Qualifier Exam for Financial Math **Name (Print):** ______________________

Summer 2023

Exam 2

19 August 2023

Time Limit: 2 hours

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	15	
3	25	
4	20	
5	20	
Total:	100	

1. (20 points) Let B_t be a standard Brownian motion and X the solution of the equation

$$dX_t = -\alpha X_t dt + \sigma dB_t.$$

Let

$$u(t,x) = \mathbb{E}_{X_t=x}\left[e^{-\int_t^T X_s ds}\right].$$

(i) Write down a partial differential equation that $u(t,x)$ satisfies.

(ii) Solve this partial differential equation and hence $u(t,x)$ in closed-form.
Hint: The solution is the exponential of an affine function of x.

2. (15 points) Recall the Put-Call parity (assume there is no dividend) and prove it by a non-arbitrage argument.

3. (25 points) Let B be a standard Brownian motion. Assume that under the risk-neutral probability measure $\mathbb{Q}$ the price of an underlying satisfies the Black-Scholes dynamics with $r = 0$ and $\sigma > 0$ constant.

$$\frac{dS_t}{S_t} = \sigma dB_t.$$

(i) Write the prices of $Call(0, s, K, T)$ and $Put(0, s, K, T)$ as expectations under $\mathbb{Q}$ where the first argument of $Call(0, s, K, T)$ and $Put(0, s, K, T)$ is the current time, the second argument is the price of the underlying at current time, third argument is the strike of the option and the last argument is the maturity of the option.

(ii)Defining the probability $\tilde{\mathbb{Q}}$ by

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = e^{-\frac{\sigma^2 T}{2} + \sigma B_T},$$

write the price $Call(0, s, K, T)$ as an expectation under $\tilde{\mathbb{Q}}$.

(iii)Using Girsanov's theorem prove that

$$Call(0, s, K, T) = Put(0, K, s, T).$$

4. (20 points) Use a stochastic representation result to solve the following terminal value problem

$$\frac{\partial F}{\partial t} + \mu x \frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} = 0$$
$$F(T, x) = \ln(x^2)$$

for $t \in [0, T]$ and $x > 0$.

5. (20 points) Let B and W be two independent standard Brownian motions. Assume that $X \geq 0$ and $Y \geq 0$ solve the equations

$$dX_t = \kappa(\theta - X_t)dt + c\sqrt{X_t}dB_t,$$

and

$$dY_t = \kappa(\theta - Y_t)dt + c\sqrt{Y_t}dW_t.$$

(i) Compute the quadratic variation of the martingale $\int_0^t \sqrt{X_s}dB_s + \int_0^t \sqrt{Y_s}dW_s$ and write this martingale as a stochastic integral with one Brownian motion.

(ii) Find $\mu \in \mathbb{R}$ so that $Z = X + Y$ solves

$$dZ_t = \kappa(\mu - Z_t)dt + c\sqrt{Z_t}d\tilde{B}_t,$$

for a Brownian motion $\tilde{B}$.