Methods of Applied Mathematics I **Qualifying Exam Fall 2023**

Solve ALL problems. Notes, books, calculators, and any other electronic devices are NOT allowed.

1. Consider a Brusselator system for the chemical reactants x, y given by

$$\begin{aligned} \dot{x} &= 1 - 4x + x^2 y \\ \dot{y} &= 3x - x^2 y \end{aligned}$$

   (a) Show that the quadrilateral bounded by the lines $x = 0$, $y = 0$, $y = x + 8$, and $y = 10 - x$ is a trapping region.

   (b) Find the equilibria and determine their linear stability.

   (c) Show that the system has at least one limit cycle.

2. Consider Rayleigh's equation

$$\begin{aligned} &\ddot{x} + \varepsilon\Big(\frac{1}{3}\dot{x}^3 - \dot{x}\Big) + x = 0 \\ &x(0) = 1 \quad \dot{x}(0) = 0. \end{aligned}$$

   with $0 < \varepsilon \ll 1$. Use two-timing or averaging to solve the equation with $O(\varepsilon)$ error. That is, fully obtain the $O(1)$ term of the asymptotic series of the solution.

3. Consider the Ornstein–Uhlenbeck process

$$dX = -X dt + dW \qquad X(0) = 1$$

   Determine the mean and variance for $X(t)$. Conjecture what the distribution for $X(t)$ approaches as $t \to \infty$.

4. Consider Griffith's model for a genetic control system where x and y are proportional to the concentration of protein and the messenger RNA from which it is translated, respectively, and $\mu > 0$ is a rate constant

$$\begin{aligned} \dot{x} &= y - \mu x \\ \dot{y} &= \frac{x^2}{1 + x^2} - y \end{aligned}$$

   (a) Show that the system has three fixed points when $\mu < \mu_c$ and one when $\mu > \mu_c$, where μ_c is to be determined.

   (b) What type of bifurcation occurs at $\mu = \mu_c$?

5. Let $\dot{x} = f(x)$ be a 1D flow.

   (a) Let $V(x)$ be the potential function for $\dot{x} = f(x)$. What is the relationship between $V(x)$ and $f(x)$?

   (b) Use the existence of the potential function $V(x)$ to show that the solutions $x(t)$ cannot oscillate.