Complex Analysis
Qualifying exam. August 2012

Instructions: Do 4 problems from part I and 4 from part II.
Part I

(1) Show that the function $f(z) = |z|^2$ has complex derivative only at $z = 0$. What is $f'(0)$?
(2) Find the radius of convergence of the series $\sum_{n=1}^{\infty} a_n z^n$ when
   (a) $a_n = 2^{-n} n!$
   (b) $a_n = \frac{n^4}{2^n + n^2}$
   Explain your answer.
(3) (a) Evaluate $\int_\gamma \bar{z}\, dz$ where γ is the circle $|z-1| = 2$ positively oriented.
   (b) Evaluate $\int_\gamma \frac{dz}{z}$ where γ is the line segment from 2 to $1 + i$.
(4) Suppose $f(z)$ has a pole of order n at z_0. Suppose $g(z)$ is holomorphic at z_0. Let
$$G(z) = g(z)\frac{f'(z)}{f(z)}.$$
Show that
$$\mathrm{res}_{z_0} G = -g(z_0)n.$$
(5) Suppose f is an entire function and there are constants A and R_0 such that
$$\sup_{|z|=R} |f(z)| \leq AR^2$$
for $R > R_0$. Show that f is of the form $az^2 + bz + c$ for constants a and b and c.

Part II

(1) Suppose that $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix with non zero determinant. Define a fractional linear transformation
$$f_M(z) = \frac{az+b}{cz+d}.$$
If M and M' are two such matrices, show that
$$f_M \circ f_{M'} = f_{MM'}.$$
(2) The Fourier transform is defined by
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx.$$
Show that if $a > 0$ and
$$P_a(x) = \frac{1}{\pi}\frac{a}{a^2 + x^2}$$
then if $\xi < 0$, $\widehat{P_a}(\xi) = e^{2\pi a \xi}$.
(3) Prove that if $\sum |z_n|^3 < \infty$, then the product $\prod_{n=1}^{\infty}(1 - z_n)e^{z_n + z_n^2/2}$ converges.
(4) (a) Show that the series
$$\sum_{n=1}^{\infty} \frac{1}{z-n}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{z+n}$$

diverge absolutely for every z not an integer.

(b) Show that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right)$$

converges absolutely every z not an integer.

(5) Show that

$$w = -\frac{1}{2}\left(z + \frac{1}{z}\right)$$

is a conformal map from the halfdisc $\{z = x + iy : |z| < 1, y > 0\}$ to the upper half-plane $\operatorname{Im} w > 0$.