Complex prelims II

Solve 4 problems:

1. Show that the series
$$\frac{1}{z} + \sum_{n\in\mathbb{Z}\setminus 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$
defines a meromorphic function on $\mathbb{C}$. Determine the set of poles, and their orders.

2. Determine the subsets A and U of $\mathbb{C}$ where the function $\zeta(z) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ (the "Riemann Zeta function") converges absolutely and uniformly, respectively.

3. Let n be an integer. The n^{th} root of any Möbius transformation T is any Möbius transformation S such that $S^n = T$. Prove:
   (a) if $T = \mathrm{Id}$, then T has infinitely many n^{th} roots;
   (b) if T is parabolic then there is a unique n^{th} root;
   (c) in all other cases T has exactly n n^{th} roots.

4. Let $\mathrm{GL}_2(\mathbb{C})$ be the group of 2×2 invertible matrices with complex entries, and $\mathcal{M}\ddot{o}b$ the group of fractional linear transformations of the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let $\pi\colon \mathrm{GL}_2(\mathbb{C}) \to \mathcal{M}\ddot{o}b$ be the usual map assigning to the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ the fractional linear transformation:
$$z \longmapsto \frac{az+b}{cz+d}\,.$$
   (a) Determine the inverse image of the identity transformation;
   (b) Let $\mathrm{SL}_2(\mathbb{C})$ be the subgroup of $\mathrm{GL}_2(\mathbb{C})$ of matrices with determinant equal to 1. Prove that the map $\mathrm{SL}_2(\mathbb{C}) \to \mathcal{M}\ddot{o}b$ given by the restriction of π to $\mathrm{SL}_2(\mathbb{C})$ is onto.
   (c) Let $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ be the corresponding objects with real entries. Prove that their images in $\mathcal{M}\ddot{o}b$ under π are *not* equal.
   (d) Identify the images of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{R})$ under π with groups of automorphisms of appropriate subsets of $\hat{\mathbb{C}}$.

5. Let $\Lambda = \Lambda(1,\tau)$ be a lattice in $\mathbb{C}$ generated by 1 and $\tau \in \mathbb{H}$. Let f be a meromorphic function on $\mathbb{C}$ with simple poles with residue equal to 1 at the lattice points, and:
$$f(z+1) = f(z) + \eta_1$$
$$f(z+\tau) = f(z) + \eta_2\,.$$
Prove that
$$2\pi i = \eta_1\tau - \eta_2\,.$$

6. Let E be an elliptic curve, which we assume isomorphic to a torus $\mathbb{C}/\Lambda$, where the isomorphism is given by $[t] \to (\wp(t), \wp'(t))$, and $\wp$ is the Weierstrass function relative to the lattice Λ. ($[t]$ denotes the congruence class of $t \in \mathbb{C}$ modulo Λ.)

   We say that a point $P \in E$ is of *order 3 in E,* if $3P = \mathcal{O}$, where $\mathcal{O}$ is the "zero" element of E. (Here $3P = P + P + P$, with respect to the group law of E.)

   Prove that there are, excluding the origin $\mathcal{O}$, at most 8 points of order 3 in E.

   Use known facts about elliptic curves and elliptic functions to conclude that they are all inflection points.