Preliminary Examination in Complex Analysis

January 5, 2004

Solve three from part I and three from part II. Indicate which one you want graded, or the first ones will be graded.

Part I

1. Let p be a real number not equal to ± 1. Compute using residues

$$\int_0^{2\pi} \frac{d\theta}{1 - 2p\cos\theta + p^2}.$$

2. Suppose the *Bernoulli polynomials* are defined by the Taylor expansion

$$\frac{ze^{wz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(w)}{k!} z^k.$$

Find the first three Bernoulli polynomials, $B_0(w)$, $B_1(w)$, $B_2(w)$.

3. Let f be a function with a simple pole at zero with residue a. Let C_R be a segment of a circle given by $z = Re^{i\theta}$ for $\theta_0 \le \theta \le \theta_0 + \alpha$. Let

$$I(R) = \int_{C_R} f(z)\, dz.$$

Show

$$\lim_{R \to 0} I(R) = \alpha i a.$$

4. Find the residues at all singularities of

$$\frac{\pi \cot \pi z}{16z^2 - 1}.$$

5. Consider the *Zhukovsky's function*

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right).$$

Prove that it is conformal and univalent on the open disk $\mathbb{D}$ of radius 1 centered at the origin. Describe the images of the circles $|z| = r$ for $0 < r < 1$.

Part II

1. Suppose f is a function analytic on some open set containing the closed disc $\{z : |z| \le 1\}$. Let Γ be the circle $|\zeta| = 1$ and suppose $|z| < 1$.

a) Show that

$$0 = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\bar{z}}{1 - \bar{z}\zeta}\, d\zeta$$

b) Use part a) and Cauchy's formula to show that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \frac{1 - |z|^2}{|1 - \bar{z}e^{it}|^2}\, dt$$

2. Suppose b_n is a sequence of complex numbers with $|b_n|$ increasing to ∞ and

$$\sum_{n=1}^{\infty} \frac{1}{|b_n|^3} < \infty.$$

Suppose $R < |b_N|$. Show that the series

$$\sum_{n=N}^{\infty} \left(\frac{1}{z - b_n} + \frac{1}{b_n} + \frac{z}{b_n^2} \right)$$

converges uniformly in $|z| < R$ to an analytic function.

3. The *order* of an elliptic function f is the multiplicity of the solution of $f(z) = \infty$ in the fundamental parallelogram P—in other words, the number of poles, counting multiplicity, inside P. Using known facts about elliptic functions, prove that there cannot exist elliptic functions of order one.

4. Let $GL(2, \mathbb{C})$ be the set of 2×2 matrices with complex entries and non-zero determinant. Let $\mathcal{M}$ be the set of Möbius (fractional linear) transformations. Consider the map π from $GL(2, \mathbb{C})$ to $\mathcal{M}$ given by

$$\pi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to T$$

where

$$T : z \to \frac{az + b}{cz + d}.$$

a. Show that

$$\pi(AB) = \pi(A) \circ \pi(B)$$

where $\circ$ indicates composition of mappings.

b. Find the inverse image of the identity under π