Preliminary Exam, Complex Analysis
Part II, August 2006

No hand calculators. 2 hours. Do four problems.

1. Prove that if $\sum |a_n| < \infty$ then the product $\prod_{n=1}^{\infty}(1+a_n)$ converges.

2. Let

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} \qquad t > 0$$

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \qquad \operatorname{Re} s > 0$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \operatorname{Re} s > 1.$$

Show that

$$\frac{1}{2}\int_0^\infty u^{(s/2)-1}[\vartheta(u) - 1]\, du = \pi^{-s/2}\Gamma(s/2)\zeta(s).$$

3. Suppose $f$ is a function analytic on some open set containing the closed disc $\{z : |z| \leq 1\}$. Let $\Gamma$ be the circle $|\zeta| = 1$ and suppose $|z| < 1$.

a) Show that

$$0 = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)\bar{z}}{1 - \bar{z}\zeta}\, d\zeta$$

b) Use part a) and Cauchy's formula to show that

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\frac{1 - |z|^2}{|1 - \bar{z}e^{it}|^2}\, dt$$

4. Which of the following limits converge to meromorphic functions in the plane? Explain your answer with proofs.

$$\sum_{n=1}^{\infty} \frac{1}{z-n} \tag{1}$$

$$\lim_{n\to\infty} \sum_{k=-n}^{n} \frac{1}{z-k} \tag{2}$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{z-n} + \frac{1}{n}\right] \tag{3}$$

$$\sum_{n=1}^{\infty} \left[\frac{1}{z-\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{z}{n}\right] \tag{4}$$

5. Suppose $\operatorname{Im}\tau > 0$ and that $F$ is the fundamental parallelogram given by

$$\{x + y\tau \,|\, 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Suppose that $f$ is an entire function with the following properties

$$f(z+1) = f(z)$$
$$f(z+\tau) = f(z)e^{-\pi i\tau}e^{-2\pi i z}.$$

and that $f$ has no zeros on the boundary of $F$. Show that $f$ has exactly one zero in $F$. (Hint: use arg. prin.)