GRV qualifying exam, 1:30–5:30 pm, August 26 2018.

Answer 5 of the following 7 questions.

General hint: You may use statements from previous parts of a question, including parts you could not prove.

1. Let G be a *simple* group of order $168 = 2^3 \cdot 3 \cdot 7$.
   (a) How many subgroups of order 7 does G have?
   (b) Prove: G is isomorphic to a subgroup of the symmetric group S_8.
   (c) Prove: G is isomorphic to a subgroup of the alternating group A_8.
2. Let G be a group and H a finite index subgroup of G.
   (a) If $g \in G$ show that there is a smallest positive integer k such that $g^k \in H$. Show that k divides every integer m such that $g^m \in H$.
   (b) If H is normal in G show that k divides $[G : H]$.
   (c) Produce a counterexample to the claim that for all subgroups H we have k dividing $[G : H]$.
3. Let $R = \mathbb{Z}[i]$, let $p = 1 + i$ and r in R, let $f = x^n - r \in R[x]$. Suppose that f is reducible in $R[x]$ and that $p|r$. Show that $2|r$.
4. Let $c, d \in \mathbb{Z}$, with d not a square, and $R = \mathbb{Z}[\sqrt{d}]$. Let $a = c + \sqrt{d}$ and let A be the absolute value of $c^2 - d$.
   (a) Show that a is prime in R if and only if A is a prime number.
   (b) Give an example (with proof) where a is irreducible but not prime.
   (c) Prove that for any such example, there must exist an ideal I in R that is not principal and that contains a.
5. Let R be a PID and let M be a submodule of R^n. Show that there exists a submodule N of R^n and a non-zero element $f \in R$ such that $f \cdot N \subseteq M \subseteq N$ and R^n/N is free.
6. Let $f, g \in \mathbb{Q}[x]$ be irreducible of degree n. Let $\alpha \in \mathbb{C}$ be a root of f, and $\beta \in \mathbb{C}$ be a root of g.
   (a) If f has a root in $\mathbb{Q}(\beta)$ then show that g has a root in $\mathbb{Q}(\alpha)$.
   (b) More generally, if f has an irreducible factor of degree d in $\mathbb{Q}(\beta)[x]$, then show that g has an irreducible factor of degree d in $\mathbb{Q}(\alpha)[x]$.
7. Let $f \in \mathbb{Q}[x]$ be irreducible of degree n, let K be the splitting field of f over $\mathbb{Q}$, and let $G = \mathrm{Gal}(K/Q)$. Let α_1 be a root of f in K, and let $H_1 = \{g \in G \mid g(\alpha_1) = \alpha_1\}$.
   (a) If H_1 is a normal subgroup of G then show that $H_1 = \{e\}$.
   (b) If G is abelian then show that $|G| = n$.