Financial Math Qualifier, Spring 2021

Qualifier

Notations

$\mathcal{F}$	σ-field
$\mathbb{F}$	filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$ or $\{\mathcal{F}_t\}_{t\geq 0}$
$\mathbb{P}$	probability
$\mathbb{E}$	expectation
X	stochastic process $\{X_n\}n = 0^{\infty}$ or $\{X_t\}_{t\geq 0}$
$(\Delta \cdot X)$	discrete stochastic integral of Δ with respect to X

Q#1. Consider the sample space generated by the infinite coin toss trials equipped with a probability measure $\mathbb{P}$ and the "natural" filtration $\mathbb{F} := \{\mathcal{F}_n\}_{n=0}^{\infty}$ generated by the consecutive trials, and let $\{X_n : n = 0, 1, ...\}$ be an "integrable" process on this space. We define the process A by $A_0 = 0$ and

$$A_{n+1} = A_n + |\mathbb{E}^{\mathbb{P}}[X_{n+1}|\mathcal{F}_n] - X_n| \text{ for } \ n \geq 0$$

a) Show that for any integrand $\{\Delta_n\}_{n\geq 0}$ with $|\Delta_n| \leq 1$, $(\Delta \cdot X) - A$ is a supermartingale.

b) Show that $X - A$ is a super martingale.

Recall that $(\Delta \cdot X)_n = \sum_{t=0}^{n-1} \Delta_i (X_{i+1} - X_i)$ is the discrete stochastic integral. Check all the properties in the definition of a supermartingale.

Q#2. Consider a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ equipped with a Brownian motion B.

a) Show that $I_t = \int_0^t e^{\alpha s + \beta B_s} dB_s$ is a martingale. Check all the properties in the definition of a martingale. Hint: use Itô isometry.

b) Find all values of α and β such that $f(t, x) = e^{\alpha t + \beta B_t}$ is a martingale.

Q#3. Consider a market model for the risky asset in which there is a risk-neutral probability measure, $\tilde{\mathbb{P}}$ and the price of a **European put option** with strike \$10.00 is given by \$2.30. Given that the price of a zero bond with the same maturity as the put option is \$0.97 and the price of risky asset is \$8.00, find the price of the **American call option** with the same strike and maturity as the European put option described before. Justify your answer by referring to some theorems. Write the assertion of those theorems that includes all the assumptions. Verify that all the assumptions are held in this case.

Q#4. Consider the binomial stochastic interest rate model under a risk-neutral probability in which the short rate is given by R_0 is constant and

$$R_n(\omega_1 \cdots \omega_n) = \frac{a}{n} \# H(\omega_1 \cdots \omega_n) + \frac{b}{n},$$

where a and b are two constants with $a + b > -1$ and $\#H(\omega_1 \cdots \omega_n)$ denotes the number of heads in a sequence of trials $(\omega_1 \cdots \omega_n)$. We also assume the transition probabilities are given:

$$\mathbb{P}(\omega_1 \cdots \omega_n H | \omega_1 \cdots \omega_n) = \tilde{p}_n$$
$$\mathbb{P}(\omega_1 \cdots \omega_n T | \omega_1 \cdots \omega_n) = 1 - \tilde{p}_n$$

a) Show that $\{R_n\}_{n \geq 0}$ is a Markov process.

b) Prove or give a counter example: is

$$D_n(\omega_1 \cdots \omega_{n-1}) = \\ \big(1 + R_0\big)^{-1} \big(1 + R_1(\omega_1)\big)^{-1} \cdots \big(1 + R_{n-1}(\omega_1 \cdots \omega_{n-1})\big)^{-1}, \quad n \geq 1$$

a Markov process?

c) Prove or give a counter example: are the futures price and forward price the same.

d) Consider an asset with binomial model given by

$$S_{n+1} = S_n H_{n+1}$$

where H_{n+1} takes values u and d with probabilities p_n and $1 - p_n$, respectively. Show that there is not arbitrage in this market if and only if $d < 1 + a + b < u$.

Q#5. In a continuous-time risk-neutral pricing framework with constant interest rate r, let the prices of all call options with a fixed maturity T and all strikes $K \geq 0$ are known; i.e., $C(x, K, T)$ is known for all $K \geq 0$ and assume that S_T has a smooth probability density function (pdf) $f(x)$.

a) Show that

$$e^{rT} C(x, K, T) = \int_K^\infty x f(x) dx - K(1 - F(K)),$$

where F is the cumulative distribution function (cdf) of S_T.

b) Show that if the function $C(x, K, T)$ is twice differentiable on K, then $e^{rT} \partial_{KK} C(x, K, T)$ is the pdf of S_T, the asset price at time T.

Q#6. Consider a measure space given by $\Omega := [0, 1]$ and the Borel σ-field on $[0, 1]$, and a random variable on this measure space X given by $X(\omega) = 4\omega(1 - \omega)$, for $\omega \in [0, 1]$.

a) Find the σ-field generated by X and show that it is strictly smaller thant he Borel σ-field on $[0, 1]$.

b) $Y(\omega) = (X(\omega) - \frac{1}{2})^2$. Show that the σ-field generated by Y is even strictly smaller than the σ-field generated by X.

c) Describe the σ-field generated by X and σ-field generated by Y

Q#7. Consider a filtered probability space $(\Omega, \mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ equipped with a Brownian motion B. For $t \in [0, T)$, let the process X be given by the following Itô integral:

$$X_t = \int_0^t e^{-t+s} dB_s,$$

where B is a standard Brownian motion.

a) Is X a martingale? Provide a comprehensive justification.

b) Show that X is a Gaussian process and find the mean $m_Y(t)$ and covariance $c_Y(s,t)$ functions.

Q#8. Let Y is given by

$$dY_t = \kappa(\gamma - Y_t)dt + \sigma\sqrt{Y_t}dW_t.$$

where σ, κ, and γ are positive constants.

a) For $y > 0$, let $u(y) := \mathbb{P}(\tau < \infty | Y_0 = y)$, where τ is the heating time of the process Y_t to zero. Assume that u is twice continuously differentiable. Show that $u(y)$ satisfies the ordinary differential equation

$$\frac{\sigma^2}{2} y u'' - \kappa(\gamma - y)u' = 0.$$

b) Solve the above ordinary differential equation to show that the general solution of the ODE has the form

$$u(y) = u(0) + C_1 \int_0^{\frac{2\kappa}{\sigma^2}y} z^{-\frac{2\kappa}{\sigma^2}} e^{-z} dz,$$

for some constant C_1.

c) By using the following boundary conditions,

$$\begin{cases} u(0) = 1 \\ 0 \leq u \leq 1 \end{cases}.$$

show that C_1 is nonpositive constant C_1, and, in addition, show that $u(y) > 0$ for all $y > 0$ if and only if $2\kappa\gamma < \sigma^2$. Hint: $\int_0^\infty z^{a-1} e^{-z} dz < \infty$ if and only if $a > 0$.