ALGEBRA QUALIFYING EXAM

JANUARY 11, 2015

Full credit will come from correct and complete solutions to 8 of the following 9 problems. Please value accuracy and precision: 8 approximate solutions will likely carry less weight than 4 complete ones. You may use standard results, such as classification theorems, provided you carefully state such results.

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(1) Describe all conjugacy classes of the symmetric group S_4. Prove that S_4 has no normal subgroups of order 8.

(2) (a) Show that every subgroup of index 2 of a group is normal.
(b) Is it true that every subgroup of index 3 of a group is normal? Proof or counterexample.

(3) Let G be a group of order 2015. Prove or disprove: G is necessarily simple.

(4) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n\mathbb{Z}), \mathbb{Z}/(m\mathbb{Z})) \cong \mathbb{Z}/(n,m)\mathbb{Z}$.

(5) Let F be a finite field. Show that $F[x]$ contains irreducible polynomials of arbitrarily high degree.

(6) Let R be a commutative ring with 1. Assume that for each $a \in R$ there is an integer $n > 1$ such that $a^n = a$. Prove that every prime ideal of R is a maximal ideal.

(7) Prove that two 3×3 matrices are similar if and only if they have the same characteristic and same minimal polynomials. Is this assertion true for 4×4 matrices? (Proof or counterexample.)

(8) Prove that an $n \times n$ matrix A with entries from $\mathbb{C}$ satisfying $A^3 = A$ can be diagonalized.

(9) Show that if φ is an endomorphism of a vector space V of dimension n, and if φ is nilpotent, then $\varphi^n = 0$.