ALGEBRA QUALIFYING EXAM

MAY 30, 2010 — 1:00–5:00 PM

Correct and complete solutions to six or more problems carry full credit. You may use standard results (such as a 'named' theorem), provided that you state such results in full.

(1) Show that if $|G| = pq$ for some primes p and q, then either G is abelian or $Z(G) = 1$.

(2) Prove that every group of order 2010 has a nontrivial proper normal subgroup.

(3) Let G be a finite abelian p-group, and assume that G has only one subgroup of order p. Prove that G is cyclic.

(4) Let $f \in \mathbb{Z}[x]$ be a cubic polynomial with odd leading coefficient, and such that $f(0)$ and $f(1)$ are odd. Prove that f is irreducible in $\mathbb{Q}[x]$.

(5) Assume R is a commutative ring with 1. Prove that $R^n \cong R^m$ as R-modules if and only if $n = m$.

(6) Show that a real 3×3 matrix has at least one real eigenvector.

(7) Prove that two 3×3 matrices are similar if and only if they have the same characteristic and same minimal polynomials. Is this assertion true for 4×4 matrices? (Proof or counterexample.)

(8) Let p be a prime number, let $\alpha = \sqrt[p]{2}$, let $K = \mathbb{Q}(\alpha)$, and let β be some element of K that is not in $\mathbb{Q}$.
  (a) Prove that there exists some polynomial $h(x) \in \mathbb{Q}[x]$ such that $h(\beta) = \alpha$.
  (b) Let $f(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of degree n. Suppose that $\gcd(n, p) = 1$. Prove that $f(x)$ is irreducible in $K[x]$.