Preliminary Examination in Complex Analysis

Summer 2003 — August 18, 2003

Solve six (and only six!) out of the following list of problems.

1. Let f be a complex valued function defined on the open region $D \subset \mathbb{C}$ and write $f = u + iv$, where u and v are the real and imaginary parts, respectively. Prove the following fact:

> f is analytic at $z \in D$ if u and v are differentiable at $z = x + iy$ and the Cauchy-Riemann equations $u_x = v_y$, and $u_y = -v_x$ are satisfied at z.

2. A pair of real valued *harmonic* functions u and v defined on a region $D \subset \mathbb{C}$ are said to be *conjugate* if they are the real and imaginary parts of an analytic function $f = u + iv$.

a. Prove that if u is harmonic on a simply connected region D then there exists a harmonic conjugate.

b. Give a counterexample to explain why the simple connectedness hypothesis is required (Be sure to explain what "simply connected" means.)

3. Consider the following form of *Jordan's lemma:*

> Let the function $f(z)$ be analytic in the upper half plane $\mathbb{H} = \{z\colon \operatorname{Im}(z) > 0\}$ with the possible exception of a finite number of isolated singular points, and let it tend to zero as $|z| \to \infty$, uniformly in $\arg z \in [0, \pi]$. Then for $a > 0$
>
> $$\lim_{R\to\infty} \int_{C_R} e^{iaz} f(z)\, dz = 0\,,$$
>
> where C_R is the semicircular arc $\{z\colon\ |z| = R\,,\ \operatorname{Im}(z) > 0\,\}$ in $\mathbb{H}$.

a. Use Jordan's lemma and the residue technique to compute the *improper integral*

$$I = \int_0^\infty \frac{\cos x}{x^2 + 4}\, dx\,.$$

Be sure to explain why and how you apply Jordan's lemma and justify all your main steps.

b. Prove Jordan's lemma. (HINT: Use the inequality $\sin\theta \geq \frac{2}{\pi}\theta$ for $0 \leq \theta \leq \frac{\pi}{2}$.)

4. Let $f(z) = \frac{(z-1)^2}{z^3+1} \cdot \exp\left(\frac{1}{z^2-1}\right)$ and $g(z) = \pi^2 z^2 \csc^2(\pi z)$.

a. Determine and classify all the isolated singularities of f and g on the *Riemann Sphere* $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. (You will have to explain how to define " analytic" at ∞.)

b. Determine the singular parts and residues of f and g at those points.

5. Determine the number of zeros of the polynomial $P(z) = z^{87} + z^{36} - 4z^5 + 1$ contained in the open unit disk $\mathbb{D} = \{z\colon\ |z| < 1\}$.

6. Find explicitly a conformal map from the open unit disk $\mathbb{D}$ to the half plane H containing i and bounded by the line L passing through $-1-i$, the origin and $1+i$.

7. Consider the *Zhukovsky's function*

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Prove that it is conformal and univalent on the open disk $\mathbb{D}$ of radius 1 centered at the origin. Determine the image $\Delta = f(\mathbb{D})$ and discuss the behavior of f at the boundary

8. a. Fix a real number $R > 0$. Show that for $n \in \mathbb{N}$, $n > R/2$, and $z \in D_R = \{z\colon\ |z| < R\}$,

$$\left|\frac{1}{z^2 + 4n^2}\right| \leq \frac{1}{4n^2\left|1 - \frac{R^2}{R'^2}\right|},$$

where R' is any real number such that $n > R'/2 > R/2$.

b. Prove that

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^2 + 4n^2}$$

defines a meromophic function in the complex plane. What are the poles and corresponding residues?

9. The *order* of an elliptic function f is the multiplicity of the solution of $f(z) = \infty$ in the fundamental parallelogram P—in other words, the number of poles, counting multiplicity, inside P. Using known facts about elliptic functions, prove that there cannot exist elliptic functions of order one.

10. Find the genus and branch points for the covering of $\mathbb{P}^1$ by the Riemann surface associated to the *Fermat curve* C defined by the equation $z^n + w^n = 1$, where the covering map $\pi\colon C \to \mathbb{P}^1$ is $(z, w) \mapsto z$. (Your results must be expressed in terms of the integer n.)