Preliminary Exam, Complex Analysis
Part I, January 2006

No hand calculators. 2 hours. Do 5 problems and indicate which 5 you are solving.

(1) Suppose f and g are holomorphic in $|z| \leq 1$ and $f(z) \neq 0$ for $|z| = 1$. Show that for $\epsilon > 0$ sufficiently small, f and $f + \epsilon g$ have the same number of zeros in $|z| < 1$.

(2) Let C_R be the semicircle $\{z \,|\, |z| = R, \operatorname{Im} z > 0\}$.
   a) Show that
$$\int_{C_R} \frac{e^{3iz}}{(z^2+1)^2}\, dz$$
   goes to zero as $R \to \infty$.
   b) Using residues compute
$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}\, dx.$$
   Explain your method.

(3) Compute the following contour integrals
   a)
$$\int_C \frac{f'(z)}{f(z)}\, dz \qquad f(z) = \frac{(z^2+1)^2}{z^2+3z+6}$$
   where C is the circle $|z| = 2$
   b)
$$\int_C z^7\, dz$$
   where C is the curve $z(t) = t + it^2$, $0 \leq t \leq 1$.
   c)
$$\int_C |z|^2\, dz$$
   where C is the curve in part b).

(4) Suppose $|a_n|^{1/n} > L > 0$ for infinitely many integers n. Show that the power series
$$\sum_n a_n z^n$$
diverges for $|z| > 1/L$.

(5) Suppose f is holomorphic in $|z| \leq 1$ and $|f(z)| \leq M$ for $|z| = 1$. Show that for $|z| \leq 1/2$ we have
$$|f^{(n)}(z)| \leq n! M 2^{n+1}.$$

(6) Find the singular (principal) part of
$$\frac{\sin(\pi z/2)}{(z^2-3z+2)^2}$$
at all poles.