Methods of Applied Mathematics 1 Qualifying Exam

January 2011

Name:

Please be careful with your sketches and show your work clearly. Be sure to read directions and to include the things that are requested. Books and notes are not allowed, but a calculator is okay.

1. For the system $\dot{x} = x + \frac{rx}{1+x^2}$, sketch all the qualitatively distinct phase portraits that occur as r is varied. Be sure to indicate the stability of all steady states. Determine all bifurcation points and draw a bifurcation diagram. Finally, classify the type of bifurcation or bifurcations that occur.

2. Consider the system

$$\begin{aligned} \frac{dx}{dt} &= x(3-2x-2y) \\ \frac{dy}{dt} &= y(2-x-y) \ . \end{aligned}$$

(a) Determine the x- and y-nullclines and plot them in the x,y-plane.
(b) If a trajectory starts in the first quadrant of the phase plane, can it enter any other quadrant? Why or why not?
(c) Determine all equilibria and their stability, and classify them as node, spiral, etc.
(d) Plot a phase portrait in the first quadrant of the phase plane. Include nullclines, representative trajectories, and steady states (and their stability).
(e) Determine the equation for the stable manifold of the saddle point.

3. Determine which of the two systems below is a gradient system. For the gradient system, determine the potential function $V(x,y)$ and the equation for the equipotential curves. Draw a few of these curves in the x,y-plane and plot a few trajectories.

(a)

$$\begin{aligned} \frac{dx}{dt} &= y + x^2 y \\ \frac{dy}{dt} &= -x + 2xy \ . \end{aligned}$$

(b)

$$\begin{aligned} \frac{dx}{dt} &= -2xe^{x^2+y^2} \\ \frac{dy}{dt} &= -2ye^{x^2+y^2} \ . \end{aligned}$$

4. For small positive ϵ, determine two terms in the asymptotic expansions of the roots of $x^3 - (3+\epsilon)x - 2 + \epsilon = 0$.