ALGEBRA QUALIFYING EXAM

THURSDAY, AUGUST 20TH, 1:00-5:00 PM

Please value accuracy and precision in your answers. You may use standard results, such as named theorems, provided that you identify them explicitly and state the relevant hypotheses carefully.

(1) If A and B are normal subgroups of G such that G/A and G/B are both abelian, prove that $G/(A \cap B)$ is abelian.

(2) Let G be a group of order 1715. Show that G has a normal subgroup H of order 5, and that $H \leq Z(G)$ (=center of G).

(3) Prove that $7x^3 + 6x^2 + 4x + 6$ is irreducible in $\mathbb{Q}[x]$ in at least two different ways. (One of them may involve long computations, which you may summarize.)

(4) (a) Find a polynomial $f \in \mathbb{F}_2[x]$ such that $\mathbb{F}_2[x]/(f)$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ as vector spaces.
(b) Prove that there is no polynomial $f \in \mathbb{F}_2[x]$ such that $\mathbb{F}_2[x]/(f)$ is isomorphic to $\mathbb{F}_2 \times \mathbb{F}_2 \times \mathbb{F}_2$ as rings.
(c) Prove that there *is* a polynomial $g \in \mathbb{F}_3[x]$ such that $\mathbb{F}_3[x]/(g)$ is isomorphic as a ring to $\mathbb{F}_3 \times \mathbb{F}_3 \times \mathbb{F}_3$.

(5) Let I be a nilpotent ideal in a commutative ring (with 1) R, that is, an ideal I such that $I^n = (0)$ for some n.
- Let K be an R-module, and assume $IK = K$. Prove that $K = (0)$.
- Let M and N be R-modules and let $\phi : M \to N$ be an R-module homomorphism. Show that if the induced map $\bar{\phi} : M/IM \to N/IN$ is surjective, then ϕ is surjective.

(6) Show that if φ is an endomorphism of a vector space V of dimension n, and if φ is nilpotent, then $\varphi^n = 0$.

(7) (a) In any category C, state the universal property of the coproduct.
(b) State the universal property of the free group over a set S.
(c) Let $F(S)$ denote the free group over a set S. Prove there is an isomorphism

$$F(S \amalg T) \cong F(S) * F(T)$$

where $\amalg$ denotes the coproduct in the category of sets and $*$ that in the category of groups.