Qualifying test

Solve any five problems; present a clear solution with all the relevant details and references

In what follows, m_p denotes the Lebesgue measure on $\mathbb{R}^p$.

Problem 1: Assume $\mathfrak{A}$ is a σ-algebra, and $\mu\colon \mathfrak{A} \to [0,\infty]$ with $\mu(\emptyset) < \infty$. Assume further that μ is finitely additive, and for any sequence $\{A_n\}$ of sets from $\mathfrak{A}$ we have

$$A_1 \subset A_2 \subset A_3 \subset \dots \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

Prove that μ is a measure.

Problem 2: Let $(X, \mathfrak{A}, \nu)$ be a measure space and f be a non-negative measurable function on X. For all measurable sets E define

$$\mu(E) = \int_E f d\nu.$$

Assume g is a summable function with respect to μ. Show that

$$\int_X g d\mu = \int_X fg d\nu.$$

Problem 3: Let $(X, \mathfrak{A}, \nu)$ be a measure space and $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Assume that $f_n(x) \to f(x)$ for μ-a.e. $x \in X$, and that

$$\int_X f^2 d\mu < \infty.$$

Show that

$$\lim_{n\to\infty} \int_X f_n^2 d\mu = \int_X f^2 d\mu \quad \text{if and only if} \quad \lim_{n\to\infty} \int_X |f_n - f|^2 d\mu = 0$$

Problem 4: Compute

$$\sum_{n=1}^{\infty} \int_{[0,\pi/4]} (1 - \sqrt{\sin x})^n \cos(x) dm_1(x)$$

and justify your answer.

Problem 5: Assume f is a summable function on $(\mathbb{R}, \mathfrak{M}_1, m_1)$. For every $x \in [0,1]$ define

$$g(x) = \int_{[x,1]} \frac{f(t)}{t} dm_1(t).$$

Show that g is summable on $[0,1]$ and show that

$$\int_{[0,1]} g dm_1 = \int_{[0,1]} f dm_1.$$

Problem 6: Let f be an m_1-measurable function on $[0,1]$ with

$$\int_{[0,1]} |f|^5 dm_1 < \infty.$$

For $t \in [0, \infty)$ define

$$\lambda(t) = m_1\left(x\colon |f(x)| \geqslant t\right).$$

Show that

$$\int_{[0,\infty]} \lambda(t) dm_1(t) < \infty.$$

Problem 7: Assume f is a non-decreasing continuous function on $[0,1]$, and for every $\varepsilon > 0$ we know that f is absolutely continuous on $[\epsilon, 1]$. Is it true that f is absolutely continuous on $[0,1]$?