ALGEBRA QUALIFYING EXAM

AUGUST 22, 2010 — 1:00–5:00 PM

Correct and complete solutions to six or more problems carry full credit. You may use standard results (such as a 'named' theorem), provided that you state such results in full.

(1) State the universal property satisfied by free groups.

(2) Let G be a finite group, and let N be a normal subgroup of G. Let p be a prime integer which does not divide $|G|/|N|$. Prove that the number of Sylow p-subgroups of N is the same as that of G.

(3) Prove that every finite abelian group G has subgroups of index n for every positive integer n dividing $|G|$. Is the abelian hypothesis necessary?

(4) Let R be a commutative ring (with 1).
- Prove that an ideal of R is proper if and only if it is contained in some prime ideal of R.
- Prove that if $\mathfrak{p}$ is a prime ideal of R, I is an ideal of R, and $\mathfrak{p} \supseteq I^2$, then $\mathfrak{p} \supseteq I$.
- Let I, J be ideals of R, and assume that $I + J = (1)$. Prove that $I^2 + J^2 = (1)$.

(5) Let R be an integral domain. Prove that if the following two conditions hold, then R is a PID:
(i) any two nonzero elements $a, b \in R$ have a greatest common divisor which can be written in the form $ra + sb$ for some $r, s \in R$
(ii) if $a_1, a_2, a_3, \dots$ are nonzero elements of R such that $a_{i+1}|a_i$ for all i, then there is a positive integer N such that a_n is a unit times a_N for all $n \geq N$.

(6) An R-module is 'simple' if it is not zero and it has no proper submodules. Show that a simple module is isomorphic to $R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R.

(7) Let ϕ be a linear transformation from the finite dimensional vector space V to itself, such that $\phi^2 = \phi$. Prove that there is a basis of V such that the matrix of ϕ with respect to this basis is a diagonal matrix whose entries are all 0 or 1.

(8) Prove that every matrix is similar to its transpose.