ALGEBRA QUALIFYING EXAM

August 25, 2005 — 1:00–5:00 pm

Correct and complete solutions to eight problems carry full credit. Please indicate clearly which eight problems you are solving.

(1) Let G be a finite group, let p be the smallest prime that divides the order of G, and let H be a subgroup of G of *index* p. Prove: H is normal in G.

(2) Let G be a group of order n acting nontrivially on a set with r elements. if $n > r!$, then G has a proper normal subgroup.

(3) Carefully state Sylow's theorems, and explain in detail an application (of your choice) of these results.

(4) Let A be an abelian group. Show that $A/2A$ is a finite group if A is finitely generated. On the other hand, show that A is not necessarily finitely generated if $A/2A$ is a finite group.

(5) Let R be a finite commutative ring with identity. Prove that every prime ideal of R is a maximal ideal.

(6) Let K/F be an algebraic extension and let R be a ring contained in K and containing F. Show that R is a subfield of K containing F.

(7) Give an example of a degree-3 polynomial $p(x) \in \mathbb{Q}[x]$ whose Galois group is S_3. Justify your answer.

(8) Compute $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{2005}$.

(9) A linear operator $T : V \to V$ on a finite dimensional vector space V is *nilpotent* if $T^r = 0$ for some $r > 0$.

—Prove that there exists a basis of V such that the matrix A for T in that basis is upper triangular, with all diagonal entries equal to 0.

—Prove that $\det(I + A) = 1$.

—Prove that if T is nilpotent and V has dimension n, then $T^n = 0$.

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX