# Doctoral Qualifying Examination in Complex Analysis

Spring 2003 — December 30, 2002

Solve 6 (six) out of 8 (eight) problems. Show all your work. Explain your answers.

**1.** Let $A \subset \mathbb{C}$ be open and connected. Define $A^* = \{z | \bar{z} \in A\}$. If $f\colon A \to \mathbb{C}$ is analytic, prove that $f^*\colon A^* \to \mathbb{C}$, where $f^*(z) = \overline{f(\bar{z})}$, is also analytic.

**2.** a. Show that every function analytic in a simply connected region $D$ has an antiderivative in $D$.

b. Use part a show that if $f$ is analytic and never zero in a simply connected region $D$, there is a function $g$ analytic in $D$ such that $e^g = f$.

**3.** Compute with residues

$$\int_0^{2\pi} \frac{1}{(2-\sin\theta)^2}\, d\theta$$

**4.** For $R > 0$ let $\gamma_R$ be the path from $R$ to $R + Ri$, then from $R + Ri$ to $-R + Ri$, and finally from $-R + Ri$ to $-R$ along straight line segments.

a. Show that as $R \to \infty$ the integral

$$\int_{\gamma_R} \frac{e^{iaz}}{1+iz}\, dz\,, \quad a > 0$$

goes to 0.

b. Use the above to compute $(a > 0)$

$$\int_{-\infty}^{+\infty} \frac{e^{iax}}{1+ix}\, dx$$

**5.** a. Let $a, b \in \mathbb{C}$, with $0 < |a| < |b|$ and

$$f(z) = \frac{1}{(z-a)(z-b)}\,.$$

Find two distinct Laurent series for $f$ in annuli centered at the origin.

b. The residue of a function $g$ at $\infty$ is defined to be the residue of the function $-z^{-2}g(1/z)$ at $z = 0$. Find the residue of $f(z)$ at $\infty$.

**6.** Classify and find the residues at the singularities in the complex plane of the function

$$f(z) = \exp\left(\frac{1}{1-z}\right) + \frac{z}{(z^4-1)(z-i)}\,.$$

**7.** a. Show that if $n > 2|z|$ then

$$\left|\frac{1}{z-n} + \frac{1}{n}\right| \leq \frac{2|z|}{n^2}.$$

b. Show that

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n}\right)$$

converges to a function meromorphic in the plane. What are the poles and corresponding residues of this function?

**8.** Find explicitly a function giving a conformal map from the open disk $|z - i| < 1$ onto the strip $|\mathrm{Re}\, z| < 1/2$.