Algebra Qualifying Exam, Tuesday August 23, 2011.

For full credit solve at least 6 of the 8 problems. Please indicate which 6 problems you are solving.

1. Let A be a set. Write down a category $\mathcal{F}^A$ such that an initial object in $\mathcal{F}^A$ corresponds to the free group $F(A)$.

2. (a) Write down a subgroup of S_{15} that is isomorphic to $Z_5 \times Z_5 \times Z_5$.

   (b) Let H be any subgroup of S_{15} of order 5^3. Show that H is commutative and not a normal subgroup of S_{15}.

3. Let $R = \mathbb{Q}[x]$, $f_1 = (x^2-2)(x^2+1)$, and $f_2 = (x^2-8)(x^2+1)$.
   Let $S_1 = R/(f_1)$ and $S_2 = R/(f_2)$.

   (a) Are S_1 and S_2 isomorphic as rings? Explain.

   (b) Are S_1 and S_2 isomorphic as R-modules? Explain.

4. Let $R = \mathbb{Q}[x, y]$ and let M be a submodule of R^n. For each of the following, give a counter-example if the answer is "no", and give a short explanation if the answer is "yes".

   (a) Must M be a finitely generated R-module?

   (b) Must M be torsion-free?

   (c) Must M be free?

5. Suppose $f(x) = a_n x^n + \cdots + a_0 x^0 \in \mathbb{Z}[x]$ and $f(x) = g(x)h(x)$ with $g(x), h(x) \in \mathbb{Q}[x]$, and $g(x)$ monic. Must then $a_n \cdot g(x)$ have integer coefficients? Explain.

6. Let M be a 4 by 4 matrix over $\mathbb{Q}$. Suppose that $M^2 = I$. Show that the trace of M is an element of $\{-4, -2, 0, 2, 4\}$.

7. Let M be a $\mathbb{Z}$-module. Let $\text{Ann}(M) = \{r \in \mathbb{Z} | r \cdot m = 0 \text{ for all } m \in M\}$. Show that $\text{Ann}(M)$ is an ideal. Let $n \in \{0, 1, \ldots\}$ such that $\text{Ann}(M) = (n)$. Show that $|M| \geq n$, with equality if and only if $M \cong \mathbb{Z}/(n)$.

8. The field $\mathbb{Q}(\zeta_n)$ is a Galois extension of $\mathbb{Q}$, and the Galois group is the multiplicative group $(\mathbb{Z}/(n))^*$.

   (a) Compute the minimal polynomial of $\beta := \zeta_5 + (\zeta_5)^4 = 2\text{Re}(\zeta_5)$.
   (the minimal polynomial of ζ_5 is $x^4 + x^3 + x^2 + x + 1$).

   (b) List all subfields of $\mathbb{Q}(\zeta_5)$.

   (c) Show that for any $a, b \in \mathbb{Q}$, we have $\zeta_5 \notin \mathbb{Q}(\sqrt{a}, \sqrt{b})$.