Algebra Qualifying Exam, January 6, 2006.

For full credit solve at least 6 of the 8 problems. Please indicate which 6 problems you are solving.

1. Let G be a group of order 87.
   (a) Write down Sylow's theorem.
   (b) Prove that the Sylow subgroups of G are normal subgroups.
   (c) Prove that G is cyclic.

2. An element a of a ring R is called nilpotent if there exists a positive integer m such that $a^m = 0$.
   (a) Let R be a commutative ring. If a and b are nilpotent, prove that $a + b$ is also nilpotent.
   (b) Is the same also true in general for non-commutative rings? (try to give a proof or a counter example).

3. Let $f(x)$ be a polynomial with rational number coefficients. Suppose that the Galois group has odd order. Prove that all roots of $f(x)$ are real numbers.

4. Let G be a finite abelian group, and let p be a prime number. Let q be the number of elements $g \in G$ for which g^p is the identity. Prove that $q = p^n$ for some non-negative integer n.

5. Let G be a finite group and let p be a prime number that does not divide the order of G. Let $g \in G$. Prove that there exists $h \in G$ with $h^p = g$.

6. Let R be a commutative ring with identity. Prove that if I and J are ideals of R satisfying $I + J = R$, then $IJ = I \cap J$.

7. Let A be a 3×4 matrix with rank 3.
   (a) Prove that there exists a 4×3 matrix B for which $AB = I$, where I is the 3×3 identity matrix.
   (b) Is such a matrix B unique?

8. Let I be the ideal (x, y) in the ring $k[x, y]$, where k is a field. Prove or disprove the assertion that I is a projective $k[x, y]$-module.