ALGEBRA QUALIFYING EXAM

AUGUST 23, 2014

Full credit will come from correct and complete solutions to 8 of the following 9 problems. Please value accuracy and precision: 8 approximate solutions will likely carry less weight than 4 complete ones. You may use standard results, such as classification theorems, provided you carefully state such results.

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(1) Let G be a finite group which possesses an automorphism σ such that $\sigma^2 =$id and $\forall g \in G$, $\sigma(g) = g \iff g = 1$.
- Prove that for all g in G there exists an x in G such that $g = x^{-1}\sigma(x)$.
- Prove that the group G is abelian.

(2) Prove that for $a, b \in \mathbb{Z}$, the group $\mathbb{Z}\times\mathbb{Z}/\langle (a, b)\rangle$ is cyclic if and only if a and b are relatively prime.

(3) Show that a finite abelian group that is not cyclic contains a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ for some prime p.

(4) Let $F(S)$ denote the free group over a set S. Use universal constructions and the properties of the category Grp of groups to show

$$F(S \coprod T) \cong F(S) * F(T),$$

where $\coprod$ denotes the disjoint union of sets and $*$ the coproduct of groups.

(5) Let I be an ideal in a commutative ring R with 1, and let M and N be R-modules. Let $\phi : M \to N$ be an R-module homomorphism, and let $\bar{\phi} : M/IM \to N/IN$ be the induced homomorphism. Prove:
- If $I^2 = 0$ and $\bar{\phi}$ is surjective, then ϕ is surjective.
- If I is nilpotent (that is, $I^n = 0$ for some $n > 0$) and $\bar{\phi}$ is surjective, then ϕ is surjective.

(6) Let α, β be complex numbers. Prove that if α and β are algebraic over $\mathbb{Q}$, then $\alpha + \beta$ is algebraic over $\mathbb{Q}$. If α, β have degrees d, e respectively over $\mathbb{Q}$, what can you say about the degree of $\alpha + \beta$? (Justify your answer.)

(7) Prove or disprove: if R is a commutative ring such that every finitely generated R-module is free, then R is a field or the zero ring.

(8) Let V be a vector space and W be a subspace. Prove that there exists a subspace U of V such that $U + W = V$ and $U \cap W = 0$.

(9) Let A be a nilpotent matrix with entries in a field. Show that $\det(I + A) = 1$.