Real Analysis Preliminary Exam

25 August 2005

1. Let F_1, F_2, F_3 be independent σ-fields of subsets of Ω. Show that F_1 and $\sigma(F_2 \cup F_3)$ are independent.

2. Let (Ω, F, μ) be a measure space and $f : \Omega \to \mathbb{R}$ Borel measurable. Show if f^2 is integrable, then f is integrable in the case that $\mu(\Omega) < \infty$, but not in general.

3. Let $\{X_n\}$ be a sequence of random variables on $((0,1], B(0,1], \lambda)$. Here $B(0,1]$ are the Borel subsets and λ is Lebesgue measure. Suppose that $\{X_n\}$ converges in probability to X. Define $X'_n(t) = X_n(t)/t$ and $X'(t) = X(t)/t$. Show that $\{X'_n\}$ converges in probability to X'.

4. Prove that Fatou's lemma and the Monotone Convergence Theorem are equivalent, that is, one implies the other.

5. Calculate the following limit and justify your answer.

$$\lim_{n\to\infty} \int_0^{1/2-1/n} (\Sigma_{k=1}^{n} x^{2k+1}) \tan^{-1}(nx)dx.$$