Measure and Integration Preliminary Exam Department of Mathematics Florida State University

August 22, 2010

Please work five problems out of seven. Clearly indicate which problems are to be graded.

1. Calculate the integral. Justify all steps.

$$\lim_{n\to\infty}\int_0^\infty \frac{dx}{(1+\frac{x}{n})^n}.$$

2. Let $f(x,y) = \frac{xy}{(x^2+y^2)^2}$ and $S = \{(x,y) | |x| < 1, |y| < 1\}$. Show that f is not integrable over S with respect to 2-dimensional Lebesgue measure yet the iterated integrals over S exist and are equal.

3. Let $1 \leq p < \infty, \delta > 0$ and suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous with compact support. Define

$$f_\delta(x) = \frac{1}{\delta}\int_x^{x+\delta} f(t)dt.$$

Show that

$$\lim_{\delta\to 0} \|f - f_\delta\|_p = 0.$$

Here $\|\cdot\|_p$ is the norm in $L^p(\mathbb{R})$.

4. Assuming the notation, hypotheses and result of Problem 3, show that

$$\|g_\delta\| \leq \|g\|_p.$$

(Hint : Minkowski's inequality.) Using the density of the continuous compactly supported functions in L^p, show that the result of Problem 3 extends to the L^p spaces.

5. Let $F : \mathbb{R} \to \mathbb{R}$ be continuous, increasing with $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$. Show that

a) $\int_{-\infty}^{\infty} F(x)dF(x) = 1/2$, (Stieltjes integral)

b) $\int_{-\infty}^{\infty}(F(x+c) - F(x))dx = c$.

6. Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable spaces and $T : \Omega \to \Omega'$. Define $T^{-1}\mathcal{F}' = \{T^{-1}A' | A' \in \mathcal{F}'\}$ and $T\mathcal{F} = \{A' | T^{-1}A' \in \mathcal{F}\}$. Show that $T^{-1}\mathcal{F}'$ and $T\mathcal{F}$ are σ-fields. Also show that measurability $\mathcal{F}/\mathcal{F}'$ of T is equivalent to $T^{-1}\mathcal{F}' \subset \mathcal{F}$ and to $\mathcal{F}' \subset T\mathcal{F}$.

7. Let $f \in L^1(\mathbb{R})$. Show that there exists continuous, integrable functions g_n such that $g_n(x) \to f(x)$ except on a set of Lebesgue measure zero. (Hint : use the fact that continuous compactly supported functions are dense in $L^1(\mathbb{R})$.)