Applied Math Prelim

January 2006

Methods of Applied Mathematics I

Instructions: Work out Problem #3 and, either one of Problem #5 and #6 (but not both). In addition, work out two of the remaining three problems). Show your work. Manage your time wisely, so that you are able to demonstrate your knowledge and make sufficient progress on all four problems chosen. No calculators are allowed that have capabilities for graphics or symbolic computation, or that are programmable. However, you may use a simple scientific calculator.

1. Consider the nonlinear system in R^2

$dx/dt = x\,(2 - x - y)$

$dy/dt = x - y$

(a) Find the fixed points.
(b) Linearize the system about each of the fixed points; and, classify them, stating their stability precisely (e.g., distinguish between neutral stability and asymptotic stability). Also, classify the fixed points of the nonlinear system.
(c) Sketch the nullclines of the nonlinear system, and the vector field on them. Then, complete the vector field on the phase plane.
(d) Sketch the phase plane trajectories for the nonlinear system.
(e) Where does the trajectory that passes through the coordinates $(x, y) = (-1, 0)$ end up (i.e., as $t \to \infty$)?

2. Analyze the nonlinear system: $d^2x/dt^2 + (dx/dt)^2 + x = 0$

Explicitly consider: (a) the fixed points (and classify them); (b) the symmetry; (c) whether the system is conservative; and, (d) whether the system has a separatrix (separating different dynamical regimes), and, if it does, then find an equation for it.

3. Use a two-timing method to find an asymptotic approximation (to zeroth order in ε, for $0 < \varepsilon << 1$) to the solution of the ODE initial-value problem

$d^2x/dt^2 + \varepsilon(x^2 - 1)(dx/dt) + 4x = 0$, where, $(x, dx/dt) = (1, 0)$ at $t = 0$.

What happens as $t \to \infty$?

Note: You may want to consider using complex notation for the calculation. The following identity may be useful.

$$\frac{1}{x(1-x^2)} = \frac{1}{x} + 0.5\left(\frac{1}{1-x} - \frac{1}{1+x}\right)$$

4. Consider the first order system: $dx/dt = \sinh(x) - rx$, where r is a system parameter that can be varied.

a) Draw all the qualitatively different vector fields of the nonlinear system.
b) Sketch the (global) bifurcation diagram (i.e., for any real value of x and r), include the nonlinear stability.
c) Describe the significance of the bifurcation diagram to the solution of the nonlinear ODE.

5. Using a Lyapunov function of the form $V = ax^2 + by^2$, determine (if possible) the stability of the origin of the phase plane for the nonlinear dynamical system

$dx/dt = x^3 - y^3$

$dy/dt = 2x y^2 + 4x^2y + 2y^3$

6. Find an equation of the form $dx/dt = f(x)$, possessing only three fixed points located at $x = -1$, 0, and 2, such that they are semi-stable, asymptotically stable, and unstable, respectively.