Analysis Qualifying Exam

Part II: Measure and Integration, January 2018

Do all problems. Give complete proofs or explanations, stating any theorems you use. Write on one side of the paper and begin each problem on a new sheet.

Notation: $(\mathbb{R}^d, \mathcal{L}, m)$ denotes the usual d-dimensional Lebesgue measure space.

1. (a) Construct a sequence $\{f_n\}$ of non-negative functions on $[0,1]$ such that

$$\int_0^1 f_n(x)\,dx \to 0 \text{ as } n \to \infty,$$

but the sequence $\{f_n(x)\}$ converges for no $x \in [0,1]$.

(b) Construct a non-negative real-valued function on $[0,\infty)$ such that

$$\int_0^\infty f(x)\,dx < \infty$$

but $f(x)$ does not converge to zero as $x \to \infty$.

2. Let E be a Lebesgue measurable subset of $\mathbb{R}^d$ and for $n = 1, 2, 3, ...$ let $f_n : E \to \mathbb{R}$ be non-negative and Lebesgue measurable.

Suppose $\sum_{n=1}^\infty \int_E f(x)\,dx < \infty$. Prove that $\sum_{n=1}^\infty f_n(x)$ converges to a finite sum for a.e. x.

3. Suppose that K is a compact subset of the unit square $[0,1]^2$ in $\mathbb{R}^2$ and the two-dimensional Lebesgue measure of K is greater than $1/2$. Show that there are infinitely many lines L in $\mathbb{R}^2$ such that $m_L(K \cap L) > 1/2$, where m_L is the one-dimensional Lebesgue measure on the line L.

4. Suppose f is a nonnegative Lebesgue measurable function on $\mathbb{R}$. For each integer k, define $E_k = \{x \in \mathbb{R} : f(x) \geq 2^k\}$.

Prove:

$$\int_{\mathbb{R}} f(x)\,dx < \infty \text{ if and only if } \sum_{k=-\infty}^{\infty} 2^k\, m(E_k) < \infty.$$

Hint: Let $F_k = \{x \in \mathbb{R} : 2^k \leq f(x) < 2^{k+1}\}$ and show that

$$\sum_{k=-\infty}^{\infty} 2^k\, m(F_k) \leq \int_{\mathbb{R}} f(x)\,dx \leq 2 \sum_{k=-\infty}^{\infty} 2^k\, m(F_k).$$