying the axioms, then there is a homeomorphism $\psi : L \rightarrow L$, isotopic to the identity by an isotopy fixing ∂L pointwise, such that $\psi(\lambda) = \lambda^g$, for each $\lambda \in \Lambda_\pm$ and $(\psi(\Lambda_+), \psi(\Lambda_-))$ is the Handel-Miller geodesic bilamination.*

Corollary 10.16. *If $f : L \rightarrow L$ is an endperiodic automorphism, the geodesic bilamination $(\Lambda_+^g, \Lambda_-^g)$ associated to f is independent, up to ambient isotopy, of the choice of standard hyperbolic metric g .*

Proof. Let g and g' be two standard hyperbolic metrics on L and let $(\Lambda_+^g, \Lambda_-^g)$ be the Handel-Miller geodesic bilamination associated to f and corresponding to g , $(\Lambda_+^{g'}, \Lambda_-^{g'})$ the one corresponding to g' . Relative to the metric g , the lamination Λ_\pm^g is pseudo-geodesic (Lemma 10.1). The pair $(\Lambda_+^{g'}, \Lambda_-^{g'})$ forms a pseudo-geodesic bilamination which clearly satisfies Axioms 1, 2, and 4 as these are metric independent. By Corollary 4.72, we can assume we are using the same choice of set \mathcal{N} of f -junctures in Axiom 4 for both laminations Λ_\pm^g and $\Lambda_\pm^{g'}$.

It remains to prove Axiom 3. Assume that $\partial L = \emptyset$. The proof in the case $\partial L \neq \emptyset$ is similar. As in the proof of Lemma 10.1, the identity map $\text{id} : L \rightarrow L$ lifts to a map $\varphi : \Delta \rightarrow \Delta$ uniquely determined by its value at one point which extends to a map $\widehat{\varphi} : \mathbb{D}^2 \rightarrow \mathbb{D}^2$. Axiom 3 follows immediately since the endpoints on S_∞^1 of the leaves of $\widehat{\mathcal{N}}$ and of $\widehat{\varphi}^{-1}(\widehat{\mathcal{N}})$ coincide. \square

10.3. Proof of Theorem 10.15. We will prove Theorem 10.15 in a sequence of steps, noting that by Corollary 10.9, we only need to straighten the semi-isolated leaves. At each step we will modify the laminations Λ_\pm by an isotopy. That is we will replace the laminations Λ_\pm by $\varphi(\Lambda_\pm)$ where φ is a homeomorphism isotopic to the identity. We do not apply the homeomorphism φ to Λ_\pm^g which is to be thought of simply as the target of the whole process.

10.3.1. A preliminary isotopy to set up the tiling. We use the tiling \mathfrak{T}^g defined in Section 4.6.2.

Lemma 10.17. *Suppose the juncture $J_e = \text{Fr } V_e$ where V_e is a closed neighborhood of the end $e \in \mathcal{E}(L)$. Then there exists a homeomorphism ψ_e , isotopic to the identity, such that $\psi_e(\tau) = \tau^g$ for every juncture component $\tau \subset V_e$.*

Proof. Enumerate the juncture components in V_e as $\{\tau_i\}_{i=1}^\infty$. Let $\psi_0 = \text{id}$. Using Theorem 3.1, Theorem 3.2 and the remark after Theorem 3.3, inductively find sequences Φ_i of isotopies and ψ_i of homeomorphisms isotopic to the identity with $\psi_i = \Phi_i^1 \circ \psi_{i-1}$, $i \geq 1$, fixing ∂L and $\tau_1^g, \dots, \tau_{i-1}^g$ pointwise with Φ_i moving $\psi_{i-1}(\tau_i)$ to its geodesic tightening. That is, $\psi_i(\tau_i) = \Phi_i^1(\psi_{i-1}(\tau_i)) = \tau_i^g$. Then, by the remark after Theorem 3.3, the supports of at most finitely many Φ_i meet any compact set so that $\psi_i \rightarrow \psi_e$, a well defined homeomorphism isotopic to the identity and $\psi_e(\tau_i) = \psi_i(\tau_i) = \tau_i^g$, $i \geq 1$. \square

Corollary 10.18. *Suppose the juncture $J_e = \text{Fr } V_e$ is chosen, each $e \in \mathcal{E}(L)$, such that the V_e are disjoint. Then there exists a homeomorphism ψ , isotopic to the identity, such that $\psi(\tau) = \tau^g$ for every juncture component $\tau \subset \bigcup_{i=1}^k V_{e_i}$.*

Proof. Take ψ to be the composition of the ψ_e each $e \in \mathcal{E}(L)$. \square

Note that the homeomorphism ψ is such that $\psi(J_e) = J_e^g$ and $\psi(V_e) = U_e$ where $J_e^g = \text{Fr } U_e$, $e \in \mathcal{E}(L)$, and the U_e are disjoint. As usual $L = W^- \cup K \cup W^+$ where,

$$W^- = \bigcup_{e \in \mathcal{E}_-(L)} U_e \quad \text{and} \quad W^+ = \bigcup_{e \in \mathcal{E}_+(L)} U_e.$$

The set \mathcal{J}_W^g consists of all components of negative geodesic junctures in W^- and all components of positive geodesic junctures in W^+ (Definition 4.42). Cutting L apart along the geodesic juncture components in \mathcal{J}_W^g decomposes L into a set \mathfrak{T}^g of “tiles” (Definition 4.58).

Lemma 10.19. *The laminations $\psi(\Lambda_\pm)$ are transverse to the geodesic junctures in \mathcal{J}_W^g .*

Proof. This follows immediately from the fact that the laminations Λ_\pm were transverse to the junctures in \mathcal{J}_\pm . \square

We replace the laminations Λ_\pm by the laminations $\psi(\Lambda_\pm)$. This is our first step in straightening the laminations Λ_\pm to the laminations Λ_\pm^g .

10.3.2. *Preliminary isotopies in the core and the laminations \mathcal{G}_\pm^g .* As above, $\text{Fr}_+ K = \bigcup_{e \in \mathcal{E}_+(L)} J_e^g$ and $\text{Fr}_- K = \bigcup_{e \in \mathcal{E}_-(L)} J_e^g$.

Notation. Denote by

$$\begin{aligned} \mathcal{G}_\pm &= \Lambda_\pm | K \\ \mathcal{G}_\pm^g &= \Lambda_\pm^g | K, \end{aligned}$$

the laminations induced on K by Λ_\pm and Λ_\pm^g .

A leaf $\alpha = [x_1.x_2] \in \mathcal{G}_\pm$ is contained in a leaf $\gamma_\alpha \in \Lambda_\pm$ and has endpoints x_i in geodesic juncture components $\sigma_i \subset \text{Fr}_\pm K$, $i = 1, 2$. Let $\tilde{\alpha} = [\tilde{x}_1, \tilde{x}_2] \subset \tilde{\gamma}_\alpha$ be lifts of $\alpha = [x_1, x_2] \subset \gamma_\alpha$ with \tilde{x}_i in the lift $\tilde{\sigma}_i$ of σ_i , $i = 1, 2$. The geodesic tightening $\tilde{\gamma}_\alpha^g$ of γ_α has lift $\tilde{\gamma}_\alpha^g$ sharing endpoints on S_∞^1 with $\tilde{\gamma}_\alpha$. There is a unique arc $\tilde{\alpha}^g = [x_1^g, x_2^g] \subset \tilde{\gamma}_\alpha^g$ with $x_i^g \in \tilde{\sigma}_i$, $i = 1, 2$.

Notation. The covering projection of the arc $\tilde{\alpha}^g$ is in \mathcal{G}_\pm^g and will be denoted $\alpha^g = [x_1^g, x_2^g]$. The correspondence $\alpha \leftrightarrow \alpha^g$ induces a one-one correspondence between \mathcal{G}_\pm and \mathcal{G}_\pm^g .

Recall from Lemma 4.53 that each isotopy class of leaves of \mathcal{G}_\pm^g contains two extremal leaves (in the degenerate case where the isotopy class has one leaf we consider that leaf to be an extremal leaf). For reasons that will become obvious, we will refer to a leaf $\alpha \in \mathcal{G}_\pm$ as an extremal leaf if the corresponding geodesic segment $\alpha^g \in \mathcal{G}_\pm^g$ is an extremal leaf of \mathcal{G}_\pm^g .

By Lemma 4.52, the leaves of \mathcal{G}_\pm^g fall into finitely many isotopy classes. Thus, \mathcal{G}_\pm has finitely many extremal leaves.

Proposition 10.20. *There is a preliminary isotopy, supported arbitrarily near K and leaving that surface invariant, that straightens each extremal leaf $\alpha \in \mathcal{G}_\pm$ to its corresponding leaf $\alpha^g \in \mathcal{G}_\pm^g$.*

Proof. Each of the finitely many extremal leaves in \mathcal{G}_+ has its endpoints in two (not necessarily distinct) geodesic juncture components in $\text{Fr}_+ K$. An application

of Lemma 4.57 to each of these in turn will straighten each of them in turn as required.

Each of the finitely many extremal leaves in \mathcal{G}_- has its endpoints in two (not necessarily distinct) geodesic juncture components in $\text{Fr}_- K$ and will possibly be divided into finitely many segments by the geodesics containing the already straightened extremal leaves in \mathcal{G}_+ . Finitely many applications of Lemma 4.57 to each of these segments in turn will straighten each of the extremal leaves in \mathcal{G}_- in turn without moving the already straightened extremal leaves of \mathcal{G}_+ . \square

Thus, it can be assumed that each of the extremal leaves $\alpha \in \mathcal{G}_\pm$ is identical with its corresponding leaf $\alpha^g \in \mathcal{G}_\pm^g$. They each lie in the geodesic γ_α^g , the lifts of which have the same endpoints as the lifts of the leaf γ_α of Λ_\pm containing α . Note that at this stage it is not yet true that $\gamma_\alpha = \gamma_\alpha^g$, only that $\alpha = \alpha^g$.

Looking in the universal cover we see that,

Corollary 10.21. *Each leaf of \mathcal{G}_+ (respectively \mathcal{G}_-) lies in a rectangle R with a pair of opposite edges extremal leaves of \mathcal{G}_+ (respectively \mathcal{G}_-) and a pair of opposite edges contained in $\text{Fr}_+ K$ (respectively $\text{Fr}_- K$).*

Notation. The set of rectangles $R \subset K$ bounded by extremal leaves of \mathcal{G}_+ will be denoted by \mathcal{R}_+ . The analogous set of rectangles bounded by extremal leaves of \mathcal{G}_- will be denoted by \mathcal{R}_- . Let $\mathcal{R} = \mathcal{R}_+ \cup \mathcal{R}_-$.

Lemma 10.22. *If $R_+ \in \mathcal{R}_+$ and $R_- \in \mathcal{R}_-$, then the components of $R_+ \cap R_-$ are rectangles with one pair of opposite sides in opposite sides of R_+ and the other pair in opposite sides of R_- .*

Proof. This follows from Axiom 2 which implies that a side of R_+ and a side of R_- cannot intersect so as to form one or more digons. \square

If α is a leaf of \mathcal{G}_+ (respectively of \mathcal{G}_-) lying in $R \in \mathcal{R}$, then α connects two positive (respectively negative) juncture components lying in $\text{Fr}_+ K$ (respectively $\text{Fr}_- K$). Let $R \in \mathcal{R}$. The geodesic arcs $R \cap \text{Fr} K$, taken in either order, will be called the bottom and top edges of R and the extremal geodesic arcs will be called the left and right edges. Fix these choices.

Notice that $|\mathcal{R}|$ has both concave and convex corners. Of course a given $R_- \in \mathcal{R}_-$ can cross an $R_+ \in \mathcal{R}_+$ more than once. Since the extremals have already been straightened, both \mathcal{G}_\pm and \mathcal{G}_\pm^g can be viewed as laminations of $|\mathcal{R}|$. If some of the rectangles R degenerate to arcs, this language is a bit unorthodox, but harmlessly so.

Notation. Denote by $\partial_* \mathcal{R}$ the closure of $\text{Fr} \mathcal{R} \setminus \text{Fr} K$.

In other words, $\partial_* \mathcal{R}$ is the part of $\partial \mathcal{R}$ lying in $|\Lambda_+| \cup |\Lambda_-|$.

The leaves of \mathcal{G}_+ are properly embedded arcs having both endpoints in $\text{Fr}_+ K$. All isotopies of these leaves are to fix $\partial_L K \cup \text{Fr}_- K$ pointwise where $\partial_L K = K \cap \partial L$. As already indicated, the isotopies leave $\text{Fr}_+ K$ componentwise invariant, but not pointwise. Similar remarks apply to \mathcal{G}_- .

The leaves of \mathcal{G}_\pm fall into finitely many isotopy classes. There are extremal leaves in each isotopy class which, together with arcs in $\text{Fr}_\pm K$ cut off rectangles containing all the leaves in that isotopy class. If the isotopy class has only one leaf, we consider that leaf as a degenerate rectangle.

10.3.3. *Straightening all leaves of \mathcal{G}_\pm .* This is the delicate part of the argument.

By Corollary 10.9, we only need to straighten the semi-isolated leaves of Λ_\pm . In this subsection, we will construct a sequence $\{\Phi_i\}_{i=1}^\infty$ of ambient isotopies which progressively “locally straighten” each leaf $\alpha \in \mathcal{G}_\pm$ which is a segment of a semi-isolated leaf $\lambda \in \Lambda_+ \cup \Lambda_-$ to the corresponding geodesic segment $\alpha^g \in \mathcal{G}_\pm^g$, taking care that once a leaf segment has been straightened by Φ_i , it remains invariant under Φ_j , $j > i$.

We will need to prove that the infinite composition $\psi = \cdots \circ \Phi_i^1 \circ \cdots \circ \Phi_2^1 \circ \Phi_1^1$ is a well defined homeomorphism, isotopic to the identity. This is a delicate point since we cannot ensure that the supports of these isotopies form a locally finite family of sets. But then, since the semi-isolated leaves have been straightened, the homeomorphism ψ performs as advertised.

Throughout this process the homeomorphism $\psi_i = \Phi_i^1 \circ \cdots \circ \Phi_2^1 \circ \Phi_1^1$ distorts the laminations Λ_\pm to laminations Λ_\pm^i .

The goal of this section is to construct an isotopy straightening all the leaves of \mathcal{G}_\pm . Note that $|\mathcal{G}_\pm| \subset |\mathcal{R}|$. Recall that there is a canonical correspondence between the leaves α of \mathcal{G}_\pm and leaves α^g of \mathcal{G}_\pm^g . We will progressively “tighten” a countable dense set of leaves $\alpha \in \mathcal{G}_\pm$ one at a time to their corresponding geodesic leaves $\alpha^g \in \mathcal{G}_\pm^g$ by ambient isotopies compactly supported in a neighborhood of K . Already tightened leaves of \mathcal{G}_\pm and components of $\partial_L K \cup \text{Fr}_\mp K$ are pointwise fixed under the ambient isotopy where $\partial_L K = K \cap \partial L$. The components of $\text{Fr}_\pm K$ and already tightened leaves of \mathcal{G}_\mp will be invariant but generally not fixed pointwise by the isotopy.

Lemma 10.23. *There are mutually transverse C^0 foliations \mathcal{F}_\pm of $|\mathcal{R}|$ such that $\mathcal{G}_+ \subset \mathcal{F}_+$ and $\mathcal{G}_- \subset \mathcal{F}_-$.*

Proof. By Axiom 1, (Λ_+, Λ_-) is a bilamination. By Lemma 4.8, applied to this bilamination, each of the rectangles making up $|\mathcal{R}_+| \cap |\mathcal{R}_-|$ can be bifoliated as required. The rest of $|\mathcal{R}|$ consists of rectangles that can be bifoliated as laminated charts for either Λ_+ or Λ_- . The usual process of gluing foliations along transverse and tangential boundary components allows the foliations in these various rectangles to be matched up. \square

Lemma 10.24. *There are mutually transverse C^0 foliations \mathcal{F}_\pm^g of $|\mathcal{R}|$ by geodesic arcs such that $\mathcal{G}_+^g \subset \mathcal{F}_+^g$ and $\mathcal{G}_-^g \subset \mathcal{F}_-^g$.*

Proof. In each rectangle R making up $|\mathcal{R}_+|$, the geodesic lamination induced by \mathcal{G}_+^g extends to a geodesic foliation \mathcal{F}_+^g by the method of proof of Proposition 4.28. Similarly, we extend \mathcal{G}_-^g to a geodesic foliation \mathcal{F}_-^g of $|\mathcal{R}_-|$. On the intersection $|\mathcal{R}_+| \cap |\mathcal{R}_-|$ these foliations, being geodesic, are necessarily transverse. \mathcal{F}_\pm^g is easily extended to the remaining subrectangles of $|\mathcal{R}_\mp|$ to be transverse to \mathcal{F}_\mp^g there. \square

Lemma 10.25. *There exists a natural map $\mu : (\mathcal{F}_+, \mathcal{F}_-) \rightarrow (\mathcal{F}_+^g, \mathcal{F}_-^g)$. The map μ takes leaves $\alpha \in \mathcal{G}_\pm$ to their corresponding geodesic leaves $\alpha^g \in \mathcal{G}_\pm^g$ and gaps of \mathcal{G}_\pm to the corresponding gaps of \mathcal{G}_\pm^g .*

Proof. We define the map μ on \mathcal{F}_+ . The definition of μ on \mathcal{F}_- is analogous. One first defines $\mu(\ell) = \ell^g$ for $\ell \in \mathcal{G}_+$ and then extends μ over the gaps of \mathcal{G}_+ . There are two types of gaps to consider. First let V be a gap of \mathcal{G}_+ lying in a rectangle $R \in \mathcal{R}_+$, σ the bottom edge of V and τ the bottom edge of the corresponding gap W of \mathcal{G}_+^g . There is a map $\nu : \sigma \rightarrow \tau$, linear in the hyperbolic metric. If $\lambda \subset V$ is a

leaf of \mathcal{F}_+ and $\lambda \cap \sigma = \{p\}$, let $\mu(\lambda) \subset W$ be the leaf of \mathcal{F}_+^g with $\mu(\lambda) \cap \tau = \{\nu(p)\}$. Next, let V be a gap of \mathcal{G}_+ lying entirely in a rectangle $R \in \mathcal{R}_-$, crossing it from left edge to right edge, σ the left edge of V , and τ the left edge of the corresponding gap W of \mathcal{G}_+^g . As in the first case, there is a map $\nu : \sigma \rightarrow \tau$, linear in the hyperbolic metric. If $\lambda \subset V$ is a leaf of \mathcal{F}_+ and $\lambda \cap \sigma = \{p\}$, let $\mu(\lambda) \subset W$ be the leaf of \mathcal{F}_+^g with $\mu(\lambda) \cap \tau = \{\nu(p)\}$. \square

For $\lambda \in \mathcal{F}_\pm$, we will denote $\mu(\lambda)$ by $\lambda^g \in \mathcal{F}_\pm^g$. The following lemma is clear.

Lemma 10.26. *The map μ induces a homeomorphism we will denote by*

$$|\mu| : |\mathcal{R}| \rightarrow |\mathcal{R}|.$$

Lemma 10.27. *There exist countable sublaminations $\mathcal{H}_\pm \subset \mathcal{F}_\pm$ and $\mathcal{H}_\pm^g \subset \mathcal{F}_\pm^g$ that correspond under the map μ and such that $|\mathcal{H}_\pm|$ and $|\mathcal{H}_\pm^g|$ are each dense in $|\mathcal{R}|$. The set \mathcal{H}_\pm contains all the semi-isolated leaves of \mathcal{G}_\pm . The rest of the leaves of \mathcal{H}_\pm lie in gaps of \mathcal{G}_\pm . .*

Proof. We first construct \mathcal{H}_+ . The construction of \mathcal{H}_- is similar. Start by putting the semi-isolated leaves of \mathcal{G}_+ into \mathcal{H}_+ . Then for each gap V of \mathcal{G}_+ , choose a countable subset of \mathcal{F}_+ with union dense in V . Since there are countably many gaps, each bounded by two semi-isolated leaves, the set \mathcal{H}_+ is countable. Define the sets $\mathcal{H}_\pm^g = \{\mu(\lambda) \mid \lambda \in \mathcal{H}_\pm\}$. \square

Enumerate the elements of the set $\mathcal{H}_+ \cup \mathcal{H}_-$ as ℓ_1, ℓ_2, \dots . Note that we are not distinguishing whether ℓ_n lies in \mathcal{H}_+ or \mathcal{H}_- . Working in the rectangle $R \in \mathcal{R}$ which contains ℓ_1 , we produce an isotopy Φ_1 , compactly supported in R and leaving $\partial_* \mathcal{R}$ fixed pointwise, which carries ℓ_1 to ℓ_1^g . As usual, this is a sliding isotopy along the bottom and top of R and then an application of Theorem 3.2. Let $\psi_0 = \text{id}$ and $\psi_1 = \Phi_1^1$.

Proposition 10.28. *For $n = 1, 2, \dots$, there exist isotopies Φ_n and homeomorphisms ψ_n defined on $|\mathcal{R}|$, fixing $\partial_* \mathcal{R}$ pointwise, such that,*

- (1) $\psi_n = \Phi_n^1 \circ \psi_{n-1}$.
- (2) ψ_{n-1} takes ℓ_k to ℓ_k^g , $1 \leq k \leq n-1$.
- (3) Φ_n straightens $\psi_{n-1}(\ell_n)$ to ℓ_n^g while leaving the already straightened ℓ_k^g invariant, $1 \leq k \leq n-1$.

Proof. We proceed inductively. The assertions of the proposition for $n = 1$ are satisfied tautologically or vacuously. Inductively, suppose Φ_n , ψ_n and ψ_{n-1} have been defined satisfying the conditions of the proposition, for some $n \geq 1$. The next leaf to be straightened is ℓ_{n+1} . To simplify the discussion, assume $\ell_{n+1} \in \mathcal{G}_+$. Let $R \in \mathcal{R}_+$ be the rectangle containing ℓ_{n+1} and let β_1, \dots, β_r be the other already straightened leaves of \mathcal{G}_+ in R . Let X be the rectangular component of $R \setminus \bigcup_{i=1}^r \beta_i$ that contains $\psi_n(\ell_{n+1})$. Then $\ell_{n+1}^g \subset X$ and the support of Φ_{n+1} will be X . Let $\alpha_1, \dots, \alpha_q$ be the arcs crossing X in which the already straightened leaves of \mathcal{G}_- meet X . Note that the endpoints of $\psi_n(\ell_{n+1})$ lie in two geodesic segments $X \cap \text{Fr}_+ K$ which we denote α_0, α_{q+1} . The segments $\alpha_1, \dots, \alpha_q$ divide X into subrectangles X_i with two opposite edges α_i, α_{i+1} , $0 \leq i \leq q$.

The isotopy Φ_{n+1} is defined as a composition of the following very straightforward isotopies. One first defines isotopies supported in small neighborhoods of the α_i , $0 \leq i \leq q+1$, that slides the point of intersection of $\psi_n(\ell_{n+1})$ with α_i along the arc α_i until it coincides with the point of intersection of ℓ_{n+1}^g with α_i . Then

Theorem 3.2 is used to define isotopies supported on the X_i , $0 \leq i \leq q$, to move the image of the arc $\psi_n(\ell_{n+1})$ under the first set of sliding isotopies to the arc ℓ_{n+1}^g .

Composing these finitely many isotopies defines an isotopy Φ_{n+1} , straightening $\psi_n(\ell_{n+1})$ to ℓ_{n+1}^g while leaving the already straightened leaves ℓ_k^g invariant, $1 \leq k \leq n$. Set $\psi_{n+1} = \Phi_{n+1}^1 \circ \psi_n$. \square

Remark. We emphasize that, if $\ell_n \in \mathcal{G}_\pm$, then $\ell_n^g \in \mathcal{G}_\pm^g$ is the corresponding geodesic tightening of ℓ_n . Also, $\psi_n(\mathcal{H}_\pm)$ is transverse to $\psi_n(\mathcal{H}_\mp)$. By abuse of notation, we denote this lamination again by \mathcal{H}_\pm .

Let $\mathcal{W}_n = \{\ell_1^g, \dots, \ell_n^g\}$. Then $|\mathcal{R}| \setminus |\mathcal{W}_n|$ consists of finitely many rectangles. Let $\rho : L \times L \rightarrow [0, \infty)$ denote the (complete) hyperbolic metric.

Definition 10.29 (mesh). The quantity $\text{mesh}(\mathcal{W}_n)$ is the diameter of the largest rectangular component of $|\mathcal{R}| \setminus |\mathcal{W}_n|$, measured in the hyperbolic metric ρ .

Let δ_n denote $\text{mesh}(\mathcal{W}_n)$. Then, by compactness and the fact that $|\mathcal{H}_\pm^g|$ is dense in \mathcal{R} we have,

Lemma 10.30. $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n \rightarrow 0$.

Let $\{\psi_n\}_{n=0}^\infty$ and $\{\Phi_n\}_{n=0}^\infty$ be the sequences of homeomorphisms and isotopies as in Proposition 10.28. On the space \mathcal{C} of continuous functions $s : \mathcal{R} \rightarrow \mathcal{R}$ we define a metric d by setting

$$d(s, r) = \max_{x \in \mathcal{R}} \rho(s(x), r(x)).$$

As is well known, the topology associated to this metric is the compact-open topology. The group $H \subset \mathcal{C}$ of homeomorphisms $h : |\mathcal{R}| \rightarrow |\mathcal{R}|$ is a topological group under the metric d .

Lemma 10.31. *Relative to the metric d , the sequence $\{\Phi_{n+k}^1 \circ \Phi_{n+k-1}^1 \circ \dots \circ \Phi_n^1\}_{n=0}^\infty$ converges to the identity uniformly for $k \geq 0$ and the sequence $\{\psi_n\}_{n=0}^\infty$ is Cauchy.*

Proof. By (3) of Proposition 10.28, $\Phi_{n+k}^1 \circ \Phi_{n+k-1}^1 \circ \dots \circ \Phi_n^1$ sends the rectangles of $|\mathcal{R}| \setminus |\mathcal{W}_{n-1}|$ into themselves. Therefore, by Lemma 10.30,

$$\rho(\Phi_{n+k}^1 \circ \Phi_{n+k-1}^1 \circ \dots \circ \Phi_n^1(x), x) \leq \delta_{n-1} \rightarrow 0,$$

uniformly in $k \geq 0$ and $x \in |\mathcal{R}|$ as $n \rightarrow \infty$. Since

$$\psi_{n+k} = \Phi_{n+k}^1 \circ \Phi_{n+k-1}^1 \circ \dots \circ \Phi_n^1 \circ \psi_{n-1},$$

it follows immediately that the sequence $\{\psi_n\}_{n=0}^\infty$ is Cauchy. \square

By a standard argument, we get the following.

Corollary 10.32. *The pointwise limit $\psi = \lim_{n \rightarrow \infty} \psi_n$ exists and equals the homeomorphism $|\mu|$.*

Recall that the group H of homeomorphisms of $|\mathcal{R}|$ is a topological group under the metric d . The homeomorphism ψ will be isotopic to the identity if there is a continuous path s in H starting at id and ending at ψ . We construct such a path. It will be convenient to parametrize it on the one point compactification $[1, \infty)$ of $[1, \infty)$. The isotopies Φ_n^t , $0 \leq t \leq 1$, send the rectangles of $|\mathcal{R}| \setminus |\mathcal{W}_{n-1}|$ into themselves and $\Phi_n^0 = \text{id}$. Recall that $\psi_0 = \text{id}$. Define $s : [1, \infty) \rightarrow H$ by

$$s(t) = \begin{cases} \Phi_n^{t-n} \circ \psi_{n-1}, & n \leq t \leq n+1, \quad 1 \leq n < \infty \\ \psi, & t = \infty. \end{cases}$$

Lemma 10.33. *The path $s : [1, \infty] \rightarrow H$ is continuous, hence ψ is isotopic to the identity.*

Proof. Continuity of $s(t)$ for $1 \leq t < \infty$ is clear. Since $\Phi_{n+k}^t \circ \Phi_{n-k-1}^1 \circ \cdots \circ \Phi_n^1$ sends the rectangles of $|\mathcal{R}| \setminus |\mathcal{W}_{n-1}|$ into themselves, $0 \leq t \leq 1$ and $k \geq 0$, we see by Lemma 10.30 that $d(s(t_1), s(t_2))$ is uniformly as small as desired, for all $t_1, t_2 \in [n, \infty)$ and n sufficiently large. Since $\lim_{n \rightarrow \infty} s(n) = \psi$ in the metric d , it follows that $\lim_{t \rightarrow \infty} s(t) = \psi$ in that metric, proving continuity at $t = \infty$. \square

Since the homeomorphism ψ isotopic to the identity is pointwise fixed on $\partial_* \mathcal{R}$, it can be extended to a homeomorphism isotopic to the identity, again denoted by ψ , which is the identity on $K \setminus \mathcal{R}$. We then have,

Proposition 10.34. *The homeomorphism ψ of K , isotopic to the identity, sends each leaf of $\ell \in \mathcal{G}_\pm$ to its corresponding geodesic tightening in $\ell^\mathfrak{g} \in \mathcal{G}_\pm^\mathfrak{g}$.*

10.3.4. *Extending the isotopy to L .* Since $\psi(\text{Fr}_\pm K) = \text{Fr}_\pm K$ and ψ is isotopic to the identity there, it is easy to extend ψ into a small collar of $\text{Fr}_\pm K$ on the side facing away from K , damping the extension off to the identity in this collar. One then extends by the identity over the rest of L , obtaining a homeomorphism $\psi_K : L \rightarrow L$ isotopic to the identity by an isotopy compactly supported as near to K as desired. Since $\psi_K|_K = \psi$, Proposition 10.34 implies that $\psi_K(\gamma_\alpha)$ and $\gamma_\alpha^\mathfrak{g}$ cross $\text{Fr}_\pm K$ in exactly the same points.

Enumerate the tiles of $\mathfrak{T}^\mathfrak{g}$ in any convenient way as $P_0, P_1, P_2, \dots, P_n, \dots$. We can take $P_0 = K$ and assume that, for each $n \geq 1$, P_n has at least one edge in common with at least one tile P_i , $0 \leq i < n$.

Next proceed to P_1 and produce the homeomorphism ψ_{P_1} , isotopic to the identity, in the same way as above, noting that the isotopy will fix pointwise the interface of P_1 with K . Remark that the procedure is easier here since, if P_1 lies in a neighborhood of a positive end, there will only be rectangles $R \in \mathcal{R}_+$ in P_1 , and if P_1 lies in a neighborhood of a negative end, there will only be rectangles $R \in \mathcal{R}_-$ in P_1 . One obtains a compactly supported homeomorphism $\psi_{P_1} : L \rightarrow L$ isotopic to the identity by an isotopy which is the identity on K and outside a neighborhood of P_1 which is only slightly larger than P_1 . Proceed in this way with each P_i in turn. The supports of the homeomorphisms $\psi_K, \psi_{P_1}, \dots, \psi_{P_n}, \dots$ and associated isotopies form a locally finite family of compact sets. Thus, the infinite composition

$$\psi_L = \cdots \circ \psi_{P_n} \circ \psi_{P_{n-1}} \circ \cdots \circ \psi_{P_1} \circ \psi_K$$

is a well defined homeomorphism on L , isotopic to the identity, which carries Λ_\pm to $\Lambda_\pm^\mathfrak{g}$. The proof of Theorem 10.15 is complete.

10.4. **Good choice of junctures.** In Section 10.1 we axiomatize a system consisting of five elements $(f, \mathcal{N}, \mathcal{J}, \Lambda_+, \Lambda_-)$ where $f : L \rightarrow L$ is an endperiodic automorphism, \mathcal{N} is a set of f -juncture components chosen as in Definition 4.17, \mathcal{J} is a set of juncture components associated to \mathcal{N} (Definition 10.5), and Λ_\pm are pseudo-geodesic laminations.

The Isotopy Theorem (Theorem 10.15) provides a homeomorphism ψ of L , isotopic to the identity, so that $\psi(\Lambda_\pm) = \Lambda_\pm^\mathfrak{g}$.

Theorem 10.35. *If the system $(f, \mathcal{N}, \mathcal{J}, \Lambda_+, \Lambda_-)$ satisfies the axioms, the entire theory for the Handel-Miller geodesic laminations developed in Sections 4 - 9 carries over verbatim to the system $(f, \mathcal{N}, \mathcal{J}' = \psi^{-1}(\mathcal{J}^\mathfrak{g}), \Lambda_+, \Lambda_-)$.*

Denote by ι^g the geodesic tightening map of Definition 4.20.

Definition 10.36 (ι'). Define the map $\iota' : \mathcal{N} \rightarrow \mathcal{J}$ by $\iota' = \psi^{-1} \circ \iota^g \circ \psi$.

The map $\iota' : \mathcal{N} \rightarrow \mathcal{J}$ serves the role of the map $\iota : \mathcal{N} \rightarrow \mathcal{J}$ of Definition 10.5 for the system $(f, \mathcal{N}, \mathcal{J}' = \psi^{-1}(\mathcal{J}^g), \Lambda_+, \Lambda_-)$ and as such replaces the geodesic tightening map of Definition 4.20. It can be extended to a map $\iota' : \mathcal{N}^\dagger \rightarrow \mathcal{J}$ where \mathcal{N}^\dagger is the set of pseudo-geodesics γ such that there exists an f -juncture component $\sigma \in \mathcal{N}$ whose lifts have the same endpoints on S_∞ as the lifts of γ . Then define $\iota'(\gamma) = \iota(\sigma)$. This definition then extends in a natural way to a function ι' with domain the finite unions of elements of \mathcal{N}^\dagger .

Proof of Theorem 10.35. First, consider the endperiodic automorphism $f^* = \psi \circ f \circ \psi^{-1} : L \rightarrow L$ with set of f^* -juncture components $\mathcal{N}^* = \{\psi(\gamma) \mid \gamma \in \mathcal{N}\}$. Since $\gamma \in \mathcal{N}$ and $\psi(\gamma) \in \mathcal{N}^*$ have the same endpoints on S_∞ , they have the same geodesic tightenings. Thus, \mathcal{J}^g is both the set of geodesic tightenings of the set \mathcal{N} of f -juncture components and the set \mathcal{N}^* of f^* -juncture components. Thus, the entire theory for the Handel-Miller geodesic laminations developed in Sections 4 - 9 is true for the system $(f^*, \mathcal{N}^*, \mathcal{J}^g, \Lambda_+^g, \Lambda_-^g)$.

Further, ψ^{-1} carries the bilamination $(\Lambda_+^g, \Lambda_-^g) = (\psi(\Lambda_+), \psi(\Lambda_-))$ to the bilamination (Λ_+, Λ_-) , the set of geodesic junctures \mathcal{J}^g to the set $\mathcal{J}' = \psi^{-1}(\mathcal{J}^g)$ of junctures, and the set \mathcal{N}^* of f^* -junctures to the set \mathcal{N} of f -junctures as well as all statements for the theory developed in Sections 4 - 9 for the system $(f^*, \mathcal{N}^*, \mathcal{J}^g, \Lambda_+^g, \Lambda_-^g)$ to the system $(f, \mathcal{N}, \mathcal{J}', \Lambda_+, \Lambda_-)$. The truth of each statements for the system $(f, \mathcal{N}, \mathcal{J}', \Lambda_+, \Lambda_-)$ then follows from the truth of the statement for the system $(f^*, \mathcal{N}^*, \mathcal{J}^g, \Lambda_+^g, \Lambda_-^g)$ since ψ^{-1} is a homeomorphism, isotopic to the identity (see examples below). \square

Example 10.37. By Theorem 6.5, Λ_+^g and Λ_-^g each have only finitely many semi-isolated leaves. Since ψ^{-1} is a homeomorphism, $\Lambda_+ = \psi^{-1}(\Lambda_+^g)$ and $\Lambda_- = \psi^{-1}(\Lambda_-^g)$ each have finitely many semi-isolated leaves. Thus, Theorem 6.5 is valid in the system $(f, \mathcal{N}, \mathcal{J}' = \psi^{-1}(\mathcal{J}^g), \Lambda_+, \Lambda_-)$.

Example 10.38. By Lemma 4.34, the laminations Γ_\pm^g and Λ_\pm^g are strongly closed. Since ψ^{-1} is a homeomorphism, it follows immediately that the laminations $\Gamma_\pm = \psi^{-1}(\Gamma_\pm^g)$ and $\Lambda_\pm = \psi^{-1}(\Lambda_\pm^g)$ are strongly closed. Thus, Lemma 4.34 is valid in the system $(f, \mathcal{N}, \mathcal{J}' = \psi^{-1}(\mathcal{J}^g), \Lambda_+, \Lambda_-)$. It is much more difficult to prove Lemma 4.34 for the system $(f, \mathcal{N}, \mathcal{J}' = \psi^{-1}(\mathcal{J}^g), \Lambda_+, \Lambda_-)$ directly from the axioms.

Example 10.39. By Theorem 4.54, there exists an endperiodic automorphism h , isotopic to f^* and permuting the elements of each of the sets Λ_+^g , Λ_-^g , \mathcal{J}_+^g , and \mathcal{J}_-^g . That is $h = \varphi \circ f^*$ with φ a homeomorphism isotopic to the identity and $h(\Lambda_+^g) = \Lambda_+^g$, $h(\Lambda_-^g) = \Lambda_-^g$, $h(\mathcal{J}_+^g) = \mathcal{J}_+^g$, and $h(\mathcal{J}_-^g) = \mathcal{J}_-^g$. Then,

$$\begin{aligned} h' &= \psi^{-1} \circ h \circ \psi \\ &= \psi^{-1} \circ \varphi \circ f^* \circ \psi \\ &= \psi^{-1} \circ \varphi \circ \psi \circ f \circ \psi^{-1} \circ \psi \\ &= \psi^{-1} \circ \varphi \circ \psi \circ f \end{aligned}$$

is an endperiodic automorphism isotopic to f and $h'(\Lambda_+) = \psi^{-1} \circ h \circ \psi(\psi^{-1}(\Lambda_+^g)) = \psi^{-1}(\Lambda_+^g) = \Lambda_+$. Thus h' permutes the elements of Λ_+ . Similarly, h' permutes the

elements of Λ_- , \mathcal{J}'_+ , and \mathcal{J}'_- . Thus, Theorem 4.54 is valid in the system $(f, \mathcal{N}, \mathcal{J}' = \psi^{-1}(\mathcal{J}^g), \Lambda_+, \Lambda_-)$.

Remark. We think of \mathcal{J}' as a “good” choice of juncture components in the sense that Theorem 10.35 holds. The good choice is not always preferable when one is trying to verify the axioms. Thus, in the proof of the transfer theorem (Theorem 12.7), the set \mathcal{J} of junctures that we will construct for the transferred laminations is definitely not “good”, but satisfies Axiom 4.

11. SMOOTHING h AND THE LAMINATIONS

The goal of this section is to prove the following.

Theorem 11.1. *Given an endperiodic automorphism f , there exists a smooth Handel-Miller pseudo-geodesic bilamination (Λ_+, Λ_-) associated to f and a smooth endperiodic automorphism h isotopic to f and preserving (Λ_+, Λ_-) .*

Recall that the bilamination (Λ_+, Λ_-) is called a Handel-Miller pseudo-geodesic bilamination (Λ_+, Λ_-) associated to f if it satisfies the four axioms (Definitions 10.7).

Remark. Geodesic laminations Λ_{\pm}^g associated to f and satisfying the axioms are probably not generally smooth. In fact, geodesic laminations may not even be C^1 or, if C^1 they may fail to be C^2 (Example 4.11). The problem is that if, in local laminated charts, there are infinitely many gaps clustering on plaques of the lamination, the mean value theorem generally obstructs attempts to extend the lamination to a C^1 foliation across these gaps and there are similar higher order obstructions to C^r smoothness, $r \geq 1$. Furthermore, as in the Nielsen-Thurston theory, there seem to be obstacles to choosing the endperiodic automorphism h^g , isotopic to f and preserving $(\Lambda_+^g, \Lambda_-^g)$, to be a diffeomorphism.

We first prove Theorem 11.1 for the case in which there are no principal regions. The changes that have to be made to handle the case in which there are principal regions are not substantive and are outlined in Section 11.5.

Temporary Hypothesis. Until Section 11.5, we assume there are no principal regions.

Strategy. We give a heuristic sketch of the proof. One begins with a Handel-Miller geodesic bilamination $(\Lambda_+^g, \Lambda_-^g)$ associated to f and an endperiodic automorphism h^g , isotopic to f and preserving $(\Lambda_+^g, \Lambda_-^g)$. We produce a Markov partition for h^g by geodesic quadrilaterals, somewhat different from the ones described in Section 9, and define a diffeomorphism on the union of these rectangles. Using standard techniques in differential topology, we extend this to an endperiodic diffeomorphism $h : L \rightarrow L$ which is isotopic to h^g . Positive iterates of h “stretch” the Markov quadrilaterals deeper and deeper into the positive ends, limiting in a natural way on a smooth lamination Λ_+ in L . Similarly, negative iteration produces a smooth bilamination Λ_- , (Λ_+, Λ_-) being an h -invariant bilamination. We verify the axioms, thereby proving Theorem 11.1 for the case that there are no principal regions. We then give a detailed outline of how to modify this construction to accommodate the presence of principal regions.

Fix a set \mathcal{N} of f -juncture components as in Definition 4.17. This determines a set of geodesic juncture components \mathcal{J}^g .

To begin with, since the intersection points of junctures with ∂L do not cluster except at ends of L , the construction of h^g in Section 4.6 can be carried out so that the following holds.

Lemma 11.2. *The automorphism $h^g : L \rightarrow L$ is a diffeomorphism in a neighborhood of ∂L .*

Fix a core K . Let $\mathcal{J}_0^- \subset \mathcal{J}_-^g$ be the set of juncture components comprising $\text{Fr}_- K$ and $\mathcal{J}_0^+ \subset \mathcal{J}_+^g$ be the set of juncture components comprising $\text{Fr}_+ K$. Recall the decomposition $L = W^- \cup K \cup W^+$. Then $|\mathcal{J}_0^-| = W^- \cap K = \text{Fr } W^-$ and $|\mathcal{J}_0^+| = W^+ \cap K = \text{Fr } W^+$. Let $\mathcal{J}_n^- = (h^g)^n(\mathcal{J}_0^-)$, $\mathcal{J}_n^+ = (h^g)^n(\mathcal{J}_0^+)$, and $W_n = (h^g)^n(W^-)$, $n \in \mathbb{Z}$.

The following is clear.

- Lemma 11.3.**
- (1) $\bigcup_{n=-\infty}^{\infty} \mathcal{J}_n^- = \mathcal{J}_-^g$;
 - (2) $\bigcup_{n=-\infty}^{\infty} \mathcal{J}_n^+ = \mathcal{J}_+^g$;
 - (3) $\text{Fr } W_n = |\mathcal{J}_n^-|$.

Definition 11.4 (quadrilateral and geodesic quadrilateral). We will call a geodesically convex figure with four edges that are geodesics a *geodesic quadrilateral*. When h has been defined, the images of a geodesic quadrilateral under applications of h and its powers will be called quadrilaterals.

Remark. We defined a rectangle (Definition 6.1) to have a pair of opposite edges in Γ_- and a pair of opposite edges in Γ_+ . In a geodesic quadrilateral, the edges may be, but need not be, in Γ_{\pm} .

Lemma 11.5. *For $n > 0$ large enough, the set of components of $K \setminus \text{int } W_n$ consists of a finite family $\{R_1, \dots, R_k\}$ of geodesic quadrilaterals with one pair of opposite edges in $\text{Fr}_+ K$ and the other pair of opposite edges components of $|\mathcal{J}_n^-| \cap K$ in $\mathcal{X}_-^g|K$.*

Proof. By Lemma 4.52 the leaves of $\Lambda_+^g \cap K$ are properly embedded, boundary incompressible arcs with endpoints in $\text{Fr}_+ K$ which fall into finitely many isotopy classes. By Lemma 4.53 each of these isotopy classes has two extreme arcs which together with two arcs on $\text{Fr}_+ K$ bound rectangles $\{R'_1, \dots, R'_\ell\}$ such that $\bigcup_{i=1}^{\ell} R'_i \supset |\Lambda_+^g| \cap K$. Since there are no principal regions, both edges of R'_i in $|\Lambda_+^g| \cap K$ are approached outside of R'_i by arcs of $|\mathcal{X}_-^g| \cap K$. Since the negative junctures can only cluster on $|\Lambda_+^g|$, the strongly closed property implies that there are only finitely many components of $\bigcup_{n=0}^{\infty} |\mathcal{J}_n^-| \cap K$ that do not lie in one of these isotopy classes and that $R'_i \subset R''_i$, a geodesic quadrilateral with a pair of opposite edges in $\text{Fr}_+ K$ and a pair of opposite edges in $\bigcup_{n=0}^{\infty} |\mathcal{J}_n^-| \cap K$ such that $\bigcup_{i=1}^{\ell} R''_i$ contains every arc of $\bigcup_{n=0}^{\infty} |\mathcal{J}_n^-| \cap K$ that lies in the same isotopy class as an arc of $|\Lambda_+^g| \cap K$.

If n is large enough, then the components of $\text{Fr } W_n \cap K$ all lie in one of the isotopy classes and all are components of $|\mathcal{X}_-^g| \cap K$. Thus, the components of $K \setminus \text{int } W_n$ are geodesic quadrilaterals $\{R_1, \dots, R_k\}$ that are subquadrilaterals of the geodesic quadrilaterals $\{R''_1, \dots, R''_\ell\}$. \square

Remark. By choosing $n > 0$ large enough, as in Lemma 11.5, we guarantee that the set $\{R_1, \dots, R_k\}$ of components of $K \setminus \text{int } W_n$ has the property that the geodesic quadrilateral $h^g(R_j)$ completely crosses any R_i that it meets, $1 \leq j \leq k$. The components of the $h^g(R_j) \cap R_i$ are the components of $K \setminus \text{int } W_{n+1}$. Now take

the R_i to be the components of $K \setminus \text{int } W_{n+1}$. As in the proof of Proposition 9.9, any $h^g(R_j)$ meets any R_i at most once. Thus, the set of geodesic quadrilaterals $\{R_1, \dots, R_k\}$ making up $K \setminus \text{int } W_{n+1}$ satisfy Properties II, III, and IV of Definition 9.6 but is not a Markov system as defined there because the edges of R_i are not subarcs of leaves of the laminations Λ_{\pm}^g .

Notation. From now on in Section 11 we denote by $N > 0$ this integer $n + 1$.

Definition 11.6 (Markov chain). A sequence

$$\iota = (i_0, i_1, i_2, \dots, i_n, \dots)$$

in which $i_n \in \{1, 2, \dots, k\}$ and $R_{i_n} \cap h^g(R_{i_{n+1}}) \neq \emptyset$, $n \geq 0$, will be called a Markov chain.

This language is borrowed from symbolic dynamics. Our system of geodesic quadrilaterals is a Markov partition as commonly defined in dynamics, but different from that defined in Section 9. The following statement and proof are closely analogous to those of Proposition 9.12. Since our construction of the smooth bilamination is modelled on this, we give details.

Proposition 11.7. *If ι is a Markov chain, then there is a unique leaf $\ell_{\iota}^g \in \Lambda_{+}^g$ such that if*

$$\ell_{i_n}^g = (h^g)^n(R_{i_n}) \cap (h^g)^{n+1}(R_{i_{n+1}}) \cap \dots, \quad n \geq 0$$

then

$$\ell_{\iota}^g = \bigcup_{n=0}^{\infty} \ell_{i_n}^g,$$

an increasing union of compact arcs.

Proof. Choose a lift $C_0^g = \tilde{R}_{i_0}$ of R_{i_0} to $\tilde{L} \subset \Delta$. This lift determines lifts C_n^g of $(h^g)^n(R_{i_n})$ for all $n > 0$. For $n \geq 0$, let $\tilde{\lambda}_n^g, \tilde{\mu}_n^g \in \tilde{X}_{-}^g$ contain opposite sides of these lifts C_n^g of $(h^g)^n(R_{i_n})$. The sequences $\{\tilde{\lambda}_n^g\}, \{\tilde{\mu}_n^g\}$ converge monotonically to $\tilde{\lambda}^g, \tilde{\mu}^g \in \tilde{\Lambda}_{+}^g$. Let δ_0^g be an edge of R_{i_0} contained in $\text{Fr}_{+} K$. Let $\delta_1^g \subset h^g(\text{Fr}_{+} K)$ be the edge of $h^g(R_{i_1})$ bordering the piece of $h^g(R_{i_1})$ that exits R_{i_0} through δ_0^g . Inductively, let $\delta_n^g \subset (h^g)^n(\text{Fr}_{+} K)$ be the edge of $(h^g)^n(R_{i_n})$ bordering the piece of $(h^g)^n(R_{i_n})$ that exits $R_{i_{n-1}}$ through δ_{n-1}^g . For $n \geq 0$, let δ_n^g be the edge of the lift C_n^g of $(h^g)^n(R_{i_n})$ that covers δ_n^g and let $\tilde{\sigma}_n^g$ be the geodesic in \tilde{L} that contains δ_n^g . Note that the $\tilde{\sigma}_n^g$ are lifts of components of $(h^g)^n(\text{Fr}_{+} K)$. By Theorem 4.24, the $\tilde{\sigma}_n^g$ nest on a point $a \in S_{\infty}^1$ which is a common endpoint of $\tilde{\lambda}^g$ and $\tilde{\mu}^g$. Similarly, starting with the other edge γ_0^g of R_{i_0} which lies in $\text{Fr}_{+} K$, one gets a family of geodesics in \tilde{L} that nest on a point $b \in S_{\infty}^1$ which is the other common endpoint of $\tilde{\lambda}^g$ and $\tilde{\mu}^g$. It follows that $\tilde{\lambda}^g = \tilde{\mu}^g$ which projects to ℓ_{ι}^g . \square

Remark. For a similar characterization of the leaves $\ell_{\kappa}^g \in \Lambda_{-}^g$, first note that applying $(h^g)^{-N}$ to the geodesic quadrilaterals R_i , gives similar geodesic quadrilaterals R_i^* with a pair of opposite edges in $\text{Fr}_{-} K$ and the other pair of edges components of $|\partial_{-N}^+| \cap K$. We use sequences

$$\kappa = (\dots, i_{-n}, \dots, i_{-2}, i_{-1}, i_0)$$

in which $R_{i_{-n}}^* \cap (h^g)^{-1}(R_{i_{-n-1}}^*) \neq \emptyset$. It is actually unnecessary to go to the geodesic quadrilaterals R_i^* since one gets the same result by interchanging the roles of R_i

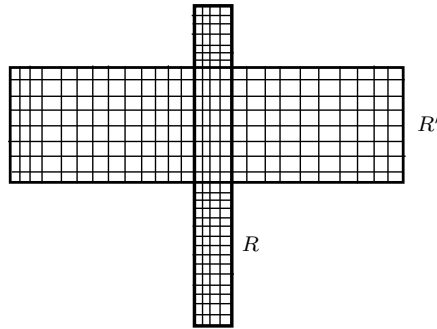


FIGURE 24. Compatible grids

and $h^g(R_j)$ and using negative powers of h^g . Notice that this would not work if an edge of R_i bordered a principal region.

As remarked earlier, our approach to the proof of Theorem 11.1 in the case that there are no principal regions will be to isotope h^g to an endperiodic diffeomorphism h such that $h(R_i) = h^g(R_i)$, $1 \leq i \leq k$, and define the new smooth bilamination (Λ_+, Λ_-) by a construction analogous to the above, defining \mathcal{X}_\pm in a fairly obvious way. Verifying the axioms will complete the proof. The proof in the presence of principal regions is similar, but notationally more complicated.

11.1. Geodesic grids. Given a geodesic quadrilateral R , let α_0^g and α_1^g be one pair of opposite sides, β_0^g and β_1^g the other. Orient the α_i^g from β_0^g to β_1^g and the β_i^g from α_0^g to α_1^g . Let $\varphi : \alpha_0^g \rightarrow \alpha_1^g$ and $\psi : \beta_0^g \rightarrow \beta_1^g$ be orientation preserving diffeomorphisms. Thus φ pairs the endpoints of β_0^g and β_1^g and ψ likewise pairs the endpoints of α_0^g and of α_1^g . Since R is geodesically convex, the geodesics joining $t \in \alpha_0^g$ to $\varphi(t) \in \alpha_1^g$ lie in R , for all $t \in \alpha_0^g$. Since the geodesic quadrilateral is simply connected, we may as well be working in the hyperbolic plane where geodesic arcs depend smoothly on their endpoints. Thus, our geodesics depend smoothly on $t, \varphi(t)$ and cannot intersect each other. Indeed, since φ is orientation preserving, any proper intersections would produce geodesic digons. Thus these geodesic arcs are the leaves of a smooth foliation \mathcal{F}_v of R such that $\beta_i^g \in \mathcal{F}_v$, $i = 0, 1$. Similarly, use ψ to obtain a smooth, geodesic foliation \mathcal{F}_h of R , necessarily transverse to \mathcal{F}_v and incorporating the α_i^g as leaves. We call this a “geodesic grid”.

Typically, we will have one geodesic quadrilateral completely crossing another once as in Figure 24. We can put geodesic grids on each which agree on the intersection. In Figure 24, define the “vertical” geodesic foliation \mathcal{F}_v first on the tall geodesic quadrilateral R and the “horizontal” geodesic foliation \mathcal{F}_h on the wide one R' . In order to extend $\mathcal{F}_h|_{R \cap R'}$ across the unfoliated geodesic subquadrilaterals of R , one needs to choose diffeomorphisms from the left side to the right that match smoothly at the common endpoints of domains with the ones already defined by $\mathcal{F}_h|_{R \cap R'}$. This is a matter of making the two ∞ -jets coincide and is standard. Similarly, \mathcal{F}_v is extended smoothly.

Another situation is that the quadrilateral R with the geodesic grid may be contained in a slightly larger quadrilateral A as in Figure 25. We will leave it to

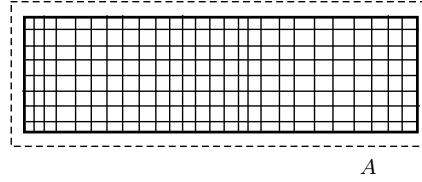


FIGURE 25. Expanded geodesic quadrilateral

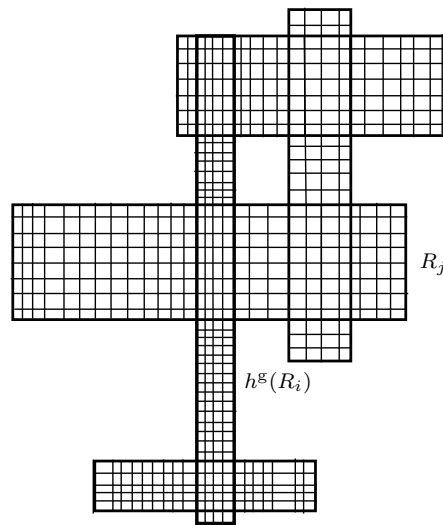


FIGURE 26. Compatible grids

the reader to adapt the previous discussion to extend the geodesic grid on R to a geodesic grid on A .

In the remark preceding Definition 11.6, we constructed families $\{R_1, \dots, R_k\}$ and $\{h^g(R_1), \dots, h^g(R_k)\}$ of pairwise disjoint geodesic quadrilaterals. Furthermore, $h^g(R_i)$ will completely cross some of the R_j 's, but never more than once (cf. Definition 9.4). As above, we define smooth geodesic grids on $\mathcal{R} = \{R_1, \dots, R_k\} \cup \{h^g(R_1), \dots, h^g(R_k)\}$ which are compatible on overlaps. In Figure 26, the vertical lines are to be leaves of \mathcal{F}_v , the horizontal ones leaves of \mathcal{F}_h . Note that the foliation \mathcal{F}_h incorporates as leaves the edges of R_i which are subarcs of $\text{Fr}_+ K$, while \mathcal{F}_v incorporates the other pair of opposite edges.

Remark. Ultimately, the smooth bilamination (Λ_+, Λ_-) that we are going to construct will be such that $\mathcal{F}_v|_{R_i}$ is an extension of $\Lambda_+|_{R_i}$ and \mathcal{F}_h is an extension of $\Lambda_-|_{R_i}$. This will establish the smoothness of the laminations in each R_i and will extend to global smoothness by iteration of the diffeomorphism h that we are going to construct.

Remark. If finitely many geodesic leaves are preassigned in any of the geodesic quadrilaterals, there is no problem choosing the foliations to incorporate them. If one wants to preassign infinitely many geodesic leaves, say arcs of Λ_{\pm}^g , serious smoothness issues arise. This is exactly why, in producing the smooth laminations Λ_{\pm} and diffeomorphism h out of the geodesic data we will not generally get back the geodesic laminations.

11.2. Smoothing in the geodesic quadrilaterals. Let $\mathcal{R} = \bigcup_{i=1}^k R_i$. In order to smooth h^g on \mathcal{R} , we first thicken each R_i to a slightly larger quadrilateral A_i (see Figure 27). We can do this so that the A_i 's are pairwise disjoint.

We want to extend the geodesic grids of Section 11.1 to a neighborhood of $\mathcal{R} \cup h^g(\mathcal{R})$ of the form

$$V = A_1 \cup \dots \cup A_k \cup h^g(A_1) \cup \dots \cup h^g(A_k).$$

The problem is that $h^g(A_i)$ will not generally be a geodesic quadrilateral, the edges failing to be geodesics. To correct this, we perform a small isotopy of h^g .

Lemma 11.8. *If A_i approximates R_i sufficiently well, there is a homeomorphism φ_i , isotopic to the identity by an isotopy supported near $h^g(A_i)$ and away from $h^g(A_j)$, $j \neq i$, taking $h^g(\text{Fr}_+ K)$ to itself componentwise, fixing $h^g(R_i)$ pointwise, fixing the vertices of $h^g(A_i)$, and such that $\varphi_i(h^g(A_i))$ is a geodesic quadrilateral.*

Proof. Let $D \subset L$ be a smoothly embedded disk containing $h^g(A_i)$, having the four vertices of that image on its boundary, but such that the edges of $h^g(A_i)$ are properly embedded in D . If the vertices of A_i have been chosen sufficiently near the corresponding vertices of R_i , D can be chosen so that the geodesic arcs joining the vertices of $h^g(A_i)$ to form a geodesic quadrilateral are also properly embedded in D . Also, choose D so that it meets no $h^g(A_j)$, $j \neq i$. By our usual application of Theorem 3.2, we can find an ambient isotopy Φ supported in D , such that Φ fixes $h^g(R_i)$ pointwise and takes the bottom two edges of $h^g(\partial A_i)$ to the geodesics joining the corresponding pairs of vertices. For the remaining two edges, simple sliding isotopies along the components of $\text{Fr}_+ K$ crossed by these edges must first be performed and the Theorem 3.2 completes the isotopy. Take $\varphi = \Phi^1$. \square

We can assume that the homeomorphisms φ_i are isotopic to the identity by isotopies which have pairwise disjoint supports and set $\varphi = \varphi_1 \circ \dots \circ \varphi_k$. Thus, replacing h^g by $\varphi \circ h^g$ we can assume that $h^g(A_i)$ is a slightly larger geodesic quadrilateral than $h^g(R_i)$, $1 \leq i \leq k$.

We construct the geodesic grid on V as above so that it restricts to the geodesic grid already constructed on \mathcal{R} . The extended foliations are again denoted by \mathcal{F}_h and \mathcal{F}_v .

These foliations have trivial holonomy, so it is easy to construct a smooth, transverse invariant measure μ_h for \mathcal{F}_h on V and another μ_v for \mathcal{F}_v on V . The measure μ_h can be viewed as a smooth measure on the leaves of \mathcal{F}_v , invariant by holonomy translations along the leaves of \mathcal{F}_h and, similarly, μ_v is such a measure on the leaves of \mathcal{F}_h .

Remark. By a smooth measure μ on a smooth 1-manifold B , we mean that if θ is Lebesgue measure on B , then the Radon-Nikodym derivative $d\mu/d\theta = f$ is a smooth function on B . Thus, for any Borel set $X \subseteq B$, $\mu(X) = \int_X f d\theta$.

On each A_i , using the above measures, we define smooth coordinates (x, y) such that the leaves of \mathcal{F}_h are the level sets of y and the leaves of \mathcal{F}_v are the level sets of x . Simply designate the coordinates at one point p_0 , say by (a, b) , designate an orientation of the leaves of $\mathcal{F}_v|_{A_i}$ and of the leaves of $\mathcal{F}_h|_{A_i}$ and use the holonomy invariant measures to define the desired smooth coordinates on A_i . More precisely, using the designated orientation of the leaf of \mathcal{F}_h through p_0 , each point p in that leaf to the right of p_0 defines a subarc $[p_0, p]$ and we define $x(p) = a + \mu_v[p_0, p]$. If p is to the left of p_0 , set $x(p) = a - \mu_v[p, p_0]$. By translation along the leaves of \mathcal{F}_v we extend the x coordinate to all of A_i . The y coordinate is defined analogously using the measure μ_h . These do not give global coordinates on V , but the above construction makes the following clear.

Lemma 11.9. *On overlaps $A_i \cap h^g(A_j)$, the induced coordinates (x, y) coming from A_i and (x', y') coming from $h^g(A_j)$ are related by*

$$\begin{aligned} x' &= \varepsilon_1 x + c \\ y' &= \varepsilon_2 y + d, \end{aligned}$$

where $\varepsilon_i = \pm 1$, $i = 1, 2$, and c, d are constants.

The sign is due to the choices of orientation and the additive constants are due to the fact that the coordinates are well defined up to translation.

Thus the Jacobian matrix for this coordinate change is

$$(*) \quad d\Phi = \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}.$$

Normalize the measures μ_h, μ_v so that the “width” of each A_i , measured by μ_v is 1 and similarly, the “height” of each $h^g(A_i)$ is 1.

We now choose diffeomorphisms $h_i : A_i \rightarrow h'(A_i)$ which preserve the smooth geodesic grids, $1 \leq i \leq k$. In the smooth local coordinates (x, y) on V we have,

$$h_i(x, y) = (\xi_i(x), \zeta_i(y)).$$

For technical reasons, choose h_i to satisfy,

$$0 < |d\xi_i/dx| < a < 1, \quad |d\zeta_i/dy| > b > 1,$$

for suitable constants a and b . This is possible since the width of each A_j and the height of each $h^g(A_i)$ is 1 and $h^g(A_i)$ intersects A_j , if at all, in a proper geodesic

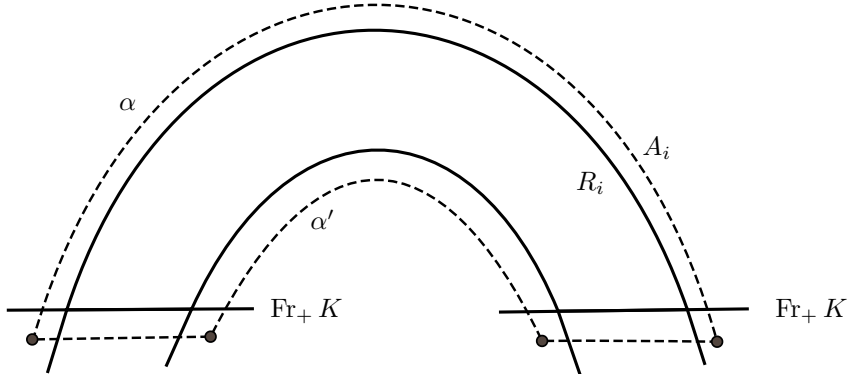


FIGURE 27. Geodesic quadrilaterals R_i and A_i (dotted)

subquadrilateral of both A_i and $h^g(A_j)$. Hence h_i strictly diminishes width and increases height. We summarize.

Proposition 11.10. *For $1 \leq i \leq k$ there are diffeomorphisms*

$$h_i = (\xi_i, \zeta_i) : A_i \rightarrow h^g(A_i),$$

preserving the geodesic grid and such that

$$0 < |d\xi_i/dx| < a < 1, \quad |d\zeta_i/dy| > b > 1$$

in the coordinates on $A_i \cup h^g(A_i)$.

Remark. Because of (*), these inequalities hold for any of our choices of local coordinates.

We begin the modification of h^g to an isotopic diffeomorphism h by replacing $h^g|_{R_i}$ by $h_i|_{R_i}$, $1 \leq i \leq k$. The fact that these maps extend to diffeomorphisms on the slightly larger geodesic quadrilaterals A_i will be useful in extending these local definitions to the global h .

11.3. Smoothing outside \mathcal{R} . We first need to “blend” h^g with the diffeomorphisms h_i so that they agree on a neighborhood of R_i . Fix a smooth, embedded disk $D_i \subset V$ with $R_i \subset \text{int } D_i$ and set $\partial D_i = S_i$. Choose these so that $D_i \cap D_j = \emptyset$, $i \neq j$. Note that $h^g|_{S_i}$ and $h_i|_{S_i}$ are homotopic. By Theorem 3.1, there is a homeomorphism φ_i , isotopic to the identity by an isotopy compactly supported outside of R_i and inside of V , such that $\varphi_i \circ h^g|_{S_i} = h_i|_{S_i}$. (Note that we are working in the complement of R_i where S_i is essential.) Again, we can assume the φ_i and φ_j , $i \neq j$, are isotopic to the identity by isotopies which have disjoint supports. In this way, we isotope h^g by a compactly supported isotopy to a homeomorphism h'' that agrees with h_i on S_i , $1 \leq i \leq k$. Use Alexander’s trick to isotope $h'' \circ h_i^{-1}|_{D_i}$ to the identity by an isotopy that fixes S_i pointwise. This shows that $h''|_{D_i}$ is isotopic to $h_i|_{D_i}$ by an isotopy throughout which they continue to agree on S_i , $1 \leq i \leq k$. We summarize.

Lemma 11.11. *After a compactly supported isotopy, we can assume that h^g agrees with h_i on a neighborhood of R_i , $1 \leq i \leq k$.*

Let L' denote the complement in L of the union of the open geodesic quadrilaterals $\text{int } R_i$. Similarly, Let $L'' = h^g(L')$ be the complement in L of the union of the open geodesic quadrilaterals $h^g(\text{int } R_i)$. We work completely in L' . By the above lemma and Lemma 11.2, $h^g|_{L'}$ is a diffeomorphism in a neighborhood in L' of $\partial L'$. It is well known that $h^g|_{L'} : L' \rightarrow L''$ is arbitrarily well approximated by a diffeomorphism $h^\#$, isotopic to $h^g|_{L'}$ by an isotopy that is fixed in a smaller neighborhood in L' of $\partial L'$. For a particularly nice proof of this which does not use the Schönflies theorem, see A. Hatcher’s unpublished note [30] which is available on the author’s website.

Since $h^\#$ agrees with h_i in a one-sided neighborhood of ∂R_i , on the side outside R_i , $1 \leq i \leq k$, and the isotopy was constant in such a neighborhood, $h^\#$ and the h_i combine to give a diffeomorphism which we again denote by $h^\# : L \rightarrow L$, isotopic to h^g and agreeing with h_i in a neighborhood of each R_i . Since $h^\#$ agrees with h^g in a neighborhood of $\partial L'$ and is isotopic to h^g , we can apply the smooth versions of Theorems 3.1 and 3.2 (cf. Theorem 3.3) to smoothly isotope $h^\#$ to a diffeomorphism h that agrees with the permutation induced by h^g on the components of $|\mathcal{J}_i^-|$, $i \leq N$,

and of $|\mathcal{J}_i^+|$, $i \geq 0$ thus preserving the structure we set up in Lemma 11.5. Again we work in L' and the isotopies are supported away from $\partial L'$. Evidently, h is endperiodic and the h^g -junctures making up $\text{Fr } K$ propagate under applications of h^n , $n \in \mathbb{Z}$. In Section 11.4 we will take these to be the set of all h -junctures. We summarize.

Lemma 11.12. *The diffeomorphism $h : L \rightarrow L$ is endperiodic, isotopic to h^g and agrees with h_i on a neighborhood of R_i , $1 \leq i \leq k$. Furthermore, if τ^g is a component of $|\mathcal{J}_i^-|$, $i \leq N$, or of $|\mathcal{J}_i^+|$, $i \geq 0$, then $h(\tau^g) = h^g(\tau^g)$.*

Remark. In particular, since $h|R_i = h_i$, we have guaranteed that h satisfies the inequalities of Proposition 11.10 on R_i , $1 \leq i \leq k$.

11.4. Proof of Theorem 11.1 in the case of no principal regions. We define the sets of positive/negative juncture components \mathcal{J}_\pm . Let \mathcal{J}_\pm be the set of arcs and circles $h^k(\gamma)$, $k \in \mathbb{Z}$, $\gamma \in \mathcal{J}_0^\pm$. Let $\mathcal{X}_\pm \subset \mathcal{J}_\pm$ be the set of nonescaping juncture components. Since the components of the junctures are either essential circles or properly embedded arcs, the following is evident.

Lemma 11.13. *The components of the junctures are pseudo-geodesics.*

Since the juncture components in \mathcal{J} are isotopic to the corresponding geodesic juncture components in \mathcal{J}^g the following is evident.

Lemma 11.14. *There is a bijection $\iota : \mathcal{N} \rightarrow \mathcal{J}$ where \mathcal{N} is the set of f -juncture components fixed on page 92.*

It is now necessary to construct the smooth, h -invariant laminations and verify the axioms. Remark that

$$R_{i_n} \cap h^g(R_{n_{i+1}}) = R_{i_n} \cap h(R_{i_{n+1}}), \quad n \geq 0.$$

Thus h and h^g define the same Markov chains $\iota = (i_0, i_1, i_2, \dots, i_n, \dots)$. In analogy with the construction of ℓ_i^g (Proposition 11.7), replacing h^g with h , we define ℓ_i by a Markov chain ι . That this is a curve requires proof.

Lemma 11.15. *The set ℓ_i is a one-one immersed smooth curve in L for each Markov chain ι .*

Proof. Set

$$R_{i_0 i_1 \dots i_n} = R_{i_0} \cap h(R_{i_1}) \cap h^2(R_{i_2}) \cap \dots \cap h^n(R_{i_n}).$$

Iterated applications of the first inequality in Proposition 11.10 to $h : R_{i_n} \rightarrow h(R_{i_n})$ show that the (geodesic) quadrilaterals in the nested sequence

$$R_{i_0} \supset R_{i_0 i_1} \supset \dots \supset R_{i_0 i_1 \dots i_n} \supset \dots$$

have widths decreasing monotonically to 0. Thus, the intersection of this nested sequence is a single leaf ℓ_{i_0} of $\mathcal{F}_v|R_{i_0}$. Likewise, the Markov chain (i_1, i_2, \dots) defines a leaf ℓ_{i_1} of $\mathcal{F}_v|R_{i_1}$ and the smooth, embedded arc $h(\ell_{i_1}) \subset L$ contains ℓ_{i_0} as a proper subarc. Set

$$\ell_{i_n} = R_{i_n} \cap h(R_{i_{n+1}}) \cap \dots$$

and obtain a strictly increasing nest

$$\ell_{i_0} \subset h(\ell_{i_1}) \subset h^2(\ell_{i_2}) \subset \dots \subset h^n(\ell_{i_n}) \subset \dots$$

of smoothly embedded curves. Since h is endperiodic, the two sequences of bottom and top edges of the quadrilaterals $\{h^n(R_{i_n})\}_{n \geq 0}$ both escape and the union ℓ_i of these arcs is a one-one, smoothly immersed copy of \mathbb{R} . \square

Remark. Since h is only a diffeomorphism, not a hyperbolic isometry, ℓ_i is not generally a geodesic, although it will repeatedly cross the quadrilaterals R_i and $h(R_j)$ in geodesic arcs which are leaves of \mathcal{F}_v .

Remark. The curves ℓ_i are in one-one correspondence with the curves $\ell_i^\mathcal{G} \in \Lambda_\pm^\mathcal{G}$ of Proposition 11.7, each being uniquely determined by the Markov chain ι . This is crucial for proving that the bilamination (Λ_+, Λ_-) that we construct satisfies the axioms. Note that, despite the notation $\ell_i^\mathcal{G}$, we have not yet proven that this curve is the geodesic tightening of ℓ_i . This will be proven in Lemma 11.23. Meanwhile, the correspondence is via the index ι .

Let $\Gamma_\pm = \Lambda_\pm \cup \mathcal{X}_\mp$.

Lemma 11.16. (Λ_+, Λ_-) and (Γ_+, Γ_-) are smooth bilaminations.

Proof. Let $\iota = (i_0, i_1, \dots, i_n, \dots)$ be a sequence with each $i_n \in \{1, 2, \dots, k\}$ and with the property that $R_{i_n} \cap h(R_{i_{n+1}}) \neq \emptyset$, for all $n \geq 0$. Since $h : \mathcal{R} \rightarrow h(\mathcal{R})$ is a diffeomorphism and preserves the smooth foliations $(\mathcal{F}_h, \mathcal{F}_v)$, every leaf ℓ_i of Λ_+ intersects a geodesic quadrilateral R_i , if at all, in plaques of \mathcal{F}_v . Via h , we define $(\mathcal{F}_h^{i_n}, \mathcal{F}_v^{i_n})$ on each $h^n(R_{i_n})$, $n \geq 0$, and remark that $\ell_i \cap h^n(R_{i_n})$ is a set of plaques of $\mathcal{F}_v^{i_n}$ in $h^n(R_{i_n})$. Since $\mathcal{F}_v^{i_n}$ and $\mathcal{F}_h^{i_n}$ are transverse smooth foliations of $h^n(R_{i_n})$, we see that the interiors of the sets $h^n(R_{i_n})$ are laminated charts for Λ_+ belonging to the maximal smooth atlas of L . That is, we have produced a smooth partial laminated atlas for Λ_+ . Similarly, we produce a smooth partial laminated atlas for Λ_- . Since the smooth foliations in each of our charts are transverse, one containing all plaques of Λ_+ meeting the chart and the other containing all plaques of Λ_- meeting the chart, we have produced a smooth partial bilaminated atlas \mathfrak{A} for (Λ_+, Λ_-) .

By the definition of \mathcal{X}_\pm we see that the intersection of any of its leaves with a chart in \mathfrak{A} is also a plaque of the appropriate foliation. That is, \mathfrak{A} consists of smooth bilaminated charts for (Γ_+, Γ_-) , but does not cover $|\Gamma_+| \cup |\Gamma_-|$. Any point of $|\Gamma_+| \cup |\Gamma_-|$ outside of $\bigcup_{U \in \mathfrak{A}} U$ either lies on a unique leaf of \mathcal{X}_+ or on a unique leaf of \mathcal{X}_- or is a point of intersection of a leaf of \mathcal{X}_+ and a leaf of \mathcal{X}_- . In all of these cases, constructing a smooth, bilaminated chart about x for (Γ_+, Γ_-) is easy since the leaves of \mathcal{X}_\pm are isolated from each other and from $|\Lambda_+| \cup |\Lambda_-|$. This gives a smooth, partial bilaminated atlas for (Γ_+, Γ_-) . \square

The pair $(\Lambda_+ \cup \mathcal{J}_-, \Lambda_+ \cup \mathcal{J}_+)$ may not be a bilamination. In fact, an escaping component of a positive juncture, if any, would coincide with an escaping component of a negative juncture. However, the following is an obvious corollary.

Corollary 11.17. $\Lambda_+ \cup \mathcal{J}_-$ and $\Lambda_- \cup \mathcal{J}_+$ are each smooth laminations.

Let \mathcal{U}_+ (respectively \mathcal{U}_-) denote the positive (respectively negative) escaping set for h . That is, $x \in \mathcal{U}_+$ if and only if $\{h^n(x)\}_{n \geq 0}$ escapes and $x \in \mathcal{U}_-$ if and only if $\{h^n(x)\}_{n \leq 0}$ escapes. These sets are open and each component of \mathcal{X}_\pm lies in \mathcal{U}_\pm . In what follows, we use $\mathcal{C}X$ to denote the complement in L of a subset X .

Lemma 11.18. $\mathcal{C}\mathcal{U}_\pm = |\Lambda_\mp|$.

Proof. We prove that $\mathcal{C}\mathcal{U}_- = |\Lambda_+|$. The proof that $\mathcal{C}\mathcal{U}_+ = |\Lambda_-|$ is similar. Let $x \in \mathcal{C}\mathcal{U}_-$. Then no point in the orbit of x is in \mathcal{U}_- and, by applying a large enough negative power of h , we can assume that $x \in K$. For all $n \geq 0$, $x \notin W_{N+n} \subset \mathcal{U}_-$.

(Recall the definition of the sets W_n just before Definition 11.4.) Thus, for each $n \geq 0$, there is a unique $1 \leq i_n \leq k$ such that $x \in h^n(R_{i_n}) \cap K$. Consequently, $x \in \ell_\iota \in \Lambda_+$ where $\iota = (i_0, i_1, \dots, i_n, \dots)$. Thus, $\mathcal{CU}_- \subseteq |\Lambda_+|$ and the reverse inclusion is obvious. \square

Lemma 11.19. *The frontier of \mathcal{U}_\mp is exactly $|\Lambda_\pm|$ and $|\overline{\mathcal{X}_\mp}| \setminus |\mathcal{X}_\mp| = |\Lambda_\pm|$.*

Proof. By Lemma 11.18, $\mathcal{U}_- \cap |\Lambda_+| = \emptyset$. On the other hand, if $x \in |\Lambda_+|$, then it lies in a segment of a leaf ℓ_ι in $h^k(R_{i_k})$ for arbitrarily large values of k . Consequently, every neighborhood of x meets a leaf of \mathcal{X}_- , hence meets \mathcal{U}_- . Thus $|\Lambda_+| \subseteq \text{Fr } \mathcal{U}_-$. By Lemma 11.18, it follows that $|\Lambda_+| = \text{Fr } \mathcal{U}_-$. A similar argument proves that $|\Lambda_-| = \text{Fr } \mathcal{U}_+$. For the second assertion, the definition of Λ_+ makes it clear that $|\Lambda_+| \subseteq |\overline{\mathcal{X}_-}| \setminus |\mathcal{X}_-|$. For the reverse inclusion, let $\{x_n\}_{n=1}^\infty$ be a sequence of points in distinct leaves of \mathcal{X}_- converging to a point $x \in L$. Since the negative h -junctures do not accumulate in \mathcal{U}_- , $x \notin \mathcal{U}_-$ and so $x \in \text{Fr } \mathcal{U}_- = |\Lambda_+|$. \square

Corollary 11.20. *The laminations Λ_\pm and Γ_\pm are closed.*

Proof. Since the frontier of a set is a closed set it follows that the laminations Λ_\pm are closed laminations. Further, the closure of $|\mathcal{X}_\mp|$ is exactly $|\Lambda_\pm| \cup |\mathcal{X}_\mp| = |\Gamma_\pm|$ so the lamination Γ_\pm is a closed lamination. \square

By construction, each leaf of \mathcal{X}_\pm is homotopic to a unique leaf of \mathcal{X}_\pm^g . In fact, the components of junctures in distinguished neighborhoods of ends are the same.

Lemma 11.21. *The leaves of Λ_+ cannot intersect those of \mathcal{J}_+ so as to form digons. The leaves of Λ_- cannot intersect those of \mathcal{J}_- so as to form digons.*

Proof. First note that, if a leaf of Λ_+ intersects a leaf of \mathcal{J}_+ so as to form a digon, then a leaf of Λ_+ intersects $\text{Fr}_+ K$ so as to form a digon. Just apply a suitable power of h . But this would imply that an edge of ∂R_i or of $\partial h(R_i)$ would intersect $\text{Fr}_+ K$ so as to form a digon, some $i \in \{1, 2, \dots, k\}$. Since these geodesic quadrilaterals are the same as the ones for the geodesic laminations, this would contradict Axiom 4 for the geodesic laminations. A similar proof works for the laminations Λ_- and \mathcal{J}_- . \square

Corollary 11.22. *The leaves of Λ_+ cannot intersect those of Λ_- so as to form digons.*

Proof. If a leaf ℓ_+ of Λ_+ intersects a leaf ℓ_- of Λ_- so as to form a digon, then a suitable locally uniform approximation of ℓ_+ by leaves of \mathcal{X}_- will give a leaf of \mathcal{X}_- that intersects ℓ_- so as to form a digon. \square

The leaves of Λ_\pm correspond one-one with the leaves of Λ_\pm^g by $\ell_\iota \leftrightarrow \ell_\iota^g$ and the leaves of \mathcal{X}_\pm correspond one-one to those of \mathcal{X}_\pm^g by homotopy. This establishes a canonical one-one correspondence $\Gamma_\pm \leftrightarrow \Gamma_\pm^g$. Denote this correspondence by $\gamma \leftrightarrow \gamma^g$. The following shows that, in conformity with the notation of Section 10, γ^g is the geodesic tightening of γ .

Lemma 11.23. *The laminations Γ_\pm are pseudo-geodesic and, under the correspondence $\gamma \leftrightarrow \gamma^g$, the completed lifts $\hat{\gamma}$ and $\hat{\gamma}^g$ have the same endpoints on S_∞ .*

Proof. By Lemma 11.13, the leaves of \mathcal{X}_\pm are pseudo-geodesics. Let

$$\iota = (i_0, i_1, i_2, \dots, i_n, \dots)$$

be a Markov chain. Let $\delta_0 = \delta_0^g$ and inductively define the edge $\delta_n \subset h^n(\text{Fr}_+ K)$ of $h^n(R_{i_n})$ exactly as the edge δ_n^g was defined in the proof of Proposition 11.7. The lift $C_0 = C_0^g = \tilde{R}_{i_0}$ of R_{i_0} to \tilde{L} determine the lifts C_n of $h^n(R_{i_n})$ with edge $\tilde{\delta}_n$ a lift of δ_n . In fact δ_n and δ_n^g are in the same lift $\tilde{\sigma}_n^g$ of a component of $h^n(\text{Fr}_+ K)$ from which it follows that ℓ_i and ℓ_i^g share the endpoint a . Similarly, starting with the other edge $\gamma_0 = \gamma_0^g \subset \text{Fr}_+ K$ of R_{i_0} one shows that ℓ_i and ℓ_i^g share the endpoint b . Thus ℓ_i is a pseudo-geodesic sharing endpoints with ℓ_i^g . \square

At this point we have verified that (Γ_+, Γ_-) is a smooth bilamination preserved by the smooth endperiodic diffeomorphism h . It remains to verify that (Λ_+, Λ_-) is a Handel-Miller pseudo-geodesic bilamination associated to h (and, thus, to h^g), that is that (Λ_+, Λ_-) satisfies the axioms:

Axiom 1. By Lemma 11.16, (Λ_+, Λ_-) is a bilamination. By Corollary 11.20, (Λ_+, Λ_-) is closed. By Lemma 11.23, the elements of Λ_\pm are pseudo-geodesics. The fact that the leaves of Λ_\pm are disjoint from ∂L is obvious from the construction.

Axiom 2. By Corollary 11.22, the leaves of $\tilde{\Lambda}_+$ and $\tilde{\Lambda}_-$ can only intersect in a single point.

Axiom 3. By Lemma 11.23, the bilamination (Λ_+, Λ_-) has the endpoint correspondence property with respect to f .

Axiom 4. By Lemma 11.14 there exists a set \mathcal{N} of f -juncture components and bijection $\iota : \mathcal{N}_\pm \rightarrow \mathcal{J}_\pm$. By Lemmas 11.13 and 11.23, $\Lambda_\pm \cup \mathcal{J}_\mp$ are sets of pseudo-geodesics. By Corollary 11.17, the elements of each of the sets $\Lambda_\pm \cup \mathcal{J}_\mp$ are disjoint. Item (1) follows. Item (2) follows since, by construction, the elements of Λ_\pm are transverse to the elements of \mathcal{J}_\mp . By Lemma 11.21, no element of \mathcal{J}_\pm meets an element of Λ_\pm to form a digon. Thus, item (3) is true.

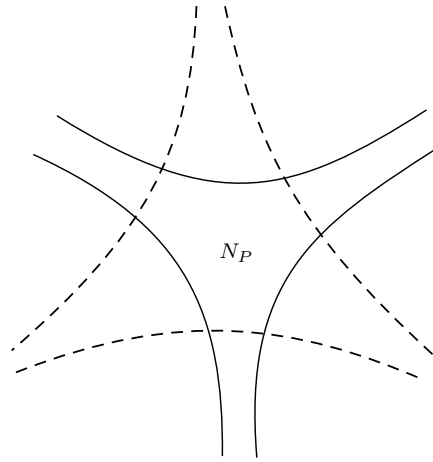
11.5. Proving Theorem 11.1 if there are principal regions. We will sketch the way the previous argument needs to be modified, leaving many details for the reader.

Let \mathcal{P} denote the union of the positive principal regions. For $N \geq 1$ large enough, the components of $K \setminus (W_N \cup \mathcal{P})$ are geodesic quadrilaterals, each having a pair of opposite edges that are subarcs of $\text{Fr}_+ K$. But the other pair of opposite edges are either both subarcs of \mathcal{J}_N^- or one is a subarc of \mathcal{J}_N^- and the other a subarc of a border leaf of a positive principal region. For the structure of principal regions and their nuclei, the reader may wish to review Section 6.4.

Let P be a positive principal region. It has a dual negative principal region P' which shares a compact nucleus $N_P = N_{P'}$ with P . See Figure 28 where the dashed curves are border curves of P' , the solid ones border curves of P .

Actually, Figure 28 depicts only the parts of P and P' including one boundary curve γ of N_P . There may be finitely many distinct components of ∂N_P . The curve γ is made up alternately of n_γ subarcs of δP and n_γ subarcs of $\delta P'$, n_γ being a positive integer depending on γ , not on P alone. In the figure we draw the case $n_\gamma = 3$. Out of γ there radiate n_γ simply connected ‘‘arms’’ of P and n_γ such arms of P' . Since we are in the case of geodesic laminations, the lifts of these arms to \tilde{L} are cusps.

The set $(P \setminus \text{int } N_P) \cap K$ may have infinitely many components, all but finitely many of which are geodesic quadrilateral components of the intersection of arms of P with K . The nonquadrilateral components of $(P \setminus \text{int } N_P) \cap K$ are each equal

FIGURE 28. The dual principal regions P and P' .

to the union of one of the components γ of ∂N_P and the “stumps” of the n_γ arms (cf. Definition 11.24). In Figure 29, we have drawn the nucleus N_P (shaded) and stumps of P cut off by one of the curves γ with $n_\gamma = 3$. The boundary curve γ of N_P is drawn in boldface.

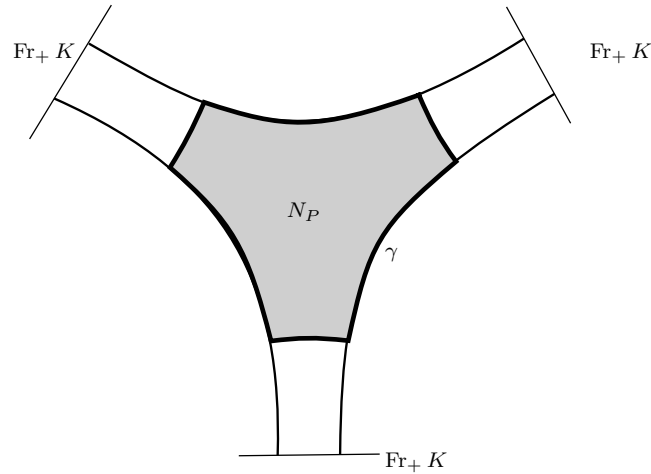


FIGURE 29. The nucleus and stumps of a crown set.

Definition 11.24 (stump). If P is a positive principal region, we will call the rectangular portion of an arm of P that is between the nucleus and $\text{Fr}_+ K$ the *stump* of the arm. The stump is bounded by the arc of $|\Lambda_-|$ separating the stump from the nucleus, two arcs of $|\Lambda_+|$, and an arc of $\text{Fr}_+ K$. It is simply connected. The stumps of the arms of the dual negative principal region P' are defined similarly.

In Figure 30, we have attached the $n_\gamma = 3$ geodesic quadrilaterals R_i^γ , each with one edge in one of the three depicted border leaves of P , the opposite edges

being arcs in \mathcal{J}_N^- . The other pair of opposite edges lie in $\text{Fr}_+ K$. These geodesic quadrilaterals and the stumps unite to produce an annulus A_γ which we have shaded. With dotted lines representing arcs in the border of P' we have cut off three other geodesic subquadrilaterals of A_γ which will be denoted as Q_j^γ . Note that the R_i^γ 's are pairwise disjoint, as are the Q_j^γ 's, but each R_i^γ overlaps two Q_j^γ 's.

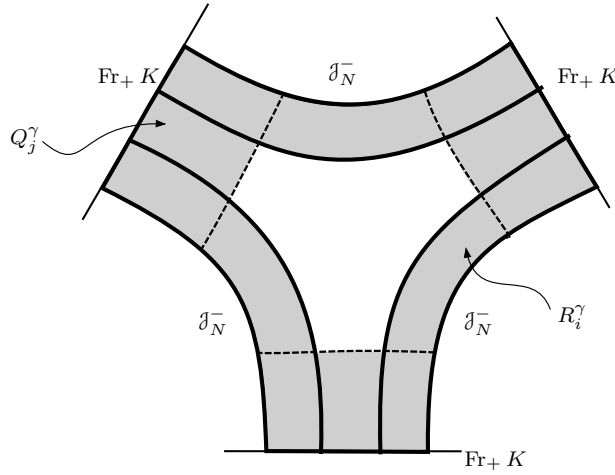


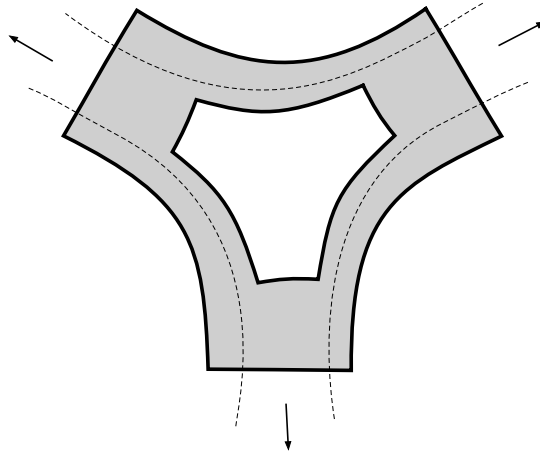
FIGURE 30. The annulus A_γ

As γ varies over boundary components of the nuclei of the finitely many principal regions in \mathcal{P} , we get a family of geodesic quadrilaterals R_i^γ , $1 \leq i \leq n_\gamma$. There are also the “normal” geodesic quadrilateral components of $K \setminus (W_N \cup \mathcal{P})$ with a pair of opposite edges in \mathcal{J}_N^- . These latter are simply indexed R_i , $1 \leq i \leq k$ as before. The geodesic quadrilaterals of both types have a pair of opposite edges that are subarcs of $\text{Fr}_+ K$. As before, if N has been chosen large enough, the h^g -images of these geodesic quadrilaterals have connected intersection (possibly empty) with each of R_i and R_i^γ , completely crossing any of these latter geodesic quadrilaterals that they meet.

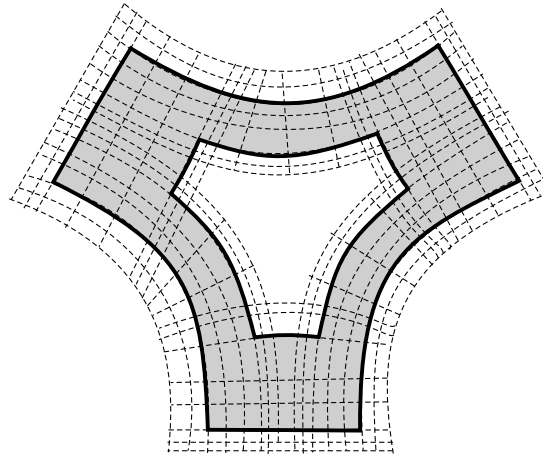
Remark. The reader should give some thought to the family of geodesic quadrilaterals consisting of Q_j^γ , $1 \leq j \leq n_\gamma$ and R_j , $1 \leq j \leq k$. These are pairwise disjoint and each has connected intersection with the geodesic quadrilaterals $h^g(Q_i^\gamma)$ and $h^g(R_i)$, completely crossing whichever of the latter that it meets. This is entirely analogous to the situation without principal regions and will be used to generate the lamination Λ_- .

While it is clear how the h^g -images of geodesic quadrilaterals intersect geodesic quadrilaterals, one also should note that there is often an annular component of $A_\gamma \cap h^g(A_{\gamma'})$, for boundary circles γ and γ' of nuclei, as indicated in Figure 31, where the dotted lines together with $\gamma = h^g(\gamma')$ bound $h^g(A_{\gamma'})$.

Set \mathcal{R} equal to the union of all R_i and all A_γ . In analogy with the former construction, we define a pair of transverse, smooth geodesic foliations \mathcal{F}_h and \mathcal{F}_v on $\mathcal{R} \cup h^g(\mathcal{R})$. The foliation \mathcal{F}_h incorporates the edges in $\text{Fr}_+ K$ and their h^g -images as leaves. The foliation \mathcal{F}_v incorporates the edges in \mathcal{J}_N^- and in $\delta\mathcal{P}$ and their h^g -images. Furthermore, the geodesic quadrilaterals and annuli and their h^g -images

FIGURE 31. An annular component of $A_\gamma \cap h^g(A_{\gamma'})$.

can be fattened to obtain a neighborhood V of $\mathcal{R} \cup h^g(\mathcal{R})$ over which the foliations are extended. Details, which are analogous to the case in which there are no principal regions, are left to the reader. Figure 32 depicts the foliated neighborhood of one of the annuli.

FIGURE 32. Geodesic foliations of a neighborhood of A_γ .

This smooth geodesic grid is again used to define a diffeomorphism which we will denote by $h_{\mathcal{R}} : \mathcal{R} \rightarrow h^g(\mathcal{R})$ and will extend over an open neighborhood of \mathcal{R} . Notice that the geodesic quadrilaterals Q_j^γ are also in the domain of $h_{\mathcal{R}}$. If R is any of the geodesic quadrilaterals R_i , R_i^γ , or Q_i^γ , the grid defines smooth coordinates (x, y) on R . Here, the leaves of \mathcal{F}_h are level sets of x and the leaves of \mathcal{F}_v are level sets of y . On overlapping geodesic quadrilaterals of $\mathcal{R} \cup h^g(\mathcal{R})$ we can again choose the coordinates so that the first derivatives of the change of coordinate functions

are ± 1 . The diffeomorphism $h_{\mathcal{R}}|_R$ has the form $h_{\mathcal{R}}(x, y) = (\xi(x), \zeta(y))$. As in Proposition 11.10, ξ and ζ can be chosen so that the following are true.

Proposition 11.25. *On $R = R_i$, $1 \leq i \leq k$,*

$$0 < |d\xi/dx| < c < 1, \quad |d\zeta/dy| > b > 1.$$

On $R = R_i^\gamma$, the first of these inequalities holds and on $R = Q_i^\gamma$ the second holds.

Lemma 11.26. *The homeomorphism $h^{\mathfrak{g}}$ is isotopic to a homeomorphism which agrees with $h^{\mathfrak{g}}$ outside a neighborhood of \mathcal{R} and with $h_{\mathcal{R}}$ inside that neighborhood.*

Proof. For each geodesic quadrilateral R_i , $1 \leq i \leq k$, and R_i^γ this is proven exactly as before. On A_γ , there remain geodesic quadrilaterals D in each stump where the isotopy has not been defined. Looking carefully at Figure 32, one sees that an application of Theorem 3.2 isotopes $h^{\mathfrak{g}}$ to agree with $h_{\mathcal{R}}$ on the boundary of a geodesic quadrilateral in V which covers D , this without affecting the isotopies already performed. An application of Alexander’s trick completes the isotopy. \square

We therefore assume that $h^{\mathfrak{g}}$ agrees with the diffeomorphism $h_{\mathcal{R}}$ in a neighborhood of \mathcal{R} . One now proceeds exactly as before, using [30] and the smooth versions of Theorem 3.1 and 3.2, to prove the following.

Lemma 11.27. *There is an isotopy of $h^{\mathfrak{g}}$, which is constant in a neighborhood of \mathcal{R} and on ∂L , to a diffeomorphism $h : L \rightarrow L$ such that $h|_{\tau^{\mathfrak{g}}} = h^{\mathfrak{g}}|_{\tau^{\mathfrak{g}}}$, where $\tau^{\mathfrak{g}}$ is a component of \mathcal{J}_i^- , $i \leq N$, or of \mathcal{J}_i^+ , $i \geq 0$. In particular, h is endperiodic.*

At this point all of the machinery has been set up for defining Λ_+ using the geodesic quadrilaterals R_i^γ and R_i and positive powers of h . One similarly produces Λ_- using the geodesic quadrilaterals R_i and Q_i^γ and negative powers of h . Throwing in the nonescaping components of junctures gives the smooth laminations Γ_\pm and verification of the axioms proceeds essentially as before. Theorem 11.1 is proven.

12. THE TRANSFER THEOREM

The principal applications of Handel-Miller theory are to the endperiodic automorphisms that arise in foliations of 3-manifolds by surfaces. See, for example, [22, 24]. We introduce here the basics, setting the stage for such applications as in [12, 16]. The main new result in this section will be Theorem 12.7, the “transfer theorem”, which is absolutely fundamental to [12, 16].

12.1. Depth one foliations. Let M be a compact, connected sutured 3-manifold with boundary. We do not require M to be orientable, but we will only be considering smooth, codimension 1 foliations \mathcal{F} of M which are transversely orientable. The boundary of M will decompose into

$$\partial M = \partial_\tau M \cup \partial_\eta M,$$

where the components of $\partial_\tau M$ (the *tangential* boundary) are compact leaves of \mathcal{F} and $\partial_\eta M$ (the *transverse* boundary) is transverse to \mathcal{F} . The components of $\partial_\eta M$ must be annuli, tori and/or Klein bottles. Möbius strips are ruled out by the transverse orientability of \mathcal{F} . The annuli will meet $\partial_\tau M$ along corners that are convex with respect to the foliation.

Definition 12.1 (taut foliation). The foliation \mathcal{F} is taut if each leaf meets either a closed transversal to \mathcal{F} or an arc transverse to \mathcal{F} with endpoints in $\partial_\tau M$.

Taut foliations are very important in the theory of 3-manifold topology. For instance, the foliation can have no Reeb components, hence well known theorems of S. P. Novikov [35] imply that the leaves are π_1 -injective and no closed transversal to \mathcal{F} is null homotopic. It follows easily that the lift $\tilde{\mathcal{F}}$ of \mathcal{F} to the universal cover \tilde{M} has all leaves simply connected and, unless $M = S^2 \times S^1$ with the product foliation, that \tilde{M} itself is contractible.

The foliation \mathcal{F} induces foliations of the components of $\partial_{\text{th}}M$. Even when \mathcal{F} is taut, $\mathcal{F}|_{\partial_{\text{th}}M}$ might have 2-dimensional Reeb components. We need to rule this out also.

Set $M^\circ = M \setminus \partial_\tau M$.

Definition 12.2 (depth one foliation). The foliation \mathcal{F} is depth one if $\mathcal{F}|_{M^\circ}$ is a fibration of M° over S^1 .

Here it is possible that $M = F \times [-\infty, \infty]$ and the foliation is transverse to the compact interval fibers. Such foliations, called foliated products of depth one, are well understood and easily classified.

Remark. Since \mathcal{F} is transversely oriented, some of the components of $\partial_\tau M$ are oriented transversely into M and some out of M . The annular components of $\partial_{\text{th}}M$ separate inwardly oriented components of $\partial_\tau M$ from outwardly oriented ones. As \mathcal{F} varies, we keep the transverse orientations of the components of $\partial_\tau M$ fixed. This then is a constraint on the transverse orientations of these foliations. In the case that M is orientable, the structure $(M, \partial_\tau M, \partial_{\text{th}}M)$ is a *sutured manifold* in the sense of D. Gabai [26]. In Gabai's notation, the sutured manifold is denoted by (M, γ) , $\partial_{\text{th}}M = \gamma = A(\gamma) \cup T(\gamma)$ and $\partial_\tau M = R(\gamma) = R_+(\gamma) \cup R_-(\gamma)$. Here, $A(\gamma)$ is the union of the annular components of $\partial_{\text{th}}M$, $T(\gamma)$ the union of the toroidal components, $R_+(\gamma)$ is the union of the outwardly oriented components of $\partial_\tau M$ and $R_-(\gamma)$ the union of the inwardly oriented ones.

12.2. Monodromy. It is possible to find a 1-dimensional foliation \mathcal{L} transverse to \mathcal{F} . For instance, construct a smooth, nowhere zero vector field v on M , transverse to the leaves of \mathcal{F} , tangent to $\partial_{\text{th}}M$ and oriented coherently with the transverse orientation of \mathcal{F} . This is easily done locally and the local fields are assembled into a global one by a suitable smooth partition of unity. The integral curves to v are the leaves of \mathcal{L} . This transverse foliation is smooth. It is also possible to construct transverse foliations \mathcal{L} that are only C^0 . In any event, \mathcal{L} is oriented by the transverse orientation of \mathcal{F} .

If $\partial_{\text{th}}M$ has no 2-dimensional Reeb components, the foliation $\mathcal{L}|_A$ induced on annular components A of $\partial_{\text{th}}M$ can always be taken to be the product foliation by compact intervals.

Suppose \mathcal{F} is a depth one foliation of M (Definition 12.2) and let L be a non-compact leaf of \mathcal{F} . That is, L is a fiber of the fibration $M^\circ \rightarrow S^1$ induced by \mathcal{F} . For each point $x \in L$, let ℓ_x be the leaf of \mathcal{L} through x . Follow ℓ_x starting at x in the direction given by its orientation until the first time a point $y \in L$ is encountered. The fact that such a point $y \in L$ exists for each $x \in L$ follows from the compactness of M [23, Lemma 4.2]. The map $x \rightarrow y$ defines the first return map

$$f : L \rightarrow L.$$

If \mathcal{L} is smooth, f is a diffeomorphism. If \mathcal{L} is only C^0 , f is a homeomorphism.

Definition 12.3 (monodromy). The automorphism $f : L \rightarrow L$ is called the monodromy induced by \mathcal{L} .

Sometimes we will denote \mathcal{L} by \mathcal{L}_f . One should note, however, that while f is determined by \mathcal{L}_f , distinct transverse foliations may determine the same monodromy.

12.2.1. *Endperiodic monodromy.*

Lemma 12.4. *Let M be a compact, connected sutured 3-manifold and \mathcal{F} a taut, transversely oriented, depth one foliation of M which induces no 2-dimensional Reeb components on $\partial_{\text{th}}M$. A leaf L of $\mathcal{F}|M^\circ$ has only finitely many ends and the monodromy $f : L \rightarrow L$, defined by any transverse, 1-dimensional foliation \mathcal{L} , is an endperiodic automorphism.*

To prove Lemma 12.4, for any end $e \in \mathcal{E}(L)$, one constructs the neighborhood U_e of e exactly as in the proof of Proposition 2.12 in Section 2.1.

12.2.2. *Realizing endperiodic monodromy.*

Lemma 12.5. *Given an endperiodic automorphism $f : L \rightarrow L$ of a surface with finitely many ends, there exists a compact 3-manifold M , a depth one foliation \mathcal{F} of M and a transverse, 1-dimensional foliation \mathcal{L} such that L is homeomorphic to each noncompact leaf of \mathcal{F} and f is the monodromy induced by \mathcal{L} . If f is a diffeomorphism, M , \mathcal{F} and \mathcal{L} are smooth. If f is only a homeomorphism, M and \mathcal{F} are smooth, but generally \mathcal{L} is not.*

The proof of Lemma 12.5 is standard. One first constructs the suspension of f ,

$$M^\circ = L \times [0, 1] / \{(x, 1) \equiv (f(x), 0)\}.$$

The fact that f is endperiodic, then enables one to glue on the compact leaves that form $\partial_\tau M$.

The annular components of $\partial_{\text{th}}M$ can arise in two ways. There might be an infinite sequence of compact components of ∂L contained in \mathcal{U} permuted simply transitively by f or a finite set of compact components of ∂L contained in the set of principal regions permuted cyclically by f . The \mathcal{L} -saturation of any one of these circles contains the others and is an annular component of $\partial_{\text{th}}M$. There might be a necessarily finite set of noncompact components of ∂L permuted cyclically by f . The \mathcal{L} -saturation of any of these contains the others and again, since there are no periodic points on such a component, we obtain an annular component of $\partial_{\text{th}}M$. In these annuli, the leaves of \mathcal{L} are compact intervals, hence \mathcal{F} induces no 2-dimensional Reeb components. The tori and/or Klein bottles in $\partial_{\text{th}}M$ are induced similarly by finite f -cycles of compact components of ∂L .

If \mathcal{F} induces no 2-dimensional Reeb components on $\partial_{\text{th}}M$, each noncompact component of ∂L joins a negative end to a positive end and the monodromy of \mathcal{F} has no fixed points on ∂L . If every component of $\partial_\tau M$ has negative Euler characteristic, L is an admissible surface. If \mathcal{F} is not a foliated product, the monodromy of \mathcal{F} is not isotopic to a translation. Thus the theory of Sections 2 - 11 applies to the monodromy of a depth one leaf of a foliation satisfying Hypothesis 7.

Hypothesis 7. Hereafter, unless we explicitly state otherwise, M is a compact, connected sutured 3-manifold with boundary, the foliation \mathcal{F} is smooth, transversely oriented, taut, depth one, not a foliated product,

induces no 2-dimensional Reeb components on $\partial_{\text{th}}M$, and every component of $\partial_{\tau}M$ has negative Euler characteristic.

12.3. The Transfer Theorem. We now turn to the main goal of Section 12. Suppose M and \mathcal{F} satisfies Hypothesis 7.

Let $f : L \rightarrow L$ be the monodromy associated to a transverse, 1-dimensional foliation \mathcal{L}_f . Fix any standard hyperbolic metric \mathfrak{g} on L and recall that, if (Λ_+, Λ_-) is a Handel-Miller pseudo-geodesic bilamination of L associated to f relative to this metric, it is Handel-Miller associated to f relative to any other choice of standard hyperbolic metric (Corollary 10.16). Thus the following definition is independent of the choice of \mathfrak{g} .

Definition 12.6 (Handel-Miller monodromy). Suppose that \mathcal{L} is a 1-dimensional foliation transverse to \mathcal{F} which induces endperiodic monodromy $h : L \rightarrow L$, isotopic to f and preserving a Handel-Miller pseudo-geodesic bilamination (Λ_+, Λ_-) associated to f . Then we say that h is a Handel-Miller monodromy for the depth one foliation \mathcal{F} .

Recall that the bilamination (Λ_+, Λ_-) is called a Handel-Miller pseudo-geodesic bilamination (Λ_+, Λ_-) associated to f if it satisfies the four axioms (Definitions 10.7).

The following is analogous to a result of Fried [21, p. 261] for pseudo-Anosov monodromy in fibrations of hyperbolic 3-manifolds and is key to proving the maximality of foliation cones defined by Handel-Miller monodromy (see Proposition 13.12).

Theorem 12.7 (Transfer Theorem). *Let \mathcal{L} be a 1-dimensional foliation transverse to \mathcal{F} and inducing Handel-Miller monodromy $h : L \rightarrow L$ on a depth one leaf L . If \mathcal{F}' is another depth one foliation transverse to \mathcal{L} and satisfying Hypothesis 7, then \mathcal{L} induces Handel-Miller monodromy $h' : L' \rightarrow L'$ on any depth one leaf L' of \mathcal{F}' .*

The proof of the transfer theorem is now our goal. We first note that by Lemma 12.4, the monodromy $h' : L' \rightarrow L'$ is an endperiodic automorphism. We show how to produce an h' -invariant bilamination (Λ'_+, Λ'_-) by transferring the bilamination (Λ_+, Λ_-) from L to L' along \mathcal{L} and then verify that this satisfies the axioms for a Handel-Miller pseudo-geodesic bilamination associated to h' .

12.4. Transferring paths. Given a parametrized path $s : [a, b] \rightarrow L$, there are countably many “transferred” parametrized paths $s' : [a, b] \rightarrow L'$ which are uniquely determined by $s'(a)$ and are obtained by projecting locally along the leaves of \mathcal{L} . This is a standard sort of local continuation process. Likewise, a parametrized path $s : (-\infty, \infty) \rightarrow L$ transfers to parametrized paths $s' : (-\infty, \infty) \rightarrow L'$, each uniquely determined by any one of its values. All of this works equally well for transfers of paths from L' to L and transfers of paths from L to L and from L' to L' . A choice of parametrization is convenient in defining the transferred paths, but the unparametrized paths underlying the parametrized transferred paths are independent of the choice. Thus, we will normally view the transfer operation as having to do with unparametrized paths.

Remark. If s is a loop it is quite possible that some or all of its transferred paths open up.

Definition 12.8 (transfer of a path). The set of paths obtained by transferring a path s in L to L' will be called simply the transfer of s to L' . Similarly, one defines the transfer of a path s in L' to L , of a path in L to L and of a path in L' to L' .

We will distinguish **the** transfer of s , which is a set of paths, from **a** transfer of s , which is a single path in the transfer of s .

Remark. The union of the paths in the transfer of s to L' is exactly the set of points of intersections of L' with leaves of \mathcal{L} which meet s . Similar remarks hold for the other three types of transfer.

Lemma 12.9. *The set of paths in the transfer of a path s on L to L' is permuted transitively by the map $h' : L' \rightarrow L'$. Similar assertions hold for the other three types of transfer.*

Indeed, h' and h are defined by flowing along \mathcal{L} .

Remark. The transfer operation on a parametrized path is a continuation process using a parametrization of s and gives a parametrization of s' . Thus, a choice of orientation of s induces canonically an orientation of any transfer s' .

Lemma 12.10. *Let s be a loop on L and s' a transferred path on L' . Then s' is a nullhomotopic loop on L' if and only if s is a nullhomotopic loop on L . A similar assertion holds for the other three types of transfer.*

Proof. Suppose that s is nullhomotopic on L . If s' opens, a classical construction (cf. [4, Lemma 3.3.7]) gives a closed transversal to \mathcal{F}' which is homotopic in M° to s . By tautness, such a loop must be essential in M and so s is essential on L . This contradiction shows that s' is a closed loop. Thus s' is homotopic to s in M along the projecting arcs of \mathcal{L} and, again by tautness, it is nullhomotopic on L' . The converse has the same proof. \square

Definition 12.11 (transfer of a set of curves). Given a set \mathcal{A} of curves on L , the union of the transfers of the elements of \mathcal{A} will be called the transfer \mathcal{A}' of \mathcal{A} .

Remark. Be careful of the logic here. The transfer of a path is a *set* of paths. The union of the transfer of a set of paths is the union of sets of paths, hence is a set of paths. As usual, we can define the transfer of a set of curves on L to L , etc.

12.4.1. *The transferred laminations.* We define laminations Λ'_\pm on L' as the transfers of the Handel-Miller pseudo-geodesic laminations Λ_\pm on L (forgetting parametrizations). There are problems here as it is not generally true that the transfer of a lamination is a lamination. It turns out that the h -invariance of Λ_\pm saves the day.

Lemma 12.12. *Let \mathcal{A} be an h -invariant family of pairwise disjoint, one-one immersed, nonparametrized curves. Then the transfer \mathcal{A}' of \mathcal{A} to L' is an h' -invariant family of pairwise disjoint, one-one immersed, nonparametrized curves.*

Proof. Let $\lambda'_1, \lambda'_2 \in \mathcal{A}'$ and suppose there is a point $x \in \lambda'_1 \cap \lambda'_2$ such that, for every simply connected neighborhood V of x in L' , the connected component of x in $V \cap (\lambda'_1 \cup \lambda'_2)$ is not an embedded curve. (It is possible that $\lambda'_1 = \lambda'_2$ and self intersects nontrivially.) By the h -invariance of \mathcal{A} , every transfer of λ'_i to L is an element of \mathcal{A} , $i = 1, 2$. Since a small enough choice of V projects along \mathcal{L} homeomorphically into L , we obtain $\lambda_1, \lambda_2 \in \mathcal{A}$ intersecting nontrivially. \square

Lemma 12.13. *For \mathcal{A}' and \mathcal{A} as in Lemma 12.12, \mathcal{A} is the transfer of \mathcal{A}' to L .*

This is an easy exercise.

Lemma 12.14. *If Λ is an h -invariant lamination of L and Λ' is its transfer to L' , then Λ' is an h' -invariant lamination of L' .*

Proof. By Lemma 12.12, we only need to show that there is a laminated partial atlas for Λ' . Let $\lambda' \in \Lambda'$ and let $x' \in \lambda'$. There is $\lambda \in \Lambda$ such that λ' is in the transfer of λ and so there is $x \in \lambda$ that is carried by the transfer operation to x' . Let (U, Y, φ) be a laminated chart associated to Λ , containing x and small enough that U projects along \mathcal{L} homeomorphically onto a neighborhood U' of x' . Let $\pi : U \rightarrow U'$ be this projection. Then $(U', Y, \varphi \circ \pi^{-1})$ is a laminated chart about x' . It is clear that the plaques in this chart are exactly the path components of the intersections of the curves in Λ' with U' . Since $x' \in |\Lambda'|$ is arbitrary, we have constructed a laminated partial atlas for Λ' , hence Λ' is a lamination. \square

Lemma 12.15. *If Λ is a closed, h -invariant lamination of L , then Λ' is a closed, h' -invariant lamination of L' .*

Indeed, a point $x' \in L' \setminus |\Lambda'|$ transfers to a point $x \in L \setminus |\Lambda|$. But x has an open neighborhood U in $L \setminus |\Lambda|$ which is small enough to project along \mathcal{L} to an open neighborhood U' of x' . Lemma 12.13 implies that U' misses $|\Lambda'|$.

Corollary 12.16. *The transferred laminations Λ'_\pm are closed, disjoint from $\partial L'$, h' -invariant and (Λ'_+, Λ'_-) is a bilamination.*

Indeed, the bilaminated charts are produced exactly as the laminated charts.

12.4.2. *Behavior of the transferred laminations at the ideal boundary E .* Since there are only thirteen nonstandard surfaces [15, Theorem 8], all of which are easily ruled out under our hypotheses as leaves of \mathcal{F} and \mathcal{F}' , L and L' can each be given a standard hyperbolic metric (cf. Definition 3.5 and the following remarks). Fix a choice of such metrics. By Corollary 10.16, the laminations satisfy the axioms relative to one choice of standard metric if and only if they do so relative to any choice of standard metric.

Remark. It is not necessary to make all of the leaves of either foliation simultaneously hyperbolic. By a well known theorem of A. Candel [3], it is possible to find a metric on M which makes all of the leaves of \mathcal{F} hyperbolic and another that makes the leaves of \mathcal{F}' hyperbolic, but nothing so sophisticated is needed here.

Let \widetilde{M}° be the universal cover of M° . Let $\widetilde{\mathcal{L}}$ be the lift of $\mathcal{L}|_{M^\circ}$ to \widetilde{M}° . Fix a lift $\widetilde{L} \subset \widetilde{M}^\circ$ of the leaf $L \in \mathcal{F}$. Similarly, fix a lift $\widetilde{L}' \subset \widetilde{M}^\circ$ of the leaf $L' \in \mathcal{F}'$. Because the foliations \mathcal{F} and \mathcal{F}' are taut, these lifts are the universal covers of L and L' , respectively, and we view $\widetilde{L}, \widetilde{L}' \subset \Delta$.

Lemma 12.17. *Each of the surfaces \widetilde{L} and \widetilde{L}' meets each leaf of $\widetilde{\mathcal{L}}$ in a single point.*

Proof. The fibration $\pi : M^\circ \rightarrow S^1$ defined by \mathcal{F} lifts to a (necessarily trivial) fibration $\widetilde{\pi} : \widetilde{M}^\circ \rightarrow \mathbb{R}$ with fibers the leaves of $\widetilde{\mathcal{F}}$. The leaves of $\widetilde{\mathcal{L}}$ are cross-sections of this bundle. Indeed, if $\widetilde{\ell}$ is a leaf of $\widetilde{\mathcal{L}}$, $\widetilde{\pi} : \widetilde{\ell} \rightarrow \mathbb{R}$ is easily seen to be a surjective immersion, hence a diffeomorphism (by the classification of 1-manifolds). The assertion for the leaves \widetilde{L} of $\widetilde{\mathcal{F}}$ follow and the same proof works for the leaves \widetilde{L}' of $\widetilde{\mathcal{F}'}$. \square

Corollary 12.18. *Projection along the leaves of $\tilde{\mathcal{L}}$ defines $\nu : \tilde{L} \rightarrow \tilde{L}'$, a homeomorphism which carries $\tilde{\Lambda}_{\pm}$ onto $\tilde{\Lambda}'_{\pm}$.*

Proof. Indeed, the fact that projection defines a homeomorphism ν is an obvious consequence of Lemma 12.17. Given a path σ on L and a lift $\tilde{\sigma}$ on \tilde{L} , it is evident that $\nu(\tilde{\sigma})$ is a lift of a transfer of σ to L' . Thus, $\nu(\tilde{\Lambda}_{\pm}) \subseteq \tilde{\Lambda}'_{\pm}$. For the reverse inclusion, note that the argument just given works equally well to prove that $\nu^{-1}(\tilde{\Lambda}'_{\pm}) \subseteq \tilde{\Lambda}_{\pm}$. Thus,

$$\nu(\tilde{\Lambda}_{\pm}) \supseteq \nu(\nu^{-1}(\tilde{\Lambda}'_{\pm})) = \tilde{\Lambda}'_{\pm}.$$

□

Remark. Note that ν generally is not the lift of a map from L to L' .

The main result of Section 12.4.2 is,

Proposition 12.19. *The map ν extends to a homeomorphism $\hat{\nu} : \hat{L} \rightarrow \hat{L}'$.*

The proof of Proposition 12.19 will be modelled on that of [15, Theorem 2], but there are important differences. Note that doubling M along $\partial_{\text{th}}M$ gives rise to foliations $2\mathcal{F}$ and $2\mathcal{F}'$ having $2L$ and $2L'$, respectively, as leaves. By this device, we can reduce the proof of Proposition 12.19 to the case that L and L' have empty boundary. Thus \tilde{L} and \tilde{L}' can be identified with Δ and $\nu : \Delta \rightarrow \Delta$ is a homeomorphism. The real reason for doubling, however, is to assure that the components of $\partial_{\tau}M$ are closed surfaces, thereby eliminating finitely many counterexamples to the conclusion of Lemma 12.21.

Remark. This doubling is only a temporary device. It will end when the proof of Proposition 12.19 is complete and nowhere during this period will it be assumed that the doubled laminations satisfy the axioms. Generally they do not, as Example 4.36 shows. We will explain shortly why this doubling is necessary.

Definition 12.20 (essentially intersecting loops). Two (parametrized) loops σ_1, σ_2 in a surface *essentially intersect* if, whenever a loop τ_1 is freely homotopic to σ_1 and a loop τ_2 is freely homotopic to σ_2 , then τ_1 and τ_2 have nonempty intersection.

In particular, loops that essentially intersect must intersect.

Lemma 12.21. *If F is a closed, connected surface other than the sphere, projective plane, or torus, it contains a pair of essentially intersecting loops each of which separates F .*

For the case of the connected sum of $n \geq 4$ tori or $n \geq 4$ projective planes, this is easy. The remaining cases are left as an exercise, but see Figure 33 for the connected sum of two tori.

Lemma 12.22. *There is a pair of essentially intersecting loops ρ and ρ_{th} in L such that,*

- (1) *Every transfer of ρ to L' is a loop ρ' ,*
- (2) *Given a transfer ρ' as above, there exists a transfer ρ'_{th} of ρ_{th} to L' which is a loop essentially intersecting ρ' .*

Proof. Let F be a component of $\partial_{\tau}M$ and consider ends e and e' of L and L' , respectively, which are asymptotic to F . Fix an h -neighborhood $U_e \subset L$ of e and

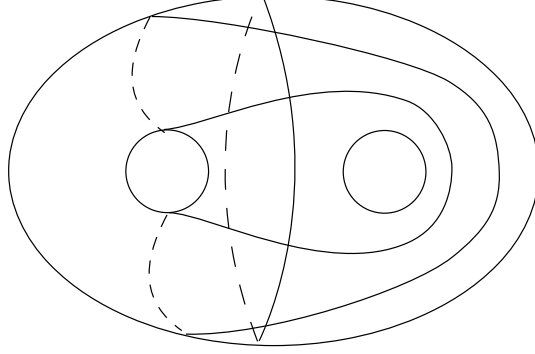


FIGURE 33. Separating loops which essentially intersect

an h' -neighborhood $U_{e'} \subset L'$ of e' which spiral on F . As in Section 2.1, let N, N' be smoothly embedded, transversely oriented 1-manifolds in F which define the respective cohomology classes corresponding to these semi-coverings. It is worth noting that, as in the cited section, these semicoverings are the restrictions of honest coverings $p : \mathcal{U} \rightarrow F$ and $p' : \mathcal{U}' \rightarrow F$, where $\mathcal{U} \subset L$ is open and h -invariant and $\mathcal{U}' \subset L'$ is open and h' -invariant. The deck transformation groups are infinite cyclic, generated by h and h' respectively.

By Lemma 12.21, we can choose a pair of essentially intersecting simple closed curves σ and $\sigma_{\mathfrak{h}}$ on F which each separate F . Let $x \in F$ be a point of intersection. Since σ and $\sigma_{\mathfrak{h}}$ are separating, the homological intersection number of each of N and N' with each of σ and $\sigma_{\mathfrak{h}}$ is zero. Thus, the curves σ and $\sigma_{\mathfrak{h}}$ lift along \mathcal{L} to essential loops ρ and $\rho_{\mathfrak{h}}$ in \mathcal{U} (and therefore L) with $\rho \cap \rho_{\mathfrak{h}}$ containing a point y which is a lift of x . Then ρ and $\rho_{\mathfrak{h}}$ intersect essentially. Similarly, the curves σ and $\sigma_{\mathfrak{h}}$ lift along \mathcal{L} to essentially intersecting loops ρ' and $\rho'_{\mathfrak{h}}$ in L' with $\rho' \cap \rho'_{\mathfrak{h}}$ containing a point z which is a lift of x . The curves ρ and ρ' and the curves $\rho_{\mathfrak{h}}$ and $\rho'_{\mathfrak{h}}$ are obviously transfers of each other. \square

Notation. Denote by \mathcal{R} the set of loops ρ on L such that,

- (1) there is a loop $\rho_{\mathfrak{h}}$ essentially intersecting ρ ;
- (2) the transfers of ρ and $\rho_{\mathfrak{h}}$ to L' are sets of loops;
- (3) for each transfer ρ' of ρ , there exists a transfer $\rho'_{\mathfrak{h}}$ of $\rho_{\mathfrak{h}}$ which is a loop essentially intersecting ρ' .

Let \mathcal{R}' denote the union of all transfers ρ' of loops $\rho \in \mathcal{R}$.

By Lemma 12.22, both \mathcal{R} and \mathcal{R}' are nonempty. Further, \mathcal{R}' is the transfer of \mathcal{R} and vice versa. Let $\tilde{\mathcal{R}}$ be the set of lifts to \tilde{L} of elements of \mathcal{R} and $\tilde{\mathcal{R}}'$ the set of lifts to \tilde{L}' of elements of \mathcal{R}' .

Lemma 12.23. (1) $\tilde{\mathcal{R}}$ is invariant under the group of deck transformations of \tilde{L} and $\tilde{\mathcal{R}}'$ is invariant under the deck transformations of \tilde{L}' .
 (2) ν induces a bijection $\tilde{\mathcal{R}} \rightarrow \tilde{\mathcal{R}}'$ which will again be denoted by ν .

- (3) Finally, every $\rho \in \mathcal{R}$ admits an essentially transverse closed loop $\rho_{\mathfrak{h}}$ such that some transfer of $\rho'_{\mathfrak{h}}$ is a closed loop essentially transverse to a given transfer ρ' of ρ .

Indeed, (1) and (3) are clear and (2) is proven exactly as Corollary 12.18.

Notation. Let,

$$\mathcal{Z} = \{z \in S_{\infty}^1 \mid z \text{ is an ideal endpoint of } \sigma \in \tilde{\mathcal{R}}\},$$

$$\mathcal{Z}' = \{z \in S_{\infty}^1 \mid z \text{ is an ideal endpoint of } \tau \in \tilde{\mathcal{R}}'\}.$$

Remark. The standard counterclockwise orientation of S_{∞}^1 induces a cyclic order on \mathcal{Z} and on \mathcal{Z}' .

Lemma 12.24. *The homeomorphism $\nu : \Delta \rightarrow \Delta$ induces a bijection $\bar{\nu} : \mathcal{Z} \rightarrow \mathcal{Z}'$ which either preserves or reverses the cyclic order.*

Proof. The bijection $\nu : \tilde{\mathcal{R}} \rightarrow \tilde{\mathcal{R}}'$ induces a bijection $\bar{\nu}$ between their sets of endpoints. If $\nu : \Delta \rightarrow \Delta$ is an orientation preserving (respectively reversing) homeomorphism, it is clear that $\bar{\nu}$ preserves (respectively reverses) cyclic order. \square

Evidently \mathcal{Z} and \mathcal{Z}' are invariant under (the extensions to S_{∞}^1 of) deck transformations for the respective covering spaces. Since L and L' have no simply connected ends, we can apply [15, Corollary 1] to obtain the following.

Lemma 12.25. *The sets \mathcal{Z} and \mathcal{Z}' are dense in S_{∞}^1 .*

Corollary 12.26. *The map $\bar{\nu} : \mathcal{Z} \rightarrow \mathcal{Z}'$ extends to a homeomorphism $\bar{\nu} : S_{\infty}^1 \rightarrow S_{\infty}^1$.*

We now define $\hat{\nu} : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ to be the bijection such that $\hat{\nu}|\Delta = \nu$ and $\hat{\nu}|S_{\infty}^1 = \bar{\nu}$. We need to prove it is a homeomorphism. Since \mathbb{D}^2 is a compact Hausdorff space and $\hat{\nu}$ is bijective, it will be enough to prove that $\hat{\nu}$ is continuous. It is only necessary to prove this at points $z \in S_{\infty}^1$.

Notation. Denote by \mathcal{G} (respectively \mathcal{G}') the set of lifts of pseudo-geodesics γ in L (respectively L') such that some, hence every, completed lift $\hat{\gamma}$ has both endpoints in \mathcal{Z} (respectively in \mathcal{Z}').

Lemma 12.27. *The homeomorphism ν defines a bijection $\nu : \mathcal{G} \rightarrow \mathcal{G}'$.*

Proof. If $\hat{\gamma}$ has an endpoint $z \in \mathcal{Z}$, we will show that $\nu(\hat{\gamma})$ limits on the point $\bar{\nu}(z) \in \mathcal{Z}'$. Applying this to both endpoints of $\hat{\gamma}$, $\tilde{\gamma} \in \mathcal{G}$, will prove that $\nu(\mathcal{G}) \subset \mathcal{G}'$. The proof for ν^{-1} is completely similar, proving the lemma.

Fix a choice of $s \in \mathcal{R}$ with a lift having z as an ideal endpoint and let σ be the closed geodesic homotopic to s . Let $s_{\mathfrak{h}} \in \mathcal{R}$ intersect s essentially and τ be the closed geodesic homotopic to $s_{\mathfrak{h}}$. Then τ intersects σ essentially and, since these loops are geodesics, the intersection is transverse. Let a be one of these intersection points. Let $\tilde{\sigma}$ be the lift of σ with z as an ideal endpoint. Let $\{a_n\}_{n \in \mathbb{Z}}$ be the lifts of a in $\tilde{\sigma}$, indexed so that $\lim_{n \rightarrow \infty} a_n = z$ monotonically along $\tilde{\sigma}$. Let $\tilde{\tau}_n$ be the lift of τ through a_n . Each $\tilde{\tau}_n$ has endpoints u_n, w_n bounding a subarc $\alpha_n \subset S_{\infty}^1$ containing the point z . The sequences $u_n \rightarrow u$ monotonically and $w_n \rightarrow w$ monotonically as $n \rightarrow \infty$. We claim that $u = z = w$. Otherwise, the closed geodesic τ in L accumulates locally uniformly on a geodesic which is impossible. The sets $V_n \subset \mathbb{D}^2$, $n \geq 0$, bounded by $\tilde{\tau}_n \cup \alpha_n$ form a fundamental system of closed neighborhoods of

z . Since z is an ideal endpoint of $\tilde{\gamma}$, it follows that, for each $n \geq 0$, there is a ray $\tilde{\gamma}_n \subset \tilde{\gamma}$ such that $\tilde{\gamma}_n \subset V_n$.

The sets $V_n \cap \Delta$ are nested with empty intersection. Thus, the homeomorphism ν carries them to a nested sequence of sets with empty intersection. Furthermore, the homeomorphism $\bar{\nu}$ carries the sequences u_n and w_n to sequences converging monotonically from opposite sides to $\bar{\nu}(z)$. By the definition of $\bar{\nu}$, it carries the ideal endpoints of $\tilde{\gamma}_n$ to the ideal endpoints of $\nu(\tilde{\gamma}_n)$. It clearly follows that $\{\hat{\nu}(V_n)\}_{n=1}^\infty$ is a fundamental system of neighborhoods of $\bar{\nu}(z)$. Furthermore, $\nu(\tilde{\gamma}_n) \subset \hat{\nu}(V_n)$ for $n \geq 0$, so $\nu(\tilde{\gamma})$ has $\bar{\nu}(z)$ as an ideal endpoint. \square

Remark. The existence of τ in the above proof depends on ρ being nonperipheral, that is, on σ not being a component of ∂L . By doubling we have assured that $\partial L = \emptyset$. Actually, we would be fine without doubling except for a few pesky cases. For example, if each of the components F of $\partial_\tau M$ is either a torus with an open disk deleted or a pair of pants, then, in the proof of Lemma 12.21, it would not be possible to find a simple separating loop $\sigma \subset F$ which is not peripheral and the proof of Lemma 12.22 would fail. Hence, in this case, we cannot construct ρ to be nonperipheral. Doubling is not necessary for the application of [15, Corollary 1] in proving Lemma 12.25 since the leaves L and L' are admissible, hence have no simply connected ends.

Proof of Proposition 12.19. We want to show that $\hat{\nu} : \mathbb{D}^2 \rightarrow \mathbb{D}^2$ is continuous at $z \in \mathbb{D}^2$. This is clear if $z \in \Delta$, so we assume $z \in S_\infty^1$. Let U be an open neighborhood of $\hat{\nu}(z)$. Since \mathcal{Z}' is dense in S_∞^1 , we can choose an arc $\alpha \subset U \cap S_\infty^1$ with endpoints $a, b \in \mathcal{Z}'$ such that $\hat{\nu}(z) \in \alpha$. Let $\hat{\gamma}$ be any curve in U with endpoints a and b and γ the projection of $\hat{\gamma} \cap \Delta$ to L . Then $\tilde{\gamma} \in \mathcal{G}'$ so by Lemma 12.27, $\nu^{-1}(\tilde{\gamma}) \in \mathcal{G}$ and has endpoints $\bar{\nu}^{-1}(a), \bar{\nu}^{-1}(b) \in \mathcal{Z}$. Then the subset V of \mathbb{D}^2 bounded by $\hat{\nu}^{-1}(\hat{\gamma})$ and $\hat{\nu}^{-1}(\alpha)$ is a closed neighborhood of z in \mathbb{D}^2 and $\hat{\nu}(V) \subset U$. This proves continuity at arbitrary $z \in S_\infty^1$. Finally, in the general case before doubling, ν restricts to the desired homeomorphism of \hat{L} onto \hat{L}' . \square

Because of the final statement in the above proof, we no longer work in the double and will not do so again. Here are two corollaries of Proposition 12.19.

Corollary 12.28. *Any transfer of a pseudo-geodesic on L to a curve on L' is a pseudo-geodesic (and vice versa).*

Proof. Let λ be a pseudo-geodesic on L and let $\tilde{\lambda}$ be a lift of λ to \tilde{L} . By the definition of ν , it is clear that $\nu(\tilde{\lambda}) = \tilde{\lambda}'$ is a lift of some transfer λ' of λ to L' . Since $\tilde{\lambda}$ has two well defined endpoints, $\hat{\nu}$ carries those endpoints to well defined endpoints of $\tilde{\lambda}'$. The transfer of λ is the h' -orbit of λ' . Since any lift \tilde{h}' extends canonically to $\partial \hat{L}'$, we see that the transfer of λ consists entirely of pseudo-geodesics. \square

Corollary 12.29. *The laminations Λ'_\pm are strongly closed.*

Proof. The corollary follows since $\hat{\nu}$ is a homeomorphism and the laminations Λ_\pm are strongly closed. \square

12.5. Proofs of Axioms 1 and 2. The positive/negative escaping sets. We fix choices of $L \in \mathcal{F}|M^\circ$ and $L' \in \mathcal{F}'|M^\circ$ and transfer the laminations Λ_\pm on L to laminations Λ'_\pm on L' .

By Corollary 12.28, the leaves of Λ'_\pm are pseudo-geodesics. By Corollary 12.16, we obtain the following.

Lemma 12.30. *Axiom 1 holds for Λ'_\pm .*

Since we have identified the lifts of Λ_\pm with those of Λ'_\pm via the homeomorphism ν (Corollary 12.18), it follows that,

Lemma 12.31. *Axiom 2 holds for Λ'_\pm .*

12.5.1. *The positive/negative escaping sets \mathcal{U}'_\pm .* By Lemma 12.4, $h' : L' \rightarrow L'$ is an endperiodic automorphism. Even though we have not verified all the axioms for Λ'_\pm , we can still define the positive and negative escaping sets \mathcal{U}'_\pm as in Definition 4.64. That is for e' an end of L' , choose an h' -neighborhood $U'_{e'}$ of e' (Definition 2.3) and define $\mathcal{U}'_{e'} = \bigcup_{n=-\infty}^{\infty} (h')^{np_{e'}}(U'_{e'})$. The union of the sets $\mathcal{U}'_{e'}$ as e' ranges over the positive (respectively negative) ends will be denoted by \mathcal{U}'_+ (respectively \mathcal{U}'_-). We will call \mathcal{U}'_+ the *positive escaping set* and \mathcal{U}'_- the *negative escaping set*.

Clearly,

Lemma 12.32. *The set \mathcal{U}'_+ (respectively \mathcal{U}'_-) consists of the set of points $x \in L'$ such that the sequence $\{(h')^n(x)\}_{n \geq 0}$ (respectively $\{(h')^n(x)\}_{n \leq 0}$) escapes (Definition 3.7).*

Lemma 12.33. *The point $x \in L'$ belongs to \mathcal{U}'_\pm if and only if the leaf ℓ_x of \mathcal{L} through x satisfies $\ell_x \cap L \subset \mathcal{U}_\pm$.*

Proof. The negative ends of L and L' are exactly the ones that wind in on inwardly oriented components of ∂M as semi-coverings, the positive ends likewise winding in on the outwardly oriented components of ∂M . Thus, the points $x \in \mathcal{U}_+$ are characterized by the fact that flowing them forward along \mathcal{L} causes them to approach the outwardly oriented components of ∂M , the points of \mathcal{U}_- being characterized analogously. Since the same characterizations hold for \mathcal{U}'_\pm , the claim follows. \square

Since all axioms hold in L and since projections along \mathcal{L} define local homeomorphisms of L to L' , Lemma 4.71 for $\Lambda_\pm \subset L$ now implies:

Corollary 12.34. *The frontier of \mathcal{U}'_\mp is $|\Lambda'_\pm|$ and, consequently, the border leaves of \mathcal{U}'_\mp accumulate locally uniformly exactly on the leaves of Λ'_\pm .*

Likewise, Corollary 4.69 for $\Lambda_\pm \subset L$ implies:

Corollary 12.35. *No leaf of Λ'_+ (respectively, Λ'_-) meets \mathcal{U}'_- (respectively, \mathcal{U}'_+).*

12.6. **Semi-isolated leaves and escaping ends in Λ'_\pm .** Let e' be an end of L' and denote its full h' -cycle by $e' = \{e' = e'_1, e'_2, \dots, e'_{p_{e'}}\}$. Set

$$\mathcal{U}'_{e'} = \bigcup_{n=-\infty}^{\infty} (h')^n(U'_{e'}) = \mathcal{U}'_{e'_1} \cup \mathcal{U}'_{e'_2} \cup \dots \cup \mathcal{U}'_{e'_{p_{e'}}}.$$

Remark that $\mathcal{U}'_{e'}$ is an open, h' -invariant set with no periodic points. The connected components $\mathcal{U}'_{e'_i}$ of $\mathcal{U}'_{e'}$ are permuted cyclically by h' . In M , the ends $e' = e'_1, e'_2, \dots, e'_{p_{e'}}$ are exactly the ends of L' asymptotic to a compact leaf $F = F_{e'} \subset \partial_\tau M$. There is an h -cycle c of ends of L asymptotic to this same compact

leaf $F = F_c$ and one similarly defines an open h -invariant subset $\mathcal{U}_c \subset L$. As in Section 2.1, one has infinite cyclic coverings

$$\begin{aligned} q' : \mathcal{U}'_{c'} &\rightarrow F \\ q : \mathcal{U}_c &\rightarrow F. \end{aligned}$$

Here, these coverings are defined by projection along the compact subarcs of leaves of \mathcal{L} issuing from points of $\mathcal{U}'_{c'}$ and of \mathcal{U}_c , respectively, and terminating on F .

The group of deck transformations for q' is generated by $h'|_{\mathcal{U}'_{c'}}$ and for q by $h|_{\mathcal{U}_c}$. Thus,

Lemma 12.36. *We can identify the surface F (which is a leaf of the foliation of M) with the surface F of Section 2.1.*

For definiteness, we consider the case that c' is an h' -cycle of negative ends of L' , c the corresponding h -cycle of negative ends of L , with $F = F_{c'} = F_c$. Since $\mathcal{U}'_{c'}$ and Λ'_- (respectively \mathcal{U}_c and Λ_-) are h' -invariant (respectively h -invariant), it follows that the induced lamination $\Lambda'_-|_{\mathcal{U}'_{c'}}$ (respectively $\Lambda_-|_{\mathcal{U}_c}$) is invariant under the group of deck transformations of $q' : \mathcal{U}'_{c'} \rightarrow F$ (respectively $q : \mathcal{U}_c \rightarrow F$). As in Section 6.7, the laminations $\Lambda'_-|_{\mathcal{U}'_{c'}}$ and $\Lambda_-|_{\mathcal{U}_c}$ descend to closed laminations of F . Since $\Lambda'_-|_{\mathcal{U}'_{c'}}$ and $\Lambda_-|_{\mathcal{U}_c}$ are transfers of each other, they descend to the same lamination of F . We have,

Lemma 12.37. *The laminations $\Lambda'_-|_{\mathcal{U}'_{c'}}$ and $\Lambda_-|_{\mathcal{U}_c}$ descend to the same well defined closed lamination Λ_F of F .*

In Section 6.7, we analyzed the structure of Λ_F and the traintrack T that carries it using well understood properties of semi-isolated leaves in Λ_+ and escaping ends in Λ_- . By Lemma 12.37, the same lamination Λ_F and traintrack are induced by the laminations Λ'_- and the covering map $q' : \mathcal{U}'_{c'} \rightarrow F$. Hence it is reasonable to expect that these properties of semi-isolated leaves also hold for Λ_+ and that these properties of escaping ends also hold for Λ'_- . Since the axioms have not all been verified for Λ'_\pm , this requires proof which we now provide.

Remark that the negative escaping set $\mathcal{U}'_- \subset L'$ is the disjoint union of the $\mathcal{U}'_{c'}$'s corresponding to h' -cycles c' of negative ends, with the parallel assertion about $\mathcal{U}'_+ \subset L'$. The following discussion will be carried out for the negative escaping sets and the cycles of negative ends, the positive case being entirely parallel.

Proposition 12.38. *The laminations Λ'_+ of L' have only finitely many semi-isolated leaves. They are periodic under h' and each contains either one h' -periodic point or one nondegenerate, compact h' periodic subarc. In the case of one h' -periodic point, it is repelling on both sides. In the case of a compact, nondegenerate h' -periodic subarc A' , the endpoints of A' are repelling on the sides not containing A' . Furthermore, $\text{int } A'$ meets no leaves of Λ'_- .*

Proof. Because the transfer operation is locally a homeomorphism, it is clear that the semi-isolated leaves of Λ'_+ are exactly the transfers of the semi-isolated leaves of Λ_+ . Since any of these latter contains at least one h -periodic point y , the leaf of \mathcal{L} through y is closed and can only intersect L' in finitely many points. Thus the laminations Λ'_+ have only finitely many semi-isolated leaves and they each contain h' -periodic points.

If $\lambda \in \Lambda_+$ is semi-isolated and contains exactly one h -periodic point y , then the leaves of \mathcal{L} passing through λ are asymptotic (in backward time) in the \mathcal{L} -saturation

of λ exactly to the closed leaf ℓ through y . Remark that ℓ meets λ only in the point y . If λ' is a transfer of λ , we claim that ℓ can only meet λ' in a single point y' . Otherwise, a transfer back to λ would send a subarc of λ' with distinct endpoints onto a nontrivial, compact curve lying in λ and having the same endpoints. This would imply that λ is compact, contrary to Axiom 1. Thus, the leaves of \mathcal{L} passing through λ' are asymptotic to ℓ in the \mathcal{L} -saturation of λ' . Since λ borders \mathcal{U}_- , y is a repelling periodic point of h on λ and is the only periodic point (cf. Section 6.2). Consequently y' is a repelling periodic point of h' on λ' . It is the only one as a second would generate another fixed point in λ .

In a similar way, the compact, nondegenerate, h -periodic subarc A of a semi-isolated leaf $\lambda \in \Lambda_+$ determines a corresponding h' -periodic subarc A' on any transfer λ' of λ , the endpoints of which are both h' -repelling on the sides not containing A' , and A' is the only h' -periodic subarc of λ' .

The final assertion transfers from the corresponding fact for Λ_+ , where the leaves of Λ_- passing through the endpoints of A lie in the arms of principal regions. \square

The following is an easy consequence of the above and Lemma 12.33.

Corollary 12.39. *If c' is a cycle of negative ends of L' , then the border $\delta\mathcal{U}'_{c'}$ has finitely many components consisting of h' -cycles of semi-isolated leaves of Λ'_+ .*

Proposition 12.40. *The ends represented by rays of leaves of Λ'_- issuing from $\delta\mathcal{U}'_-$ into \mathcal{U}'_- , either from a unique h' -periodic point on a semi-isolated leaf λ of Λ'_+ or from the endpoints of a unique nondegenerate, compact, h' -periodic subarc of such a leaf are exactly the escaping ends (Definition 6.19) of leaves of Λ'_- . Only finitely many leaves of Λ'_- have an escaping end.*

Proof. Such a ray $r' = [a', \infty)$ is clearly the transfer of such a ray r in L . In particular, $\text{int } r' = (a', \infty)$ lies in the negative h' -escaping set. We will suppose it lies in $\mathcal{U}'_{e'}$ for a negative end $e' \in \mathcal{E}(L')$. As the transfer of an h -periodic ray, r' is h' -periodic. Let p be an h' -period of r' . (If $(h')^p$ is orientation reversing on λ , replace p by $2p$.)

Let $x \in (a', \infty)$ and consider the compact arc $\alpha \subset (a', \infty)$ bounded by x and $(h')^{-p}(x)$. Let $U_{e'}$ be an h' -neighborhood of e' for the automorphism h' which by Lemma 12.4 is endperiodic. Since each point $y \in \alpha$ converges to e' under negative iterations of $(h')^p$, the compactness of α implies that there is a sufficiently large integer $n \geq 1$ such that $(h')^{-kp}(\alpha) \subset U_{e'}$, all $k \geq n$. Thus, $\bigcup_{k=n}^{\infty} (h')^{-kp}(\alpha) \subset U_{e'}$. Therefore, the ray r' represents an escaping end as asserted.

Conversely, if the ray $r' = [a', \infty)$ represents an escaping end, then r' is the transfer of a ray $r = [a, \infty)$ with $(a, \infty) \subset \mathcal{U}_-$ and $a \in |\delta\mathcal{U}_-|$. By Lemma 6.42, the ray r represents an escaping end so by Lemma 6.26 a is h -periodic. Thus, a' is h' -periodic and the converse is proven.

Proposition 12.38 now implies that only finitely many leaves of Λ'_- have an escaping end. \square

By transferring from L' to L , we obtain the following.

Proposition 12.41. *The completions of the leaves of $\Lambda'_-|\mathcal{U}'_{c'}$ are all compact arcs with endpoints in $\delta\mathcal{U}'_{c'}$, except for the rays $[a, \infty)$ corresponding to escaping ends. The compact completions fall into parallel packets permuted by h' and there are only finitely many h' -orbits of parallel packets.*

We repeat that the analogous statements for $\Lambda'_\pm|\mathcal{U}'_{c'}$, where c' is an h' -cycle of positive ends of L , are true and have completely parallel proofs.

Thus, we see exactly how $\Lambda'_\pm|\mathcal{U}'_{c'}$ descends via the covering map q' to Λ_F .

For future reference, we also note the following.

Proposition 12.42. *The union of the border components of \mathcal{U}'_{\mp} is dense in $|\Lambda'_\pm|$.*

Proof. Equivalently, the border components of \mathcal{U}'_{\mp} accumulate locally uniformly on every leaf of Λ_\pm . Since the border leaves are the transfers of the border leaves of \mathcal{U}_{\mp} , this will be true if the corresponding assertion is true for \mathcal{U}_{\mp} and Λ_\pm in L . Since the axioms are satisfied for Λ_\pm , this is given by Proposition 5.12. \square

12.7. The h' -junctures. In this subsection we choose and fix a countable set of h' -junctures.

Let c' denote an h' -cycle of positive or negative ends of L' . The covering projection $q' : \mathcal{U}'_{c'} \rightarrow F$ defines a cohomology class κ on F exactly as in Section 2.1 which we represent by a compact, properly embedded, transversely oriented 1-manifold J_κ which is “weakly groomed” (Definition 2.16). Thus, *as a cocycle*, we can replace J_κ with a transversely oriented 1-manifold $|J_\kappa|$ to which each component is assigned a positive weight. Each component σ of $|J_\kappa|$ is a properly embedded, transversely oriented arc or circle, with a positive integral weight w_σ , representing a “packet” of w_σ parallel arcs or parallel circles, coherently transversely oriented. By an abuse of notation, we generally use J_κ to signify this weighted 1-manifold also. For certain purposes, especially lifting J_κ to define a system of h' -juncture components in $\mathcal{U}'_{c'}$, it is best to view J_κ in this way, but for other purposes, such as “tunneling” (Section 2.4), it is better to view J_κ as the honest 1-manifold produced as in Lemma 2.14. For details, consult Sections 2.1 and 2.4.

Regard \mathbb{T} as a *finite*, nonorientable 1-complex having as vertices the switches. Its structure is identical with that of the traintrack for Λ_F . In particular, it has no “Reeb annuli” (see the remark on page 60), although this is not essential for the following application of standard general position arguments. Clearly,

Lemma 12.43. *An isotopy of $|J_\kappa|$ guarantees that $|J_\kappa|$ is transverse to the 1-complex \mathbb{T} , meets no vertex of \mathbb{T} and intersects \mathbb{T} so as to form no digons.*

Corollary 12.44. *The isotopy of $|J_\kappa|$ can be chosen so that this properly embedded 1-manifold meets the leaves of Λ_F transversely and so as to form no digons.*

Proof. Wherever $|J_\kappa|$ crosses \mathbb{T} , put a laminated chart for Λ_F . An isotopy makes $|J_\kappa|$ cross the chart in a transversal without introducing digons. \square

Since every escaping ray must cross junctures, we have the following.

Lemma 12.45. *Each circle in \mathbb{T} crosses at least one component of J_κ .*

Lemma 12.46. *The juncture J_κ can be chosen so that each oriented circle leaf C of Λ_F crosses J_κ only in the positive sense.*

Proof. By Proposition 2.32, one rechooses J_κ , if necessary, so that each C either crosses it everywhere in the positive sense, or everywhere in the negative sense. This introduces no new intersections, hence does not undo any features we have established. But the transverse orientation of the negative h' -junctures in $\mathcal{U}'_{c'}$ are into the neighborhood of the end of L' or L that they cut off. The orientation of an escaping ray is toward that end. Since juncture and ray will have all intersections

positive or all negative, they must have all intersections positive. Down in F , this means that all circle leaves of Λ_F have positive intersections with J_κ . \square

We fix these properties of J_κ .

As in Section 2.1, we now lift J_κ to a set $\{(h')^n(N'_{c'}) \mid n \in \mathbb{Z}\}$ of h' -juncture components in $\mathcal{U}'_{c'}$. Letting c' vary over all h' -cycles of negative (respectively positive) ends of L' , we obtain a countable set of h' -junctures in \mathcal{U}'_- (respectively in \mathcal{U}'_+) which we fix. As in Definition 4.17, we define \mathcal{N}'_- to be the set of components of these negative h' -junctures, \mathcal{N}'_+ to be the set of components of these positive h' -junctures, and $\mathcal{N}' = \mathcal{N}'_- \cup \mathcal{N}'_+$.

We define the set of juncture components \mathcal{J}' associated to \mathcal{N}' by $\mathcal{J}' = \mathcal{N}'$ and the map $\iota : \mathcal{N}' \rightarrow \mathcal{J}'$ to be the identity (see Definition 10.5). Thus the set of junctures is identical to the set of h' -junctures.

By Corollary 12.34, an element of \mathcal{J}'_- h' -escapes if and only if it is disjoint from $[\Lambda'_-]$, with a similar statement for the elements of \mathcal{J}'_+ .

Lemma 12.47. *A component of a negative juncture intersects a negative escaping end $[a, \infty)$ in at most one point.*

Proof. Indeed, the intersection must be a singleton because, by Lemma 12.46, an escaping ray can only meet a given juncture in positive intersections, hence only once. \square

Definition 12.48 (\mathcal{Y}'_\pm). The set \mathcal{Y}'_\pm consists of the nonescaping components of the elements of \mathcal{N}'_\pm .

Note that \mathcal{Y}'_\pm is a discrete, nonclosed lamination.

Lemma 12.49. *If a sequence $\{y_n\}_{n=1}^\infty$ of points in distinct leaves of \mathcal{Y}'_\mp converges to a point $y \in L'$, then $y \in \Lambda'_\pm$.*

Proof. Evidently, $y \in \overline{\mathcal{U}'_\mp}$ and, by Corollary 12.34, $\overline{\mathcal{U}'_\mp} = \mathcal{U}'_\mp \cup \Lambda'_\pm$. If $y \in \mathcal{U}'_\pm$, all but finitely many terms of the sequence lie in \mathcal{U}'_c for an h' -cycle of ends. The covering projection $q' : \mathcal{U}'_c \rightarrow F_c$ would then carry y to a point $q'(y) = y' \in F_c$ such that infinitely many strands of J_κ pass through a neighborhood of y' . Since J_κ is a compact 1-manifold, this is impossible. \square

12.8. The last two axioms. It remains that we verify Axiom 3 and Axiom 4.

Let \mathcal{N}' denote the set of h' -juncture components just constructed and let the set \mathcal{J}' of juncture components be the same set, taking for $\iota : \mathcal{N}' \rightarrow \mathcal{J}'$ the identity map. Then, the condition that $\iota(N)$ be homotopic to N for each $N \in \mathcal{N}$ is tautological. By our construction, the following has already been proven.

Lemma 12.50. *Axiom 4 holds for Λ'_\pm , with $\mathcal{J}'_\pm = \mathcal{N}'_\pm$.*

It remains to check Axiom 3, the endpoint correspondence property.

12.8.1. Convergence properties of \mathcal{Y}'_\pm . We work explicitly with the set \mathcal{Y}'_- in \mathcal{U}'_- , but everything carries through for the positive case in a completely parallel way. Let λ be a border leaf of \mathcal{U}'_e , where e is a negative end of L' , and let $[a, \infty)$ be an escaping end issuing from $a \in \lambda$. There may or may not be a second escaping end $[a', \infty)$ issuing from $a' \in \lambda$. Let $\{\sigma_n\}_{n=-\infty}^\infty$ be an enumeration of the elements of \mathcal{Y}'_- intersecting $[a, \infty)$. By Lemma 12.47, $\sigma_n \cap [a, \infty)$ is a singleton, x_n , $-\infty < n < \infty$. The indexing is chosen so that $\{x_n\}_{-\infty < n < \infty}$ is monotonically increasing in $[a, \infty)$.

Definition 12.51 (assemblage). The connected set

$$\lambda \cup [a, \infty) \cup \bigcup_{n=-\infty}^{\infty} \sigma_n \subset L'$$

will be called an *assemblage*.

Remark. We can also view an assemblage as contained in the internal completion $\check{\mathcal{U}}'_-$ and define similar assemblages in $\check{\mathcal{U}}'_+$. For an end e of L , there are finitely many assemblages in $\check{\mathcal{U}}'_e$, hence in $\check{\mathcal{U}}'_\pm$. An assemblage in $\check{\mathcal{U}}'_-$ may be called a negative assemblage and one in $\check{\mathcal{U}}'_+$ a positive assemblage. As mentioned above, we are explicitly considering the negative ones, but will mention the minor adjustment necessary in the positive case.

Let p be the least positive integer such that $(h')^p$ carries λ to itself, preserving orientation. Thus $(h')^p$ carries $[a, \infty)$ to itself and permutes the set $\{x_n\}$ so as to preserve the order. The sequence $\{x_n\}_{n=-\infty}^{\infty}$ falls into finitely many $(h')^p$ -orbits. The point a is $(h')^p$ -contracting in $[a, \infty)$. If it is an isolated fixed point in λ , then a is $(h')^p$ -expanding in λ on both sides. Otherwise, there is an invariant subinterval $[a', a] \subset \lambda$ and a and a' are both $(h')^p$ -expanding in λ on the sides not containing $[a', a]$. All of this is easily deduced from the corresponding properties in L , being known there since the axioms are satisfied in L .

Lemma 12.52. *If e is the end of L' to which $[a, \infty)$ escapes, then $\lim_{n \rightarrow \infty} x_n = e$ and $\lim_{n \rightarrow \infty} x_{-n} = a$.*

Proof. The first equality is due to the fact that the x_n 's lie in the component \mathcal{U}'_e of the negative h' -escaping set. The second sequence is monotonically decreasing in $[a, \infty)$, hence converges to a point b in that ray. Clearly b is h' -periodic, hence cannot lie in \mathcal{U}'_- . Therefore, $b = a$. Alternatively, $b = a$ is a consequence of Lemma 12.49. \square

Let

$$\lambda \cup [a, \infty) \cup \bigcup_{n=-\infty}^{\infty} \sigma_n \subset L'$$

be an assemblage. A choice of lift $\tilde{\lambda}$ gives a lift \tilde{a} of a , hence a lift $[\tilde{a}, \infty)$ of $[a, \infty)$, hence lifts $\tilde{x}_n \in [\tilde{a}, \infty)$ and lifts $\tilde{\sigma}_n$ through \tilde{x}_n , $-\infty < n < \infty$. This defines a lifted assemblage

$$\tilde{\lambda} \cup [\tilde{a}, \infty) \cup \bigcup_{n=-\infty}^{\infty} \tilde{\sigma}_n \subset \tilde{L}'$$

of which there are a countable infinity. An application of ν^{-1} (Corollary 12.18) transfers this assemblage to

$$\tilde{\gamma} \cup [\tilde{b}, \infty) \cup \bigcup_{n=-\infty}^{\infty} \tilde{\tau}_n \subset \tilde{L},$$

a lift of a *particular* transfer of the assemblage in L' to L . In this transfer,

$$\gamma \cup [b, \infty) \cup \bigcup_{n=-\infty}^{\infty} \tau_n \subset L,$$

the respective transfers of the points $x_n \in [a, \infty)$ will be denoted by $y_n \in [b, \infty)$.

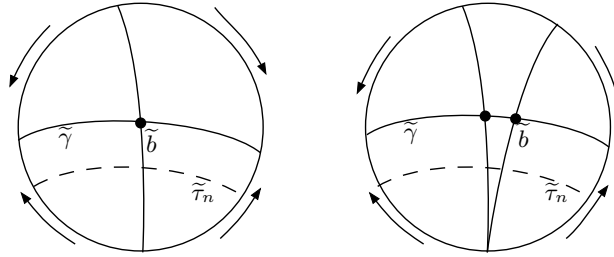


FIGURE 34. Case of juncture components meeting one or two escaping ends of a leaf of Λ_- .

Remark. There is a possible surprise here. Examples show that the curves σ_n may be loops that open up under transfer. As nonparametrized curves, there may be only finitely many distinct transferred curves τ_n , each meeting $[b, \infty)$ in a bi-infinite sequence of points each. By parametrizing these transfers so that $\tau_n(0) = y_n$, we view $\{\tau_n\}_{n \in \mathbb{Z}}$ as an infinite family of distinct parametrized curves.

There is a smallest positive integer q such that h^q carries $\gamma \cup [b, \infty)$ to itself, preserving orientation of γ . It is not necessarily true that $q = p$.

Lemma 12.53. *the map h^q permutes the set $\{y_n\}$ and also the set $\{\tau_n\}$ of parametrized curves, having finitely many distinct orbits in each.*

Proof. Denote by $\mu : [a, \infty) \rightarrow [b, \infty)$ the transfer such that $\mu(x_n) = y_n$, $-\infty < n < \infty$. Then $\mu^{-1} \circ h^q \circ \mu : [a, \infty) \rightarrow [a, \infty)$ is a transfer operation, hence is of the form $(h')^{kp}$, for some integer k . That is, $h^q \circ \mu = \mu \circ (h')^{kp}$. Thus, for each $n \in \mathbb{Z}$, there is $m \in \mathbb{Z}$ such that

$$h^q(y_n) = h^q(\mu(x_n)) = \mu((h')^{kp}(x_n)) = \mu(x_m) = y_m.$$

That is, h^q maps the set $\{y_n\}$ into itself. The same argument applies to h^{-q} , hence h^q maps $\{y_n\}$ bijectively to itself. Evidently, the same argument applies to $\{\tau_n\}$. That there are only finitely many h^q -orbits is evident. \square

Since the laminations on L satisfy the axioms, Proposition 6.9 applies. In Figure 34 we reproduce the pertinent two cases of Figure 13, relabeling with the current notation. The arrows indicate the action on S_∞ of the completion \hat{g} of a suitable lift g of h^q (one which fixes the ideal endpoints of $\tilde{\gamma}$). Since the curves $\tilde{\tau}_n$ are permuted by g , they have endpoints (finite or ideal) converging to the ideal endpoint of $[\tilde{b}, \infty)$ as $n \rightarrow \infty$ and to the ideal endpoints of $\tilde{\gamma}$ as $n \rightarrow -\infty$. Under application of $\tilde{\nu}$ (Proposition 12.19), the same convergence property holds for the lifted set in \tilde{L}' . This establishes the following.

Proposition 12.54. *Given an assemblage*

$$\lambda \cup [a, \infty) \cup \bigcup_{n=-\infty}^{\infty} \sigma_n \subset L',$$

and a lift of this assemblage to \tilde{L} , the endpoints of $\tilde{\sigma}_n$ in S_∞ converge to the ideal endpoints of $\tilde{\lambda}$ as $n \rightarrow -\infty$ and to the ideal endpoint of $[\tilde{a}, \infty)$ as $n \rightarrow \infty$.

Remark. The above discussion was for a negative assemblage. In the positive case, g should be a suitable lift of h^{-g} . This gives Proposition 12.54 in both the positive and the negative case.

12.8.2. *Verifying the endpoint correspondence property.* Let \mathcal{J}^g be the set of geodesic juncture components obtained as the geodesic tightening of the elements of \mathcal{J}' . Let \mathcal{U}^g denote the negative escaping set for $(\Lambda_+^g, \Lambda_-^g)$, the geodesic Handel-Miller bilamination associated to h' . The set \mathcal{Y}_\mp^g consists of the geodesic tightening of the elements of \mathcal{Y}'_\mp .

In order to verify Axiom 3 for (Λ'_+, Λ'_-) , we must show that Λ_\pm^g is exactly the set of geodesic tightenings of the elements of Λ'_\pm .

Lemma 12.55. *A component of \mathcal{J}' escapes if and only if its geodesic tightening escapes. Equivalently, the set \mathcal{Y}_\mp^g is exactly the set of nonescaping components of the geodesic junctures.*

Proof. “Only if” is given by Theorem 3.11. Since there are no digon intersections of elements of \mathcal{J}'_\pm with leaves of Λ'_\pm , any such intersection is essential, hence the same is true of the geodesic tightenings of such juncture components. That is, nonescaping elements of \mathcal{J}' tighten to nonescaping elements of \mathcal{J}^g , proving the “if” assertion. \square

By the construction of the Handel-Miller geodesic bilamination in Section 4, we have the following.

Corollary 12.56. *The geodesic laminations and nonescaping juncture components satisfy the following:*

- (1) $\Gamma_\pm^g = \Lambda_\pm^g \cup \mathcal{Y}_\mp^g$;
- (2) Γ_\pm^g is a closed geodesic lamination;
- (3) $|\mathcal{Y}_\mp^g|$ is dense in $|\Gamma_\pm^g|$.

Notation. Let X_\pm^g denote the set of pairs $\{x, y\} \subset S_\infty^1$ such that there exists $\lambda \in \Lambda_\pm^g$ having a lift $\tilde{\lambda}$ with endpoints x, y .

Proposition 12.57. *The following two statements are equivalent,*

- (1) *There exists a sequence $\{\sigma_n\}_{n \geq 1} \subset \mathcal{Y}'_\mp$ and lifts $\tilde{\sigma}_n$ having endpoints $x_n, y_n \in S_\infty$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$;*
- (2) $\{x, y\} \in X_\pm^g$.

Proof. This follows immediately from the facts that lifts of $\sigma_n \in \mathcal{Y}'_\mp$ and $\sigma_n^g \in \mathcal{Y}_\mp^g$ have the same endpoints on S_∞ and that the geodesic laminations Γ_\pm^g have the strongly closed property (Definition 4.33). \square

Notation. Let $X'_\pm \subset S_\infty^1$ denote the set of pairs $\{x, y\} \subset S_\infty^1$ such that there exists $\lambda \in \Lambda'_\pm$ having a lift $\tilde{\lambda}$ with endpoints x, y . We have,

Proposition 12.58. (1) *For every pair $\{x, y\} \in X'_\pm$, there exists a sequence $\{\sigma_n\}_{n \geq 1} \subset \mathcal{Y}'_\mp$ and lifts $\tilde{\sigma}_n$ having endpoints $x_n, y_n \in S_\infty$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$;*
 (2) $X'_\pm \subset X_\pm^g$.

Proof. By Proposition 12.57, (1) \Rightarrow (2). To prove (1), we prove that for every $\{x, y\} \in X'_\pm$ there exists a sequence $\{\sigma_n\} \subset \mathcal{Y}'_\mp$ and lifts $\tilde{\sigma}_n$ having endpoints $x_n, y_n \in S_\infty$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$. The other case is analogous.

If x, y are the endpoints of the lift of a border leaf of \mathcal{U}'_- in Λ'_+ , then such a sequence exists by Proposition 12.54.

If x, y are the endpoints of $\tilde{\lambda}$, where $\lambda \in \Lambda'_+$ is not a border leaf of \mathcal{U}'_- , then Proposition 12.42 implies that there is a sequence $\{\tilde{\lambda}_n\}$ of lifts of border leaves that strongly converges to $\tilde{\lambda}$. Since Λ'_+ is strongly closed (Corollary 12.29), the lifts $\tilde{\lambda}_n$ have endpoints $a_n, b_n \in S_\infty^1$ such that $a_n \rightarrow x$ and $b_n \rightarrow y$. By Proposition 12.54, there exists a sequence $\{\sigma_n\} \subset \mathcal{Y}'_-$ and lifts $\tilde{\sigma}_n$ having endpoints $x_n, y_n \in S_\infty$ such that $d(x_n, a_n) < 1/n$ and $d(y_n, b_n) < 1/n$. Here the distance is Euclidean distance on the closed unit disk. Thus, $x_n \rightarrow x$ and $y_n \rightarrow y$ as required. \square

Corollary 12.59. *If $\ell \in \tilde{\Lambda}'_\pm$ is the leaf with endpoints $\{x, y\}$ and a side of ℓ is specified, the lifts $\tilde{\sigma}_n$ in part (1) of Proposition 12.58 can be chosen to lie on that side unless ℓ is a lift of a border leaf of a positive principal region P and the side in question borders a lift \tilde{P} .*

Proof. If the specified side of the leaf ℓ is not isolated, the argument in the proof of (1) of Proposition 12.58 goes through on that side of ℓ . If the specified side of ℓ is isolated and borders \mathcal{U}'_- on that side, we explicitly handled that case in the proof of (1) of Proposition 12.58. If the side of ℓ is isolated and the leaf ℓ borders a positive principal region P on that side, there are no negative junctures meeting P , by definition. \square

Proposition 12.60. $X_\pm^g \subset X'_\pm$.

Proof. We prove that $X_+^g \subset X'_+$. The proof that $X_-^g \subset X'_-$ is analogous. Suppose that $\{x, y\} \in X_+^g$. By Proposition 12.57, there exists a sequence $\{\sigma_n\} \subset \mathcal{Y}'_-$ and lifts $\tilde{\sigma}_n$ having endpoints $x_n, y_n \in S_\infty$ such that $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$. By definition, $\{x, y\} \in X_+^g$ implies that there exists $\lambda^g \in \Lambda_+^g$ having a lift $\tilde{\lambda}^g$ with endpoints x, y . Since $x_n \rightarrow x$ and $y_n \rightarrow y$ as $n \rightarrow \infty$, it follows that the sequence of hyperbolic geodesics $\tilde{\sigma}_n^g$ converges strongly to the hyperbolic geodesic $\tilde{\lambda}^g$. In particular, the sequence $\{\sigma_n^g\}$ does not escape so, by Theorem 3.11, the sequence $\{\sigma_n\}$ does not escape.

Remark that the points $x, y \notin \partial\tilde{L}'$, hence lie in E . Let $B_1 \subset B_2 \subset \dots \subset B_j \subset \dots$ be an exhaustion of \tilde{L}' by compact sets homeomorphic to disks. Since the sequence $\{\sigma_n\}_{n \geq 1}$ does not escape, for some $j_0 \geq 1$ and all large enough values of n , $\tilde{\sigma}_n \cap B_{j_0} \neq \emptyset$. Thus, there exists points $z_n \in \tilde{\sigma}_n \cap B_{j_0}$. These points cluster on at least one point $z \in B_{j_0}$. By passing to a subsequence, we can assume that $z_n \rightarrow z$. The point z projects to a point $w \in L'$ on which the sequence of negative juncture components $\{\sigma_n\}$ accumulates. By Lemma 12.49, there is a leaf $\ell \in \Lambda'_+$ containing w . Thus, there is a lift $\tilde{\ell} \in \tilde{\Lambda}'_+$ containing z . Let $a, b \in S_\infty^1$ be the endpoints of $\tilde{\ell}$.

Without loss we can assume that the $\tilde{\sigma}_n$ all lie on the same side of $\tilde{\ell}$. By Corollary 12.59, there exists a sequence $\{\tau_k\}_{k \geq 1} \subset \mathcal{Y}'_-$ and lifts $\tilde{\tau}_k$ having endpoints $a_k, b_k \in S_\infty$ such that $a_k \rightarrow a$ and $b_k \rightarrow b$ as $k \rightarrow \infty$ with the $\tilde{\tau}_k$ all lying on the same side of $\tilde{\ell}$ as the $\tilde{\sigma}_n$'s. For each $k \geq 1$, the lifts $\tilde{\sigma}_n$ must be caught between $\tilde{\tau}_k$ and $\tilde{\ell}$, for all large enough values of n . This implies that the endpoints x_n, y_n converge to the endpoints a, b of $\tilde{\ell}$. Choose notation so that $x_n \rightarrow a$ and $y_n \rightarrow b$ as $k \rightarrow \infty$. Therefore $\{x, y\} = \{a, b\} \in X'_+$. \square

By Propositions 12.58 and 12.60, $X_\pm^g = X'_\pm$. Thus, the laminations Λ'_\pm satisfy the endpoint correspondence property and we have,

Corollary 12.61. *Axiom 3 holds for Λ'_\pm .*

All four axioms have been verified for (Λ'_+, Λ'_-) , hence the proof of Theorem 12.7 is complete.

13. FOLIATIONS

In Section 13.2, we give a heuristic, detailed outline of the authors' theory of "foliation cones" [12, 16] for taut, depth one foliations of compact 3-manifolds. More generally, the monodromy of a proper leaf in an open saturated foliated subset of a C^2 foliation of a compact 3-manifold M is an endperiodic automorphism. If such a leaf is not at depth one it will have infinite endset. In Section 13.3, we give a sketch of the structure of open saturated foliated sets and in Section 13.4 we show how the theory of endperiodic automorphisms can be extended to surfaces with infinite endset.

13.1. A Technical result about isotopic monodromies. Let $(M, \mathcal{F}) = (M_0, \mathcal{F}_0)$ be a depth one (Definition 12.2), compact, foliated 3-manifold with $\partial_7 M$ as sole compact leaves. As usual we set $M^\circ = M \setminus \partial_7 M$. Let $f = f_0 : L \rightarrow L$ be a monodromy for this foliation and let $f_1 : L \rightarrow L$ be an endperiodic automorphism isotopic to f_0 by an isotopy f_t , $0 \leq t \leq 1$. If f_0 and f_1 are diffeomorphisms, the isotopy can be taken to be smooth. It will not be necessary to assume that f_t is endperiodic, $0 < t < 1$. Let (M_1, \mathcal{F}_1) be the depth one foliated manifold with monodromy f_1 given by Lemma 12.5.

We are going to realize $M^\circ = M_0^\circ$ as $L \times [-1, 1]/\{(x, 1) \equiv (f_0(x), -1) \mid x \in L\}$. The factors $\{x\} \times [-1, 1]$ hook together to form a one dimensional foliation \mathcal{L}_0 transverse to $\mathcal{F}|M^\circ$ inducing the monodromy $f = f_0$ on $L \times \{0\}$. We modify \mathcal{L}_0 on $L \times [-1, 0]$ by replacing the segment $(f_0(x), t)$, $-1 \leq t \leq 0$, with the segment $(f_{t+1}(x), t)$, for all $x \in L$. Denote the new transverse foliation by \mathcal{L}_1 and note that it induces monodromy f_1 on $L \times \{0\}$. In the smooth case, \mathcal{L}_1 may have corners at $L \times \{-1\}$ and $L \times \{0\}$, but these can be smoothed by standard techniques. Thus, in constructing (M_1, \mathcal{F}_1) by Lemma 12.5, we can assume that $M^\circ = M_0^\circ = M_1^\circ$ canonically. Since the inclusions $M^\circ \hookrightarrow M_0$ and $M^\circ \hookrightarrow M_1$ are homotopy equivalences, we obtain the following result.

Lemma 13.1. *If $f_0, f_1 : L \rightarrow L$ are isotopic endperiodic automorphisms and (M_0, \mathcal{F}_0) , (M_1, \mathcal{F}_1) are the corresponding depth one foliated 3-manifolds given by Lemma 12.5, Then there is a canonical identification $M_0^\circ = M_1^\circ$, $\mathcal{F}_0|M_0^\circ = \mathcal{F}_1|M_1^\circ$ and a canonical homotopy equivalence of M_0 to M_1 .*

In particular, M_0 and M_1 have the same homology and cohomology. This will be critical in what follows.

Remark. It seems certain that $M_0 = M_1$ by a canonical homeomorphism (diffeomorphism if f_0 and f_1 are diffeomorphisms), but a proof seems surprisingly delicate. Lemma 13.1 is adequate for our current purposes.

13.2. Foliation cones. We give a brief synopsis of the results of [12, 16], indicating how Handel-Miller theory plays a key role in the proofs.

Remark. The theory of endperiodic automorphisms is basic to our theory of foliation cones. The development in [12, Section 5], particularly the proof of the transfer theorem [12, Theorem 5.8], is not adequate. We have rectified these problems in

this paper and [16]. In addition we have replaced the proof of [12, Lemma 4.10] by an elementary proof of [16, Theorem 4.1].

Let (M, \mathcal{F}) be a depth one foliated 3-manifold satisfying Hypothesis 7. That is, M is a compact connected 3-manifold that is not a product such that every component of $\partial_\tau M$ has negative Euler characteristic and the foliation \mathcal{F} is smooth, transversely oriented, taut, depth one, and induces no 2-dimensional Reeb components on $\partial_{\text{th}} M$.

Remark. If M has a tangential boundary component that is a toral leaf T , then each end of a leaf of \mathcal{F} that approaches T has a neighborhood of the form $S^1 \times [0, \infty)$ which spirals in on T . The torus $T \subset M$ has a neighborhood of the form $T \times [0, 1]$ with $T \times \{0\} = T$ and $T \times \{1\} = T'$, a torus transverse to \mathcal{F} . Then $M' = M \setminus T \times [0, 1]$ has T' as transverse boundary and $\mathcal{F}|_{T'}$ is a foliation of T' by circles. The 3-manifold M has at most finitely many tangential toral boundary components all of which can be converted to transverse toral boundary components in this way to yield a 3-manifold M' that is either fibered by $\mathcal{F}|_{M'}$ or is such that $\mathcal{F}|_{M'}$ is a depth one foliation satisfying Hypothesis 7.

Since (M, \mathcal{F}) can be assumed to be of class C^∞ , we will carry out all of our constructions in the smooth category. Let \mathcal{L}_f be a smooth, transverse, 1-dimensional foliation inducing an endperiodic diffeomorphism $f : L \rightarrow L$ on a depth one leaf. Let \mathfrak{X}_f denote the compact sublamination of \mathcal{L}_f consisting of the leaves that do not meet $\partial_\tau M$. This is the \mathcal{L}_f -saturation of the maximal compact f -invariant subset $X_f \subset L$. That is, \mathfrak{X}_f is the union of the leaves of \mathcal{L}_f which intersect L in points of X_f . Since we assume that f is not a translation, $X_f \neq \emptyset$.

Definition 13.2. The lamination \mathfrak{X}_f will be called the core (sub)lamination of \mathcal{L}_f .

The Schwartzmann-Sullivan theory of asymptotic cycles [41, 39] associates to the core lamination \mathfrak{X}_f a closed convex cone with compact base in the infinite dimensional space of closed de Rham 1-currents. For example, every closed leaf of \mathfrak{X}_f is a homology cycle asymptotic to \mathfrak{X}_f . More generally, any leaf $\ell \in \mathfrak{X}_f$ defines “long almost closed orbits” which converge to asymptotic cycles. One takes a sequence of longer and longer subarcs P_k of ℓ , divides by the length $\|P_k\|$ of the subarc, obtaining a sequence of de Rham 1-currents containing subsequences that converge to asymptotic cycles called “homology directions”. All asymptotic cycles are limits of sequences of linear combinations of homology directions with positive coefficients. One also allows 0 to be considered an asymptotic cycle so that the cone has a vertex. The cone of asymptotic cycles passes to a closed convex cone $\mathfrak{C}'_f \subset H_1(M; \mathbb{R})$ with compact base. The dual cone $\mathfrak{C}_f \subset H^1(M; \mathbb{R})$ consists of exactly those classes which take all values ≥ 0 on \mathfrak{C}'_f . This cone is closed and convex, has nonempty interior, but does not necessarily have a compact base.

Remark. There are extreme cases in which \mathfrak{C}'_f reduces to a single ray issuing from the origin, in which case \mathfrak{C}_f is an entire half space.

Definition 13.3 (foliated form). A 1-form $\eta \in A^1(M^\circ)$ is a foliated form if it is closed, nowhere vanishing and becomes unbounded at $\partial_\tau M$ in such a way that the corresponding foliation \mathcal{F}_η° that it defines on M° extends by adjunction of $\partial_\tau M$ to a transversely oriented C^∞ foliation \mathcal{F}_η of M , C^∞ -flat at $\partial_\tau M$.

The following is a compilation of [16, Theorem 4.9 and Theorem 7.1], the last assertion about the structure of the foliations being rather elementary.

Proposition 13.4. *The open cone $\text{int } \mathfrak{C}_f$ consists of classes in $H^1(M)$ that can be represented by foliated forms transverse to $\mathcal{L}_f|M^\circ$. The ray $\langle \eta \rangle$ issuing from the origin through any of these classes $[\eta]$ (called a foliated ray) determines the foliation \mathcal{F}_η uniquely up to a C^0 ambient isotopy. Those foliated rays passing through nontrivial elements of the integer lattice $H^1(M; \mathbb{Z})$ (called rational rays) determine foliations of depth one, while the irrational foliated rays define foliations which, in M° , are dense leaved without holonomy.*

Definition 13.5. If $\mathcal{G} = \mathcal{F}_\eta$ for a foliated form η , we denote the foliated ray $\langle \eta \rangle$ by $\langle \mathcal{G} \rangle$.

Definition 13.6. The cones \mathfrak{C}_f , obtained from smooth, one dimensional foliations \mathcal{L}_f transverse to \mathcal{F} , are called the *foliation cones* associated to \mathcal{F} .

One wants to find a monodromy h for \mathcal{F} with “tightest” dynamics in the sense that the cone \mathfrak{C}'_h is contained in every cone \mathfrak{C}'_g as g varies over the smooth monodromies for \mathcal{F} . Equivalently, \mathfrak{C}_h will contain every \mathfrak{C}_g . As the reader will have guessed, h will be a Handel-Miller monodromy (Definition 12.6). We are going to sketch the proof after some preliminary considerations.

To begin with, by Theorem 11.1, there is a smooth Handel-Miller automorphism h , that is a smooth endperiodic automorphism $h : L \rightarrow L$ isotopic to a monodromy automorphism $f : L \rightarrow L$ for \mathcal{F} and preserving a Handel-Miller pseudo-geodesic bilamination (Λ_+, Λ_-) associated to f . Using Lemma 12.5, construct a depth one foliated manifold (M_h, \mathcal{F}_h) with monodromy h . By Lemma 13.1, we have $M^\circ = M_h^\circ$. Given any smooth one dimensional foliation \mathcal{L} transverse to \mathcal{F} , its core sublamination \mathfrak{X} lives in $M^\circ = M_h^\circ$. The restriction of \mathcal{L} to M_h° is integral to a smooth vector field which, near $\partial_\tau M_h$ and outside a neighborhood of \mathfrak{X} can be smoothly modified so as to be defined and nonsingular on M_h and transverse to \mathcal{F}_h there. This provides a one dimensional foliation \mathcal{L}' of M_h , transverse to \mathcal{F}_h and having \mathfrak{X} as its core sublamination. By the equalities $H_1(M) = H_1(M_h)$ and $H^1(M) = H^1(M_h)$, we see that every foliation cone associated to \mathcal{F} is identical with a foliation cone associated to \mathcal{F}_h . Reasoning similarly, every foliation cone associated to \mathcal{F}_h is also a foliation cone associated to \mathcal{F} . We summarize.

Lemma 13.7. *Let h be a smooth Handel-Miller automorphism isotopic to a monodromy of \mathcal{F} . Then, under the canonical identification $H^1(M; \mathbb{R}) = H^1(M_h; \mathbb{R})$, the foliation cones associated to \mathcal{F} are identical with those associated with \mathcal{F}_h .*

Temporary Hypothesis. From now on in Section 13.2, we assume that \mathcal{F} itself has a smooth Handel-Miller monodromy h .

Let \mathcal{L}_h be a smooth, transverse, 1-dimensional foliation defining h , \mathfrak{X}_h the compact sublamination of \mathcal{L}_h which is the \mathcal{L}_h -saturation of the maximal, compact, h -invariant subset $X_h \subset L$. In general, X_h consists of $|\Lambda_+| \cap |\Lambda_-| \cup N$, where N is a compact surface, generally not connected, which is the union of the nuclei of principal regions. The components of N are permuted by h . An isotopy of h supported in N puts $h|N$ in Nielsen-Thurston canonical form [21] without destroying the fact that h leaves the laminations invariant. This means that N is partitioned into connected subsurfaces N_i , each of which is invariant under h^p , some $p \geq 1$, such that $h^p|N_i$ is either pseudo-Anosov or periodic. These subsurfaces are bordered by annuli in N and $h^p|N$ is smooth except at finitely many multi-pronged singularities.

Definition 13.8. If h is Handel-Miller and $h|N$ is in Nielsen-Thurston canonical form, h is said to be a tight Handel-Miller monodromy automorphism for \mathcal{F} .

Remark. The transfer theorem (Theorem 12.7) easily extends to show that tight Handel-Miller monodromy h on L transfers to tight Handel-Miller monodromy h' on L' [16, Theorem 6.20]. Since $\mathfrak{X}_h = \mathfrak{X}_{h'}$, we see that $\mathfrak{C}_h = \mathfrak{C}_{h'}$.

Remark that h is not uniquely determined by its isotopy class. We state without proof Theorem 6.18 of [16].

Proposition 13.9. *The cone $\mathfrak{C}_h \subset H^1(M)$ is independent of the choice of the representative h of tight Handel-Miller monodromy automorphism for \mathcal{F} .*

Because of this proposition, we will often denote \mathfrak{C}_h by $\mathfrak{C}_{\mathcal{F}}$.

Definition 13.10 (Handel-Miller foliation cone). The cone $\mathfrak{C}_h = \mathfrak{C}_{\mathcal{F}}$ is called the *Handel-Miller foliation cone* associated to \mathcal{F} .

The Handel-Miller foliation cones have a nice geometric structure which often makes explicit computations possible.

Proposition 13.11. *Each Handel-Miller foliation cone $\mathfrak{C}_{\mathcal{F}} \subset H^1(M; \mathbb{R})$ is polyhedral.*

Outline of the proof. Let $X_h^* = |\Lambda_+| \cap |\Lambda_-|$, called the *meager* invariant set, and let \mathfrak{X}_h^* denote the \mathcal{L}_h -saturation of the meager invariant set. For each Nielsen-Thurston component N_i , let \mathfrak{X}_{N_i} denote the \mathcal{L}_h -saturation. If A is the union of annuli bordering these components, as well as any annular or Möbius nuclei of principal regions, let \mathfrak{X}_A denote the \mathcal{L}_h -saturation of A . We can ignore nuclei that are disks since their \mathcal{L}_h -saturation contribute no asymptotic cycles not already contributed by \mathfrak{X}_h^* . The Markov partition of Section 9 determines finitely many minimal period h -orbits in X_h^* which correspond to minimal loops in \mathfrak{X}_h^* . Homologically, all loops in X_h^* are linear combinations of minimal loops with coefficients nonnegative integers. In turn, homology directions corresponding to leaves of \mathfrak{X}_h^* are limits, homologically, of sequences of nonnegative linear combinations of loops, while every asymptotic cycle for \mathfrak{X}_h^* is in the closure of the linear span with nonnegative coefficients of the homology directions. Bottom line, the finitely many minimal loops span the homology cone represented by asymptotic cycles of \mathfrak{X}_h^* . Similarly, if N_i is a pseudo-Anosov component of the Nielsen-Thurston decomposition of N , invariant under h^p , there is a Markov partition for $h^p|N_i$ and the same reasoning shows that the homology cone represented by asymptotic cycles of \mathfrak{X}_{N_i} is spanned by finitely many minimal loops. If N_i is a periodic component, \mathfrak{X}_{N_i} is a compact 3-manifold which is Seifert fibered by \mathcal{L}_h . The cone has a single generator. It is elementary that \mathfrak{X}_A has finitely many generating cycles, at most 2 for each annulus or Möbius strip. All of this gives a finite spanning set, with possible redundancies, for \mathfrak{C}'_h . It follows easily that \mathfrak{C}'_h and $\mathfrak{C}_h = \mathfrak{C}_{\mathcal{F}}$ are polyhedral. \square

Proposition 13.12. *Let h be a tight Handel-Miller monodromy for \mathcal{F} . Then \mathfrak{C}_h is the maximal foliation cone \mathfrak{C}_u as u ranges over the (smooth) monodromies for \mathcal{F} .*

Sketch of the proof. If not, one easily finds a rational foliated ray $\langle \mathcal{G} \rangle$ lying in the boundary of \mathfrak{C}_h . Let g be a tight Handel-Miller monodromy for \mathcal{G} . Then $\langle \mathcal{G} \rangle \subset \text{int } \mathfrak{C}_g$ and so $\text{int } \mathfrak{C}_h$ and $\text{int } \mathfrak{C}_g$ intersect. Let $\langle \mathcal{H} \rangle$ be a rational foliated ray in the interior of both cones. By Proposition 13.4, this ray is represented by a foliated form

η transverse to $\mathcal{L}_h|M^\circ$. By [10], it follows that \mathcal{H} is isotopic to a foliation \mathcal{H}' transverse to \mathcal{L}_h by an ambient isotopy that is smooth in M° . By Theorem 12.7, \mathcal{L}_h induces tight Handel-Miller monodromy h' on the depth one leaves of \mathcal{H}' . Similarly, \mathcal{H} is isotopic to a foliation \mathcal{H}'' transverse to \mathcal{L}_g which induces tight Handel-Miller monodromy on the depth one leaves of \mathcal{H}'' . An ambient isotopy of \mathcal{H}'' to \mathcal{H}' , applied to \mathcal{G} and \mathcal{L}_g allows us to assume that \mathcal{H}' is transverse to both \mathcal{L}_h and \mathcal{L}_g , each inducing tight Handel-Miller monodromy h' and g' , respectively, on the depth one leaves of \mathcal{H}' . By Proposition 13.9, $\mathfrak{C}_g = \mathfrak{C}_{g'} = \mathfrak{C}_{h'} = \mathfrak{C}_h$, contrary to the assumption that $\langle \mathcal{G} \rangle$ lies on the boundary of \mathfrak{C}_h . \square

Remark. The use of the transfer theorem in the above argument is critical. We know of no other way to prove the maximality of the Handel-Miller foliation cones.

Finally, there are only finitely many Handel-Miller foliation cones by [12, Theorem 6.4] or [16, Theorem 6.25], hence we have classified all depth one foliations of M up to isotopy by a finite set of combinatorial data. Explicit computations of the cones for some knot complements have been made. For instance, cf. [12, Section 7].

Remark. This theory is closely analogous to the classification of smooth foliations without holonomy transverse to ∂M . These foliations are either fibrations of M over S^1 or they are dense leaved. A well known theorem of W. Thurston [42] shows that, if M has any such foliations, certain top dimensional faces of the ‘‘Thurston ball’’ (a convex polyhedron which is the unit ball of the Thurston norm) subtend polyhedral cones $\mathfrak{C}_1, \mathfrak{C}_2, \dots, \mathfrak{C}_r \subset H^1(M; \mathbb{R})$ such that the rational rays in the interior of these cones correspond one-to-one to the smooth isotopy classes of fibrations. Furthermore, combining the Laudenbach-Blank theorem [32] with a theorem of the authors [13], the irrational rays in the interiors of the Thurston cones correspond one-to-one to the C^0 isotopy classes of dense leaved foliations without holonomy.

Remark. Recently, I. Altman [1] has shown that, with some important restrictions on the sutured manifold (M, γ) , our maximal foliation cones are subtended by certain top dimensional faces of the dual Juhász polytope [31], the unit ball for a *nonsymmetric* norm defined via sutured Floer homology. For these sutured manifolds, the nonsymmetry of the norm implies that our cones are not permuted under multiplication by -1 . In the general case, this is a consequence of the fact that, as part of the definition of a sutured manifold, there is a given transverse orientation on the boundary leaves. This poses a restraint on which transversely oriented foliations are allowed. Reversing the boundary orientations changes the set of maximal foliation cones by multiplication by -1 .

13.3. Open foliated sets of relative depth one. In this subsection, we sketch how the relation between depth one foliations and endperiodic automorphisms can be extended to open foliated sets without holonomy in compact 3-manifolds. See [4, Section 5.2] for a treatment of open foliated sets without holonomy. In [17] we begin a theory of foliation cones for open foliated sets without holonomy.

13.3.1. The C^2 case. Let \mathcal{F} be a C^2 foliation of a compact 3-manifold M . We continue to require \mathcal{F} to be transversely oriented, taut and without 2-dimensional Reeb components in $\partial_{\text{th}}M$. We also require that every compact leaf of \mathcal{F} has strictly negative Euler characteristic and that no noncompact leaf is a plane, open annulus or open Möbius strip. We fix a transverse, 1-dimensional foliation \mathcal{L} .

Let $W \subseteq M$ be an open, connected \mathcal{F} -saturated set (i.e., an open, connected union of leaves). There is a notion of “transverse completion” \widehat{W} [4, Page 130] of such a set, analogous to the notion “internal completion” (Definition 5.1). Essentially, it is a generally noncompact, foliated 3-manifold with finitely many boundary leaves and perhaps infinitely many transverse boundary components. We continue to write $\partial\widehat{W} = \partial_\tau\widehat{W} \cup \partial_\hbar\widehat{W}$. Some or all of the finitely many boundary leaves may be noncompact and some or all of the components of $\partial_\hbar\widehat{W}$ may be noncompact. Again, we require that the induced foliation of $\partial_\hbar\widehat{W}$ contain no Reeb components. Also, $\partial_\tau\widehat{W}$ and $\partial_\hbar\widehat{W}$ are separated by convex corners wherever they meet. The natural inclusion $\iota : W \hookrightarrow M$ extends to a natural immersion $\widehat{\iota} : \widehat{W} \looparrowright M$ which may identify some boundary leaves pairwise. We set $\widehat{\mathcal{F}} = \widehat{\iota}^{-1}(\mathcal{F})$, and $\widehat{\mathcal{L}} = \widehat{\iota}^{-1}(\mathcal{L})$, the foliations induced on \widehat{W} by \mathcal{F} and \mathcal{L} , respectively. These foliations are C^2 .

Definition 13.13 (relative depth one). The foliation $\widehat{\mathcal{F}}$ has relative depth one if $\widehat{\mathcal{F}}|W (= \mathcal{F}|W)$ fibers W over S^1 .

Remark. If W is an open saturated set without holonomy in a foliation \mathcal{F} with \widehat{W} not foliated as a product, then $\widehat{\mathcal{F}}$ has relative depth one.

In order to properly visualize these foliations, we need the notion of an “octopus decomposition” [4, Definition 5.2.13] of $\widehat{W} = K \cup A_1 \cup \dots \cup A_r$. Here, K is a compact, connected 3-manifold with boundary and corners, foliated by $\widehat{\mathcal{F}}|K$, the corners dividing $\partial_\tau K = K \cap \partial_\tau\widehat{W}$ from $\partial_\hbar K$. We call K the nucleus of the octopus decomposition and A_i the arms. The arms are of the form $A_i = L_i \times I$, the I -fibers being leaves of $\widehat{\mathcal{L}}|A_i$ and L_i a noncompact, connected surface with boundary. The arms attach to K along annular components of $\partial_\hbar K$ and/or rectangular subsets of $\partial_\hbar K$.

As a consequence of the Generalized Kopell Lemma [4, Lemma 8.1.24], one proves that, when $\widehat{\mathcal{F}}$ has relative depth one, the junctures in $\partial_\tau\widehat{W}$ are compactly supported cohomology classes κ . This requires differentiability of class at least C^2 . This is a special case of [4, Theorem 8.1.26]. The following is an easy consequence.

Proposition 13.14. *If $\widehat{\mathcal{F}}$ has relative depth one, the nucleus of the octopus decomposition can be chosen large enough that, in the arms A_i , $\widehat{\mathcal{F}}$ is the product foliation with leaves $L_i \times \{t\}$.*

Thus, everything interesting happens in $\widehat{\mathcal{F}}|K$, which is an honest depth one foliation. Since $\partial_\tau K \subseteq \partial_\tau\widehat{W}$, we can assume that every component F of $\partial_\tau K$ has negative Euler characteristic. Indeed, either F is a full component of $\partial_\tau\widehat{W}$, in which case it is a compact leaf of \mathcal{F} and has negative Euler characteristic by hypothesis, or F has nonempty boundary and is contained in a noncompact component N of $\partial_\tau\widehat{W}$. By hypothesis, N is not a plane, open annulus or open Möbius strip. Thus, choosing K large enough guarantees that F is not a disk, closed annulus or closed Möbius strip. This shows that the depth one foliation $\widehat{\mathcal{F}}|K$ of the compact 3-manifold K satisfies all the requirements of our previous discussion.

Let L be a leaf of $\mathcal{F}|W$ and $L' = L \cap K$. Since the leaves of $\widehat{\mathcal{F}}|A_i$ are of the form $L_i \times \{t\} \subset L_i \times I$, $1 \leq i \leq r$, the fact that L is connected implies that L' is connected. By Lemma 12.4, the monodromy of L' is endperiodic and so the monodromy of L is endperiodic (in the sense of Definition 2.4) if infinite endsets are allowed.

Remark. In Section 13.4 we show how the theory of endperiodic automorphisms can be extended to surfaces with infinite endset.

Proposition 13.15. *If $f : L \rightarrow L$ is the monodromy of a leaf of $\mathcal{F}|W$, where \mathcal{F} is C^2 and $\widehat{\mathcal{F}}$ has relative depth one, then f is endperiodic.*

Remark. We refer to L' as the *soul* of L (Definition 13.31). The above construction provides motivation for the construction of the soul (Proposition 13.30) in the general situation of an endperiodic automorphism of a surface with infinite endset. In the construction of Proposition 13.30, the ends that are not periodic are pared off to obtain the soul. In the above construction these ends that are pared off of L to obtain the soul L' are the surfaces $L_i \times \{t\}$, $1 \leq i \leq r$, which attach to L' . In the above construction the choice of L' is not unique but clearly depends on how large we choose K . In the construction of Proposition 13.30 the soul also depends on choices.

13.3.2. *Finite depth foliations.* One says that a leaf of \mathcal{F} has depth 0 if it is compact. Inductively, a leaf L has depth $r \geq 1$ if $\bar{L} \setminus L$ is a union of leaves of depths $\leq r - 1$, at least one of which has depth $r - 1$. It is possible that a leaf has no well defined depth (for example, a leaf that is dense in M).

Definition 13.16 (depth r). The foliation \mathcal{F} has depth r if all leaves have finite depth and r is the least upper bound of the depths.

The leaves at depth r of a depth r foliation unite to form an open saturated set and one can establish the following.

Proposition 13.17. *If \mathcal{F} is a depth r foliation and W is a component of the union of depth r leaves, then the foliation $\widehat{\mathcal{F}}$ of \widehat{W} has relative depth one.*

It is easy to construct C^∞ depth r foliations with all leaves proper for arbitrary $r > 1$. For examples, see [6, Theorems 2, 3]. These examples have leaves of type $r - 1$ (Definition 13.18) and thus monodromy an endperiodic automorphism of a surface with infinite endset.

In many of the examples of finite depth foliations constructed by Gabai' [26], it is possible to choose all the junctures to be compact. If all junctures are compact, [11, Main Theorem] implies that the finite depth foliation can be C^∞ smoothed giving many examples of C^∞ finite depth foliations.

13.3.3. *Leaves at relative depth one with a Cantor set of ends.* Let $X \subset M$ be an exceptional minimal set of a C^2 foliation \mathcal{F} of M . This is a compact, nonempty, saturated set which does not properly contain another such and does not reduce to a single compact leaf nor to M itself. Such a set is transversely Cantor. A theorem of G. Duminy (unpublished, but cf. [14]) asserts that the semi-proper leaves in X (i.e., those which border a gap in the Cantor set) have a Cantor set of ends. If W is a connected component of $M \setminus X$ and $\widehat{\mathcal{F}}$ is of relative depth one in \widehat{W} , then the leaves of $\mathcal{F}|W$ will have a Cantor set of ends.

There are many examples of exceptional minimal sets. See, for example, Raymond's example [5, Corollary 8.4.2]. In these examples the leaves will have monodromy which is a normal endperiodic automorphism (Definition 13.21) and thus, by Propositions 13.23 and 13.25, finitely many of the ends will be attracting and finitely many will be repelling.

13.3.4. *Hyperbolic knots and the C^{0+} case.* A C^0 foliation is C^{0+} if its leaves are integrable to a C^0 2-plane field. If $\gamma \subset S^3$ is a hyperbolic knot, then $M = S^3 \setminus \gamma$ is a cusped hyperbolic 3-manifold. In general, a cusped hyperbolic 3-manifold has finitely many cusps which are topologically of the form $T^2 \times [0, \infty)$. Amputating these cusps along $T^2 \times \{0\}$ leaves behind a compact 3-manifold M' with finitely many new boundary components, all tori. A finite depth foliation \mathcal{F} of a cusped 3-manifold is to be a product foliation in each cusp $T^2 \times [0, \infty)$, the leaves there being of the form $C \times [0, \infty)$ where C ranges over a family of circles in T^2 which fiber that torus. The foliation $\mathcal{F}|_{M'}$ will be finite depth in the usual sense, meeting the new tori in $\partial M'$ in circles that fiber those tori and in general are only known to be C^{0+} . If $\mathcal{F}|_{M'}$ is C^2 , then the monodromy of the leaves of an open saturated set without holonomy will be endperiodic and the methods of Section 13.3.2 apply. If $\mathcal{F}|_{M'}$ can not be C^2 smoothed, then some junctures will not be compact ([11, Main Theorem]) and the monodromy of the leaves of an open saturated set without holonomy will not be endperiodic. An approach to the type of monodromy that occurs in these C^{0+} foliations that can not be C^2 smoothed is given in [17].

D. Gabai [28, Theorem 3.1] shows how to construct taut, finite depth foliations of a knot complements in S^3 which meet the boundary of the knot complement in circles. The construction of these foliations is intrinsically C^{0+} and not all of these foliations can be smoothed. In fact, In [9, Theorem III] the authors give an example of a knot in S^3 that has no taut, finite depth, C^2 foliation which meets the boundary of the knot complement in circles.

13.4. Endperiodic automorphisms of surfaces with infinite endsets. In this subsection we treat endperiodic automorphisms of surfaces with infinite endset. The definitions and results here are all exemplified by the foliations $\widehat{\mathcal{F}}$ of \widehat{W} of relative depth one of Section 13.3.

Recall that $\mathcal{E}(L)$ is a totally disconnected separable metric space. Define the 0th derived endset to be $\mathcal{E}^{(0)}(L) = \mathcal{E}(L)$. If, for an ordinal $\alpha \geq 0$, the α^{th} derived endset $\mathcal{E}^{(\alpha)}(L)$ has been defined and is a compact subset of $\mathcal{E}(L)$, then the $(\alpha + 1)^{\text{th}}$ derived set $\mathcal{E}^{(\alpha+1)}(L)$ is the set of cluster points in $\mathcal{E}^{(\alpha)}(L)$. If β is a limit ordinal and $\mathcal{E}^{(\alpha)}(L)$ has been defined for all $\alpha < \beta$, we define the β^{th} derived set to be $\mathcal{E}^{(\beta)}(L) = \bigcap_{\alpha < \beta} \mathcal{E}^{(\alpha)}(L)$. It can be shown that, for a first countable ordinal γ , either $\mathcal{E}^{(\gamma)}(L)$ is finite and nonempty, or $\mathcal{E}^{(\gamma)}(L) = \mathcal{E}^{(\Omega)}(L)$ is a Cantor set, where Ω is the first uncountable ordinal [36].

The following terminology has been used by the authors elsewhere and will also be found in [4, Section 4.1], but may not be considered standard.

Definition 13.18 (type of surface). Let $\gamma \geq 0$ be the first countable ordinal such that $\mathcal{E}^{(\gamma)}(L)$ is either finite or a Cantor set. If $\mathcal{E}^{(\gamma)}(L)$ is finite and nonempty, the surface L has *topological type* γ . Otherwise, L has topological type Ω . If L is compact it is said to have topological type -1 .

Definition 13.19 (type of end). An end of L is of type α if it is isolated in the α^{th} derived endset. If the end lies in $\mathcal{E}^{(\Omega)}(L)$, it has type Ω .

Remark. In C^2 foliations, leaves at finite depth k have topological type $k - 1$ and growth type exactly polynomial of degree k [7, Theorem 6.0]. This was one of the first theorems relating the topology of a leaf and its volume growth function. It is valid for codimension 1 foliations of compact n -manifolds, $n \geq 3$.

In what follows L is allowed to have any topological type $\leq \Omega$.

Lemma 13.20. *If $f : L \rightarrow L$ is endperiodic, then a neighborhood of a periodic end e' cannot lie in an f -neighborhood U_e for another periodic end e .*

Proof. For then the iterates $f^k(e')$ would all have to be distinct, contradicting the fact that e' is periodic. \square

In our applications to foliations in Section 13.3, it was necessary to consider endperiodic automorphisms $f : L \rightarrow L$ where L has topological type Ω (cf. Section 13.3.3). Without some added hypothesis, examples show that some very undesirable bizarre behavior can arise. The following condition will always be verified in the foliation setting.

Definition 13.21 (normal endperiodic automorphism). An endperiodic automorphism $f : L \rightarrow L$ is normal if the f -minimal sets in $\mathcal{E}(L)$ are the finite periodic orbits.

Of course, the finite periodic orbits are f -minimal. By the standard Zorn's Lemma argument, there are f -minimal sets. By the following, normality only becomes a restriction if L has type Ω .

Lemma 13.22. *If $f : L \rightarrow L$ is endperiodic and L has topological type $\gamma < \Omega$, then f is normal.*

Proof. Let $X \subseteq \mathcal{E}(L)$ be an f -minimal set. Let $e \in X$. If this is not a periodic end, then $\{f^k(e)\}_{k \in \mathbb{Z}}$ is infinite and clusters at every point of X , including e itself. If e has type α , then $\alpha \leq \gamma < \Omega$ and every element of X must have type $> \alpha$, contradicting the fact that e has type α . This contradiction means that the end e is periodic, so f is normal. \square

Remark. Further, if L with topological type Ω is a leaf of a C^2 foliation of an open saturated set W of relative depth one as in Section 13.3.1, then the monodromy map $f : L \rightarrow L$ is a normal endperiodic automorphism.

Proposition 13.23. *If $f : L \rightarrow L$ is endperiodic and normal, there are only finitely many periodic ends.*

Proof. Let $\mathcal{P}(L) \subseteq \mathcal{E}(L)$ be the set of periodic ends. If this set is infinite, the set $\mathcal{Q}(L) \subseteq \mathcal{E}(L)$ of cluster points of $\mathcal{P}(L)$ must be compact, f -invariant and nonempty. Thus, $\mathcal{Q}(L)$ contains an f -minimal set. Since f is normal, this is a finite periodic orbit on which $\mathcal{P}(L)$ accumulates, contradicting Lemma 13.20. \square

Example 13.24. The methods of [18, Section 5] can be used to give an example of an endperiodic automorphism $f : L \rightarrow L$ in which the endset of L is the union of one negative, isolated, nonplanar end and a Cantor set of nonplanar ends containing a countable infinite set of positive ends and no other periodic ends. Thus, Proposition 13.23 is not true without the assumption that the endperiodic automorphism is normal.

Proposition 13.25. *If L has topological type $\alpha \geq 1$, a normal endperiodic automorphism $f : L \rightarrow L$ has both positive and negative ends.*

Proof. There will be finitely many periodic ends of type α , each necessarily negative or positive. If there are no positive ends of type α replace f by f^{-1} in the following proof. Therefore we can assume there is a positive end e of type α .

The fact that $\alpha \geq 1$ implies that $\mathcal{E}(L)$ is infinite. Hence Proposition 13.23 implies that there are nonperiodic ends. Let U_e be an f -neighborhood of e and let $e_0 \in U_e$ be an end with $e_0 \neq e$. Then by Definition 2.2, e_0 is a nonperiodic end. If $e_i = f^{ip_e}(e_0)$, the set of accumulation points of $\{e_i\}_{i=0}^{-\infty}$ contains a minimal set which is a finite periodic orbit since f is normal. Clearly this orbit cannot contain e . By Lemma 2.6, the ends in this orbit cannot be positive ends since each end in the orbit has an f -neighborhood meeting an f -neighborhood of the positive end e . Thus, the set of negative ends is nonempty. \square

Remark. The next three examples, Lemma 13.22, and the remark following it, indicate that the concept “normal endperiodic automorphism” is a natural concept.

Example 13.26. In [18, Section 5], we give an example of an endperiodic automorphism $f : L \rightarrow L$ in which the endset of L is the union of one negative, isolated, nonplanar end and a Cantor set of nonplanar ends, none of which are periodic. Thus, Proposition 13.25 is not true without the assumption that the endperiodic automorphism is normal.

Example 13.27. Let $C \subset S^2$ be a Cantor set, $L = S^2 \setminus C$ be an the open, planar surface with $\mathcal{E}(L) = C$, and $\nu : C \rightarrow C$ be the homeomorphism without periodic points given in [18, Proposition 2]. Then it is easy to define a homeomorphism of S^2 which induces the homeomorphism ν on C and thus restricts to a homeomorphism $f : L \rightarrow L$ which has no periodic ends. Thus f is an endperiodic automorphism by default and has no positive nor negative ends. The endperiodic automorphism f obviously cannot be normal.

Remark. Examples 13.24, 13.26, 13.27, Lemma 13.22, and the remark following it, indicate that the concept “normal endperiodic automorphism” is a natural concept.

Definition 13.28 (escaping end of surface). An end e of L is escaping if $\{f^k(e)\}_{k \in \mathbb{Z}}$ accumulates exactly on an f -cycle of positive periodic ends as $k \rightarrow \infty$ and on an f -cycle of negative ones as $k \rightarrow -\infty$.

Lemma 13.29. *If $f : L \rightarrow L$ is a normal endperiodic automorphism, every non-periodic end of L is escaping.*

Proof. If $e \in \mathcal{E}(L)$ is nonperiodic, the set $\Omega_+(e)$ of accumulation points of $\{f^k(e)\}_{k \geq 0}$ is compact, nonempty and f -invariant. It contains a periodic point ε such that every f -neighborhood U_ε^i of ε in L is a neighborhood of some $f^k(e)$, $k \geq 0$. Since this is true as $k \rightarrow \infty$, ε must be a positive end. But once $f^k(e) \in U_\varepsilon^i$, then $f^{k+pn_\varepsilon}(e) \in U_\varepsilon^{i+p}$, for all $p \geq 0$. It follows that $\Omega_+(e)$ is exactly the finite orbit of ε . In a similar way, define $\Omega_-(e)$ and prove that it is exactly the f -cycle of a negative end. \square

Finally, we turn to the process of “paring off” the nonperiodic ends of L to reduce the “interesting” dynamics of f to its action on an f -invariant subsurface $L' \subset L$ with finite endset, the Handel-Miller situation.

Proposition 13.30. *If $f : L \rightarrow L$ is a normal endperiodic automorphism, then there exists a subsurface $L' \subset L$ with finite endset such that,*

- (1) $f|_{L'} : L' \rightarrow L'$ is endperiodic;
- (2) $\text{Fr } L'$ in L has only compact components;
- (3) The components of $L \setminus L'$ are neighborhoods of all the escaping ends of L .

Proof. Let e_0 be a negative end and consider the cycle

$$c = \{e_0, f(e_0), f^2(e_0), \dots, f^{p_{e_0}-1}(e_0)\}.$$

of negative ends. Let U_c be an f -neighborhood of e_0 and let $X_c = U_c \setminus \text{int } f^{p_{e_0}}(U_c)$. Let $\mathcal{E}_c \subset \mathcal{E}(L)$ consists of the ends of L contained in X_c .

For each $e \in \mathcal{E}_c$ choose an open neighborhood V_e of e such that $\{f^n(\overline{V_e})\}_{n \in \mathbb{Z}}$ escapes, ∂V_e consists of one simple closed geodesic, and $\overline{V_e} \subset \text{int } X_c$. The set $\{V_e \mid e \in \mathcal{E}_c\}$ is an open cover of the compact set \mathcal{E}_c so there exists a finite subcover $\{V_{e_1}, \dots, V_{e_k}\}$. Then $O_c = \bigcup_{i=1}^k V_{e_i}$ has finitely many piecewise geodesic boundary components $\gamma_1, \dots, \gamma_\ell$, the set $\{f^n(\overline{O_c})\}_{n \in \mathbb{Z}}$ escapes, and $\overline{O_c} \subset \text{int } X_c$.

Construct such an O_c for every cycle c of negative ends. Then $L' = L \setminus \bigcup_c O_c$, where the union is over all cycles of negative ends, is a standard surface, satisfying (1), (2), and (3) of the proposition and thus having no escaping ends. By Proposition 13.23, L has finitely many periodic ends. Thus, L' has finite endset and the proposition is proven. \square

Definition 13.31 (soul). The f -invariant, type 0 subsurface $L' \subset L$ will be called the *soul* of L .

Of course, the soul is not unique.

Glossary of Symbols

- ∂ , Fr: Page 3
- δ : Definition 5.2
- E : Definition 3.4
- $\mathcal{E}(L)$: Page 3
- $\mathcal{E}_\pm(L)$: Definition 2.2
- $\mathcal{E}^\alpha(L)$, $\mathcal{E}^\Omega(L)$: Page 133
- Fr: Page 3
- γ^g : Definition 4.19
- ι : Definitions 4.20, 10.5
- ι' : Definition 10.36
- \ddot{i} : Definition 5.1
- J_κ : Definition 2.15, Lemma 6.40
- J_n : Definition 4.22, 10.6
- \mathcal{J} , \mathcal{J}_+ , \mathcal{J}_- : Definitions 4.21, 10.5
- \mathcal{J}_W : Definition 4.42
- K , K_i : Definition 4.41
- Λ_F : Lemma 6.40
- N , N_+ , N_- : Definition 4.17
- p_e : Definition 2.1
- \mathbb{Q}^+ : Definition 9.3
- \mathfrak{S} : Sections 6.5, 7.2, 7.3, and Definition 6.34
- S_∞ : Definition 3.12
- S_∞^1 , Δ , \mathbb{D}^2 : Page 18
- T: Proposition 6.45
- \mathfrak{T}^g , \mathfrak{T}_*^g : Definition 4.58
- U_e : Definition 4.37
- U_e^n : Definition 4.39

$\mathcal{U}_e, \mathcal{U}_\pm$: Definitions 4.64
 \mathcal{U} : Definition 4.67
 \ddot{U} : Definition 5.1
 W^\pm, W_i^\pm : Definition 4.41
 W : Definition 4.42
 \mathcal{X}_\pm : Definition 4.26
 \mathcal{Y}'_\pm : Definition 12.48

Glossary of Terms

admissible: Definition 4.14
arm: Definition 6.17
assemblage: Definition 12.51
attracting/repelling ends: Page 2
bilamination: Definition 4.7
bilaminated chart: Definition 4.6
border component: Definition 5.2
border component of the first/second kind: Section 7.6
completely crosses: Definition 9.4
converges strongly: Definition 4.32
core and i^{th} core: Definition 4.41
core dynamical system: Definition 9.1
depth one foliation: Definition 12.2
depth r : Definition 13.16
distinguished neighborhood: Definition 4.37
dual crown sets: Definition 6.32
dual principal regions: Definition 6.29
end, endset: Page 3
endperiodic automorphism: Definition 2.4
endpoint correspondence property: Definition 10.2
escapes (junction): Definition 4.25
escapes (sequence of sets): Definition 3.7
escaping end of leaf: Definition 6.19
escaping end of surface: Definition 13.28
escaping set: Definition 4.67
escaping set (positive/negative): Definitions 4.64
essentially intersecting loops: Definition 12.20
extreme rectangle: Definition 6.2
 f -domain: Definition 2.24
 f -junction: Definition 2.9, Hypothesis 2
 f -junctions (fixed set of): Definition 4.17
 f -neighborhood of an end: Definition 2.3
first/second kind: Definition 7.6
foliated form: Definition 13.3
frontier: Page 3
geodesic lamination: Definition 4.16
geodesic tightening of a curve: Definition 3.6, 4.19
geodesic tightening map: Definitions 4.20, 10.5, 10.36
Handel-Miller (geodesic) bilaminations: Definitions 4.30
Handel-Miller pseudo-geodesic bilaminations: Definitions 10.7

Handel-Miller foliation cone: Definition 13.10
Handel-Miller monodromy: Definition 12.6
Hypothesis 1: page 6
Hypothesis 2: page 10
Hypothesis 3: page 15
Hypothesis 4: page 18
Hypothesis 5: page 29
Hypothesis 6: page 45
Hypothesis 7: page 109
ideal boundary: Definition 3.4
internal completion: Definition 5.1
invariant set: Definition 5.16
junction: Definitions 4.21, 10.5
junction intersection property: Definition 2.11
 κ -**junction:** Definition 2.15
lamination: Definition 4.2
laminated atlas, laminated partial atlas: Definition 4.3
laminated chart: Definition 4.1
locally uniform accumulation: Definition 4.5
Markov chain: Definition 11.6
Markov family, pre-Markov family: Definition 9.6
mesh: Definition 10.29
monodromy: Definition 12.3
neighborhood of an end: Page 3
normal endperiodic automorphism: Definition 13.21
nucleus: Definition 6.30
parallel packets: Definition 6.43
passes arbitrarily near: Definition 4.49
peripheral border components: Definition 7.18
positive/negative ends: Definition 2.2
principal regions (positive/negative): Definition 5.13
pseudo-anosov: Definition 8.13
pseudo-geodesic: Definition 4.18
pseudo-geodesic lamination: Definition 4.31
quadrilateral and geodesic quadrilateral: Definition 11.4
rectangle, rectangular: Definition 6.1
reducing curves: Section 6.5
relative depth one: Definition 13.13
rim: Definition 6.33
semi-isolated: Definition 5.6
simple end: Definition 2.27
sliding isotopy: Definition 4.56
smooth lamination, bilamination: Definition 4.10
soul: Definition 13.31
strongly closed property: Definition 4.33
standard metric, standard surface: Definition 3.5
stump: Definition 11.24
taut foliation: Definition 12.1

tile, tiling: Definition 4.58
transfer of a path: Definition 12.8
transfer of parametrized paths: page 110
transfer of a set of curves: Definition 12.11
translation: Definition 2.34
transversely totally disconnected: Definition 4.4
type of end: Definition 13.19
type of surface: Definition 13.18
vertex/edge: Lemma 7.3
virtually escapes: Definition 3.9
weakly groomed: Definition 2.16

Glossary of Axioms, Main Results, and Main Theorems

Axioms: Pages 81 - 82
constructing the geodesic bilaminations (Λ_+, Λ_-) : Section 4.3
core dynamics conjugate Markov shift: Theorem 9.2
defining h : Theorem 4.54
defining the reducing curves: Sections 6.5, 7.2, and 7.3
Epstein-Baer Theorems: Theorem 3.1 and 3.2
escaping ends of leaves: Section 6.3
escapes implies virtually escapes: Theorem 3.11 (see also Theorem 4.24)
finitely many semi-isolated leaves: Theorem 6.5
Isotopy Theorem: Theorem 10.15
periodic points and leaves: Section 6.2
Smoothing Theorem: Theorem 11.1
Transfer Theorem: Theorem 12.7
Uniqueness Theorem: Theorem 10.13

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