Qualifying test

Solve any five problems; present a clear solution with all the relevant details and references

In what follows, m_p denotes the Lebesgue measure on $\mathbb{R}^p$.

Problem 1:. Let $E \subset \mathbb{R}$ be a Lebesgue measurable set, and for any interval I we have

$$m_1(E \cap I) \leqslant \frac{7}{8} m_1(I).$$

Prove that $m_1(E) = 0$.

Problem 2:. Let f be a non-negative continuous function on $\mathbb{R}$, and

$$\int_{\mathbb{R}} f dm_1 < \infty.$$

Is it true that f is bounded on $\mathbb{R}$?

Problem 3:. Let f be a non-negative measurable function on $\mathbb{R}$, and

$$\int_{\mathbb{R}} f dm_1 = 2021.$$

Show that the following limit exists, and compute it:

$$\lim_{n\to\infty} \int n \cdot \log\left(1 + \frac{f^3}{n^3}\right) dm_1.$$

Hint: for any $x \geqslant 0$, we have $1 + x^3 \leqslant (1+x)^3$.

Problem 4:. Consider a measure space $(X, \mathfrak{A}, \mu)$, and $\{f_n\}_{n=1}^{\infty}$ is a sequence of real-valued measurable functions on X.

a. Assume $\mu(X) = 1$, and the sequence $\{f_n\}_{n=1}^{\infty}$ converges to zero almost everywhere. Is it true that it converges to zero in measure?

b. Assume $\mu(X) = \infty$, and the sequence $\{f_n\}_{n=1}^{\infty}$ converges to zero almost everywhere. Is it true that it converges to zero in measure?

Problem 5: Let $(X, \mathfrak{A}, \mu)$ be a σ-finite measure space, and $f\colon X \to [0, \infty)$ be a measurable function. For $a \in \mathbb{R}$ define $h(a) = \mu(\{x\colon f(x) > a\})$. Prove that h is continuous m_1-almost everywhere, and that

$$\int_X f d\mu = \int_{[0,\infty)} h(a) dm_1(a).$$

As always, m_1 denotes the Lebesgue measure on $\mathbb{R}$.

Problem 6:. Denote the total variation of a function f on an interval $[a, b]$ by $TV(f_{[a,b]})$. Assume $TV(f_{[a,b]}) < \infty$ and define $v(x) = TV(f_{[a,x]})$, $x \in [a, b]$. Show that $|f'| \leqslant v'$ almost everywhere on $[a, b]$ and deduce that

$$\int_{[a,b]} |f'| dm_1 \leqslant v(b).$$

Problem 7: Let $f(x) = x^2 \sin(1/x)$, $g(x) = x^2 \sin(1/x^2)$, and $f(0) = g(0) = 0$. Show that both f and g are differentiable everywhere on $\mathbb{R}$. Then show that $TV(f_{[-1,1]}) < \infty$ and $TV(g_{[-1,1]}) = \infty$.