ALGEBRA QUALIFYING EXAM

JANUARY 4, 2005 — 1-5 PM

Correct and complete solutions to six problems carry full credit. Please indicate clearly which six problems you are solving.

(1) Let G be a finite group, let p be the smallest prime that divides the order of G, and let H be a subgroup of G of *order* p.
Prove: if H is normal in G, then H is contained in the center of G.

(2) Let G be a group of order 2005. Show that if G is not cyclic, then G is not abelian. In this case, how many elements of order 5 are there in G?

(3) Let G be a finite group, and let N be the number of ordered pairs of commuting elements in G. Prove that $N/|G|$ is the number of conjugacy classes in G.

(4) Let R be an integral domain which contains a field k as a subring and has finite dimension as a k-vector space. Prove that R is a field.

(5) Suppose $F = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$ where $\alpha_i^2 \in \mathbb{Q}$ for $i = 1, \dots, n$. Prove that $\sqrt[3]{2} \notin F$.

(6) Prove that the trace of a nilpotent $n \times n$ matrix with entries over a field is 0. (Recall that a matrix A is *nilpotent* if $A^m = 0$ for some integer m.)

(7) Let φ be a linear operator on a finite dimensional vector space V, such that $\varphi^2 = \varphi$.
(a) Prove that image $\varphi \cap \ker \varphi = 0$.
(b) Prove that $V =$ image $\varphi \oplus \ker \varphi$.
(c) Let $r = \dim \ker \varphi$ and $s = \dim \mathrm{image} \varphi$. What is the characteristic polynomial of φ?

(8) Give an explicit example of an exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

and of an abelian group M such that the induced sequence

$$0 \to A \otimes_{\mathbb{Z}} M \to B \otimes_{\mathbb{Z}} M \to C \otimes_{\mathbb{Z}} M \to 0$$

is *not* exact.

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