# Algebra Qualifying Exam

August 25, 2019

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*Please value accuracy and precision: 8 approximate solutions will likely carry less weight than 4 complete ones.*
*You may use standard results, provided you carefully state them* ***in full.***
**Six** problems carry **full credit.**

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(1) Let $\mathcal{C} = \mathcal{M}od_R$ be the category of (left) $R$-modules, where $R$ is a ring (with 1, as always). Prove that in $\mathcal{C}$ every morphism is a mono if and only if it is injective.

(2) Let $G$ be a group. $\mathrm{Inn}(G)$ is the group of *inner automorphisms,* that is, those of the form $a \mapsto \iota_g(a) = gag^{-1}$, for some $g \in G$. Prove that $\mathrm{Inn}(G)$ cannot be a nontrivial cyclic group.

(3) Prove that a group $G$ of order 45 is abelian.

(4) Prove that the polynomial $f = x^3y^2 + x^2y^3 + x + y^3$ is irreducible in $k[x, y]$ ($k$ a field of characteristic zero). Is the ideal $I = (f)$ prime? Maximal?

(5) Prove that $R = \mathbb{Z}[\sqrt{-7}]$ is not a PID.

(6) Let $I$ be the ideal $(p, x)$ in $R = \mathbb{Z}[x]$, where $p$ is a prime.

  (a) Prove that $I$ is not a free $R$-module.
  (b) Provide a resolution.

(7) Let $V$ be a $\mathbb{Q}$-vector space of dimension 2, and let $\{e_1, e_2\}$ be a basis. Define $\mathbb{Q}[t]$-module structures on $V$ as follows:

$$t\,e_1 = e_2$$
$$t\,e_2 = e_1$$

resp.

$$t\,e_1 = e_2$$
$$t\,e_2 = e_1 + e_2$$

and denote the corresponding $\mathbb{Q}[t]$-modules by $V_{t,1}$ (resp. $V_{t,2}$). Determine whether $V_{t,1}$ and $V_{t,2}$ are isomorphic as $\mathbb{Q}[t]$-modules.

(8) Let $K/\mathbb{Q}$ be a Galois extension with group $G$ and let $\alpha \in K$. Let $d = [\mathbb{Q}(\alpha) : \mathbb{Q}(\alpha^2)]$.

  (a) Show that $d \leq 2$.
  (b) If the order of $G$ is odd then show that there exists a polynomial $h(x) \in \mathbb{Q}[x]$ for which $h(\alpha^2) = \alpha$.
  (c) If $d = 2$ then show that there exists $\sigma \in G$ for which $\sigma(\alpha) = -\alpha$.