Preliminary Examination in Complex Analysis

Fall semester 2005 — August 27

Solve four from part I and three from part II

Part I

1. Use known facts in Complex Analysis to prove the following statement:
If f is analytic on a simply connected region D of the complex plane, then f admits a primitive in D.

2. Compute the Laurent expansion of

$$f(z) = \frac{1}{z(z-1)(z-2)}$$

in the following annuli: $A(0;0,1)$, $A(0;1,2)$, and $A(0;2,\infty)$. (Notation: $A(p;r,R)$ denotes the annulus centered at p with radii r, R.)

3. Using known techniques in Complex Analysis, compute:

$$I = \int_{-\infty}^{\infty} \frac{x^2\, e^{-2\pi i\,\xi x}}{(4+x^2)^2}\, dx\,.$$

as a function of $\xi \in \mathbb{R}$. Be sure to justify your main steps.

4. Consider the *Zhukovsky's function*

$$w = f(z) = \frac{1}{2}\Big(z + \frac{1}{z}\Big)\,.$$

Prove that it is conformal and univalent on the open disk $\mathbb{D}$ of radius 1 centered at the origin. Determine the image $\Delta = f(\mathbb{D})$ and discuss the behavior of f at the boundary $\partial\mathbb{D}$.

5. Determine if the four points $0, 1, \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}, \dfrac{2i}{\sqrt{3}}$ lie on the same circle.

Part II

1. Suppose that f is a non-vanishing continuous function on the closed unit disk $\bar{\mathbb{D}}$ which is holomorphic on $\mathbb{D}$. Prove that if

$$|f(z)| = 1 \quad \text{whenever } |z| = 1,$$

then f is constant. (**Hint:** Extend to $|z| > 1$ by setting $f(z) = 1/\overline{f(1/\bar{z})}$. Why?)

2. Investigate the covering of $\widehat{\mathbb{C}}$ by $\widehat{\mathbb{C}}$ determined by the rational function

$$f(z) = \frac{-z^2+3}{z^3-2z}\,.$$

(Find the number of sheets, the branch points, the nature of the branching.)

3. Prove that the sum $\sum_{n \in \mathbb{Z}} 1/(z-n)^2$

a. converges normally on all compact subsets of $\mathbb{C} \setminus \mathbb{Z}$;

b. defines a meromorphic function of period 1.

4. Use known facts to prove that there are no elliptic functions of order 1. Justify your steps.