**Methods of Applied Mathematics I** **Qualifying Exam Fall 2022**

Solve ALL problems. Notes, books, calculators, and any other electronic devices are NOT allowed.

1. Consider Selkov's model of glycolysis, the process whereby living cells break down sugar, where $x$ and $y$ are concentrations of ADP and F6P. Show that it has a nonconstant periodic solution.

$$\dot{x} = -x + \frac{y}{10} + x^2 y$$
$$\dot{y} = \frac{1}{2} - \frac{y}{10} - x^2 y$$

[Hint: show the polygon with vertices $(0,0)$, $(0,5)$, $(\frac{1}{2},5)$, and $(\frac{11}{2},0)$ is forward invariant (i.e., a trapping region).]

2. Consider Rayleigh's equation

$$\ddot{x} + \varepsilon\left(\frac{1}{3}\dot{x}^3 - \dot{x}\right) + x = 0$$
$$x(0) = 1 \quad \dot{x}(0) = 0.$$

with $0 < \varepsilon \ll 1$.

(a) Explain why regular perturbation theory fails for this problem.

(b) Use two-timing or averaging to solve the equation with $O(\varepsilon)$ error. That is, fully obtain the $O(1)$ term of the asymptotic series of the solution.

3. Consider the Ornstein–Uhlenbeck process

$$dX = -kX\,dt + \sqrt{D}dW \qquad X(0) = 1$$

where $k, D \in \mathbb{R}$ are fixed constants.

(a) Solve the SDE.

(b) Determine the mean and variance for $X(t)$.

(c) Using your results from part (b), describe the stationary distribution for $X$. That is, what does the distribution for $X$ approach as $t \to \infty$?

4. Consider the 1D flow

$$\dot{x} = \frac{2x^2}{x^2+2} - \gamma x$$

with $\gamma > 0$.

(a) Determine the equilibria of the system and the stability of each equilibrium.

(b) Plot the bifurcation diagram with $\gamma$ treated as the bifurcation parameter. Hence, show that the system is bistable over a range of values of $\gamma$.

(c) What types of bifurcations occur as $\gamma$ is varied?

5. Consider the nonlinear dynamical system

$$\dot{x} = x(3 - x - y)$$
$$\dot{y} = y(2 - x - y)$$

(a) Find all equilibria of the system.

(b) Determine the stability of each equilibrium and draw the phase portrait.