Methods of Applied Mathematics 1 Qualifying Exam
August 2010

Name:

Please be careful with your sketches and show your work clearly. Be sure to read directions and to include the things that are requested. Books and notes are not allowed, but a calculator is okay.

1. For the equation:

$$\dot{x} = -x(x^2 - 2x - \mu)\ ;\ \mu \in (-\infty, \infty)$$

(a) Find equations for the steady states as a function of the parameter μ.
(b) Determine the stability of the steady states for the entire range of values of μ.
(c) Sketch a bifurcation diagram, indicating stable solutions with a solid curve and unstable solutions with a dashed curve. What type of bifurcations occur? Label these on the diagram.
(d) Discuss the states of the system that would be observed in practice when μ oscillates very slowly between -2 and 2 if solutions $x < 0$ are prohibited. [Assume that the system has 'noise', i.e., that small perturbations occur frequently.]

2. Show that the system

$$\begin{aligned}\frac{dx}{dt} &= y + \frac{x(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}} \\ \frac{dy}{dt} &= -x + \frac{y(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}}\end{aligned}$$

has a stable limit cycle solution, and give the equations for the limit cycle. [Hint: Differentiate $re^{i\theta} = x + iy$ with respect to time to obtain differential equations for r and θ.]

3. Find asymptotic expansions for the roots of $x^2 - (2 + \epsilon)x + 1 + \epsilon = 0$, where $\epsilon > 0$ is a small parameter. Include up to two terms in the expansions where possible.

4. Find a two-term asymptotic expansion for the solution to the initial value problem

$$\frac{d^2y}{dt^2} = 2y\epsilon - 1$$

with $y(0) = 0$ and $y'(0) = 1$.