Analysis Qualifying Exam, August 22, 2020
Part 2: Measure and Integration

Problem 1: Assume f is an increasing and continuous function on a closed interval I, and μ_f is the measure defined by f; i.e., for every interval $[a,b) \subset I$ define $\mu_f([a,b)) = f(b) - f(a)$, and take the Lebesgue–Caratheodory extension (this procedure is assumed to be known).

Take any set $E \subset I$ measurable with respect to μ_f and any number $a \in (0, \mu_f(E))$. Show that there exists a measurable set $E_a \subset E$ such that $\mu_f(E_a) = a$.

Problem 2: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of non-negative measurable functions on a measure space $(X, \mathfrak{A}, \mu)$. Assume $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a function f, and assume

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu < \infty.$$

Prove that for every measurable set $E \subset X$ we have

$$\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu.$$

Problem 3: Let $(X, \mathfrak{A}, \mu)$ be a σ-finite measure space, and $f\colon X \to [0,\infty]$ be a measurable function. Define $h(a) = \mu(\{x\colon f(x) > a\})$. Prove that h is continuous almost everywhere, and that

$$\int_X f d\mu = \int_{[0,\infty)} h(a) dm_1(a).$$

As always, m_1 denotes the Lebesgue measure on $\mathbb{R}$.

Problem 4: Let $A \subset [0,1]$ be a measurable set with respect to m_1, such that for every interval $I \subset [0,1]$ we have

$$0 < m_1(A \cap I) < m_1(I).$$

You can assume the existence of such a set.

Define

$$F(x) := m_1(A \cap [0,x]).$$

Show that F is a strictly increasing and absolutely continuous function, but $F'(x)$ is equal to zero on a set of positive measure.

Hint: use the Lebesgue differentiation theorem.