Algebra Qualifying Exam

Fall 2015

Six problems carry full credit. Please clearly mark the problems you are solving. Moreover, justify all your answers. You may use standard results, provided that you state such results in full.

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1. Let S be a set. Write down a category $\mathcal{C}_S$ such that an initial object in $\mathcal{C}_S$ corresponds to the free group $F(S)$ over S.

2. Let G be a group of order $n > 2$ acting nontrivially on a set with r elements, where $n \geq r!$. Prove that G is not simple.

3. Let G be a finite simple group, and let p be a prime. Suppose that p divides $|G|$, but p^2 doesn't. Show that G has at least $p^2 - 1$ elements of order p.

4. Prove that the polynomial $f(x, y) = y^2 - x(x^2 - 1) \in k[x, y]$ is irreducible over any field k. Is $(f(x, y))$ prime? If prime, is it maximal? Fully justify your answers.

5. Let $R = k[x]/(x^n)$. Compute a free resolution of the R-module $R/(\bar{x}^m)$, for all $m \leq n$ (where $\bar{x}$ is the class of x).

6. Let R be a ring. Suppose the R-module I has the following property: for every diagram of R-modules

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\varphi"] \arrow[d, "\psi"'] & M \\ & I & \end{tikzcd}$$

   where the horizontal line is exact, that is $\varphi\colon L \to M$ is a monomorphism, there exists $\tilde{\psi}\colon M \to I$ such that $\tilde{\psi} \circ \varphi = \psi$. Such a module is called *injective*.

   (a) Prove that every short exact sequence $0 \to I \to M \to N \to 0$ with I injective splits.

   (b) Prove that $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module.

7. Let A be an abelian group. Prove that if A is finitely generated, then $A/2A$ is finite. Show that $A/2A$ is not necessarily finite if A is not necessarily finitely generated.

8. Give an example of a ring R and a finitely generated R-module M for which the following statement is true:

   "Every set of generators for M contains a basis for M."

   Then give an example of a ring R and a finitely generated R-module M for which the above statement is false.

   Be sure to justify both your answers as much as possible.

9. Prove that a square matrix A over any field is similar to its transpose.