**Algebra Qualifying Exam, August 2008.**

For full credit solve at least 6 of the 8 problems. Please indicate which 6 problems you are solving.

1. Let $\phi : G \to H$ be a homomorphism of groups. Suppose $H$ is abelian, and $N$ is a subgroup of $G$ that contains $\ker\phi$. Prove that $N$ is normal in $G$.

2. The permutation group $S_{20}$ has an abelian subgroup of order $5^4$, namely

   $$< (1\,2\,3\,4\,5), (6\,7\,8\,9\,10), (11\,12\,13\,14\,15), (16\,17\,18\,19\,20) >$$

   Show that every other subgroup of $S_{20}$ of order $5^4$ is abelian as well.

3. Let $R$ be a commutative ring. Show that every cyclic left $R$-module (i.e., a module generated by a single element) is isomorphic as a left $R$-module to $R/J$ for some ideal $J$ of $R$.

4. Let $F$ be a field, and let $f$ be a polynomial in $F[x]$ that has at least two distinct irreducible factors in $F[x]$. Show that there exists a polynomial $g \in F[x]$ with $0 < \text{degree}(g) < \text{degree}(f)$ for which $g^2 \equiv g \bmod (f)$.

5. Let $E$ be a finite field extension of a field $F$. Suppose $[E : F]$ is odd, and $\alpha \in E$ is such that $E = F(\alpha)$. Prove that $E = F(\alpha^2)$.

6. Suppose $K$ is a finite extension of a field $F$. Prove or disprove: if $R$ is a subring of $K$ that contains $F$, then $R$ is a field.

7. Let $A$ be an $n$ by $n$ matrix with entries in $\mathbb{Z}$. Show that every eigenvalue in $\mathbb{Q}$ is an element of $\mathbb{Z}$.

8. If $A$ is an $n$ by $n$ matrix with entries in $\mathbb{Z}$ and odd determinant then show that for some positive number $k$, all entries of the matrix $A^k - I$ are even. Hint: work over the finite field $\mathbb{Z}/(2)$.