Financial Math Qualifier, Spring 2020

Final

Problem 1. Consider the sample space generated by the infinite coin toss trials and let $\{M_n : n = 0, 1, ...\}$ be a submartingale on this space . For each $n \geq 0$, let $\Delta_n(\omega_1 \cdots \omega_n)$ be a function that depends only the first n trials. Show that the discrete stochastic integral defined by

$$I_0 = 0 \text{ and }, I_n = I_{n-1} + \Delta_{n-1}(M_n - M_{n-1})$$

is a submartingale if and only if $\Delta_n \geq 0$ almost surely for all $n = 0, 1, ...$

Problem 2. A portfolio manager is given $\$x$ to invest on an asset which follows the n-period binomial market model and risk-free money market. The contract between the investor and the manager asserts that if the value of the portfolio is above level $\ell > 0$, the the manager receives a payment; otherwise, no payment is made. In addition, there is a restriction that the wealth should not fall below zero at any time. Therefore, the manager's goal is to maximize $\mathbb{P}(X_N \geq \ell)$ where X_N is the total amount of wealth from investment at time N. We also assume no arbitrage, the parameters of the binomial tree satisfy $d < 1 + r < u$.

a) Show that the wealth at all times, X_n for all $n = 0, 1, ..., N$, is nonnegative almost surely if and only if the wealth at time N, X_N, is so.

b) Show that the optimal wealth X_N^* is given by

$$X_n^* = \mathrm{I}\Big(\frac{\lambda Z}{(1+r)^N}\Big)$$

where

$$\mathbb{E}\left[\frac{Z}{(1+r)^N}\mathrm{I}\Big(\frac{\lambda Z}{(1+r)^N}\Big)\right] = x,$$

and the function I is given by

$$\mathrm{I}(y) = \begin{cases} \gamma & 0 < y \leq \frac{1}{\gamma} \\ 0 & y > \frac{1}{\gamma} \end{cases}.$$

Problem 3. Consider a market model for the risky asset in which there is a risk-neutral probability measure, $\tilde{\mathbb{P}}$. Show that in such market, any convex payoff g with $g(0) = 0$ generates the same price for the American option as for the European option.

Problem 4. Consider the binomial stochastic interest rate model under a risk-neutral probability in which the short rate is given by

$$R_n(\omega_1\cdots\omega_n) = \frac{r_H}{n}\#H(\omega_1\cdots\omega_n) + \frac{r_T}{n}(n - \#H(\omega_1\cdots\omega_n)),$$

where r_H and r_T are two given positive constants and $\#H$ denotes the number of heads in a sequence of trials. We also assume the transition probabilities are given:

$$\mathbb{P}(\omega_1\cdots\omega_n H|\omega_1\cdots\omega_n) = p(\omega_1\cdots\omega_n)$$
$$\mathbb{P}(\omega_1\cdots\omega_n T|\omega_1\cdots\omega_n) = 1 - p(\omega_1\cdots\omega_n)$$

a) Show that $\{R_n\}_{n\geq 0}$ is a Markov process.

b) Through an example of this model, show that the discount process given by

$$D_n(\omega_1\cdots\omega_{n-1}) = \big(1+R_0\big)^{-1}\big(1+R_1(\omega_1)\big)^{-1}\cdots\big(1+R_{n-1}(\omega_1\cdots\omega_{n-1})\big)^{-1},\ n\geq 1$$

is not Markov.

c) Check if in your example the futures price and forward price are the same. If different, explain why.

d) Is there any stochastic interest rate binomial model with Markov discount process? Justify your answer.

Problem 5. In a risk-neutral pricing framework, let the prices of all call options with a fixed maturity T and all strikes $K \geq 0$ are known; i.e., $C(x,K,T) = \tilde{\mathbb{E}}[e^{-rT}(S_T - K)_+|S_0 = x]$ is known for all $K \geq 0$. Show that if the function $C(x,K,T)$ is twice differentiable on K, then $e^{rT}\partial_{KK}C(x,K,T)$ is the probability density function (pdf) of S_T, the asset price at time T.

Problem 6. Consider a measure space be given by $\Omega := \mathbb{R}$ and the Borel σ-field on $\mathbb{R}$ and a random variable on this measure space X given by $X(\omega) = \lfloor\omega\rfloor$, where $\lfloor x\rfloor$ is the largest integer less than or equal to x.

a) Find the σ-field generated by X.

b) $Y(\omega) = \omega^2$. Show that the σ-field generated by Y is strictly smaller than the σ-field generated by X.

Problem 7. For $t \in [0,T)$, let the process X be given by the follwing Itô integral:

$$X_t = \int_0^t \frac{dB_t}{T-u},$$

where B is a standard Brownian motion.

a) Show that $\lim_{t\to T}(T-t)X_t = 0$ almost surely.

b) Find the mean $m_Y(t)$ and covariance $c_Y(s,t)$ functions of the Gaussian process

$$Y_t = \begin{cases} (T-t)X_t & t < T \\ 0 & t = T \end{cases}.$$

Problem 8. Let the two dimensional process (X,Y) is given by

$$\begin{cases} X_t &= rX_t + \sigma Y_t dB_t \\ Y_t &= -\gamma Y_t dt + \sqrt{Y_t} dW_t \end{cases},$$

where r, σ, and γ are positive constants

a) For $y > 0$, let $u(y) := \mathbb{P}(\tau < \infty | Y_0 = y)$, where τ is the heating time of the process Y_t to zero. Show that $u(y)$ satisfies the ordinary differential equation

$$y^2 u'' - \gamma y u' = 0.$$

b) Solve the above ordinary differential equation using the following boundary conditions:

$$\begin{cases} u(0) = 1 \\ 0 \leq u \leq 1 \end{cases}.$$