Preliminary Exam, Complex Analysis
Part I, August 2006

No hand calculators. 2 hours. Do all problems.

(1) Consider a function $f = u + iv$ of a complex variable $z = x + iy$. The existence of a complex derivative f' implies a relation between the partial derivatives of u and v. Explain this, with proof.

(2) Evaluate

$$\frac{1}{2\pi i}\int_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz$$

counterclockwise around the circle $|z| = 3$. (Your answer should be in terms of t, which can be any complex number.)

(3) Suppose $f(z)$ is holomorphic in the complex plane except for poles at $z = 1$ with residue 1 and at $z = -1$ with residue -1. Suppose also that

$$\lim_{z\to\infty} \frac{f(z)}{z^2} = 1.$$

Show that

$$f(z) = \frac{z^4 - z^2 + 2}{z^2 - 1} + az + b$$

for constants a and b.

(4) Show that

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\,dx = \pi\frac{e^{-a}}{a}$$

for all $a > 0$.