Preliminary Exam, Complex Analysis
August 2007

No hand calculators. Do 4 problems from part I and 3 problems from part II.

Part I

(1) State the Cauchy Riemann equations and using them show that an analytic function satisfying $|f(z)| = 1$ for all z must be a constant. (Assume the domain is connected.)

(2) Use the method of residues to find an expression for the integrals

$$\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} \text{ where } -1 < a < 1$$

and

$$\int_{-\infty}^{\infty} \frac{\cos x\, dx}{x^2 + a^2} \text{ where } a > 0.$$

(3) Let $f(z)$ and $g(z)$ be analytic in the closed disk $|z| \leq 1$. Suppose that $|f(z)| > |g(z)|$ for $|z| = 1$. Use the argument principle to show that $f(z)$ and $f(z) + g(z)$ have the same number of zeros in the region $|z| < 1$. (This is a version of Rouché's Theorem).

(4) Find the first three terms of the Laurent series expansion of

$$\frac{e^z}{z(z^2+1)}$$

in each of the regions $0 < |z| < 1$ and $0 < |z + i| < 1$.

(5) Let f and g be holomorphic in the unit disk $|z| \leq 1$. Evaluate, in terms of f, g and z the integral

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \left(\frac{f(\zeta)}{\zeta - z} + \frac{zg(\zeta)}{z\zeta - 1} \right) d\zeta$$

for $|z| > 1$ and for $|z| < 1$.

Part II

(1) Prove that if $\sum |a_n| < \infty$ then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges.

(2) Prove that the transformation $x + iy = \sin(u + iv)$ maps the semi-infinite strip $0 \leq u \leq \pi/2, v \geq 0$ one to one onto a region of the x, y plane. Also find the region.

(3) Show that an analytic function which is meromorphic in the extended complex plane is the quotient of two polynomials (i. e., a rational function).

(4) For $x > 0$, the gamma function can be defined by

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt.$$

Prove the functional equation $\Gamma(z + 1) = z\Gamma(z)$ and show that $\Gamma(z)$ can be extended to a function meromorphic on $\mathrm{Re}\, z > -1$ with pole at $z = 0$ with residue 1.