Qualifier Exam for Financial Math
Summer 2023
Exam 1
19 August 2023
Time Limit: 2 hours

Name (Print): ____________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	15	
3	10	
4	20	
5	20	
6	15	
Total:	100	

1. (20 points) Let $S_n = X_1 + \cdots + X_n$, where X_i are i.i.d. and $\mathbb{P}(X_1 = 1) = 1/3$ and $\mathbb{P}(X_1 = -1) = 2/3$, where $S_0 = 0$.

   (i) Find a value c_1 such that $S_n - c_1 n$ is a martingale.

   (ii) Fine a value $c_2 > 0$ such that $(c_2)^{S_n}$ is a martingale.

2. (15 points) Let X_t be the solution to the following stochastic differential equation:

$$d\log(X_t) = \theta(\mu - \log(X_t))dt + \sigma dB_t, \qquad X_0 = x,$$

where B_t is a standard Brownian motion starting at $B_0 = 0$ and $\theta, \mu, \sigma > 0$.

(i) Solve this stochastic differential equation.

(ii) What is the mean of X_t?

(iii) What is the variance of X_t?

3. (10 points) Suppose the stock price at time zero is $S_0 = \$20$, and the stock does not pay dividends. Also assume the risk free rate is $r = 0$. Consider an Asian call option with payoff $\left(\frac{1}{N}\left(S_1 + S_2 + \cdots + S_N\right) - K\right)^+$ and an Asian put option payoff $\left(K - \frac{1}{N}\left(S_1 + S_2 + \cdots + S_N\right)\right)^+$ at maturity, where $K = \$30$ is the strike price. Suppose the time 0 price of this Asian call option is \$1. What is the time 0 price of this Asian put option?

4. (20 points) Consider a non-dividend-paying stock with zero risk-free rate. Assume the stock price follows a binomial tree model with probability of moving up being 1/2 in each period in the real world. Suppose $S_0 = 50$, $u = 1.1$ and $d = 0.8$. For a one-month time step, consider a two-month European put option with strike $K = 35$.

   (i) What is the real world probability that the European put option is in the money at maturity?

   (ii) What is the risk-neutral probability that the European put is in the money at maturity?

5. (20 points) Suppose you flip a biased coin four times, where the biased coin gives you the tail with probability $3/4$ and the head with probability $1/4$.

   (i) What is the probability that you will not get two consecutive heads?

   (ii) What is the variance of the number of heads you will get?

6. (15 points) Consider the stochastic differential equation:

$$dX_t = \theta X_t dt + \sigma X_t dB_t,$$

where $\theta \in \mathbb{R}$ and $\sigma > 0$ are constants.

(i) By applying Itô's formula to $\log(X_t)$ to solve this stochastic differential equation.

(ii) Compute $\mathbb{E}[X_{s_2} X_{s_1}]$ for any $s_2 \geq s_1 \geq 0$.

(iii) Compute $\mathbb{E}\left[\left(\int_0^t X_s ds\right)^2\right]$.